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HIGHER ORDER EXPANSIONS IN FINITE ION SIZE VIA POISSON-NERNST-PLANCK SYSTEMS WITH BIKERMAN'S LOCAL HARD-SPHERE POTENTIAL

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract Finite ion sizes play significant roles in characterizing ionic flow properties of interest, such as the selectivity of ion channels. As an extension of the work done in [Discrete Contin. Dyn. Syst. Ser. B, **21** (2016), 1775-1802], we further investigate the higher order (in the volume of the cation), mainly the second order, contributions from finite ion sizes to ionic flows in terms of both the total flow rate of charges and the individual fluxes. This is particularly important since the first-order terms approach zero as the left boundary concentration is close to the right one for the same ion species. The interaction between the first-order terms and the second-order terms is characterized in detail. Moreover, several critical potentials are identified, and they play critical roles in examining the qualitative properties of ionic flows. Some can be estimated experimentally. The analysis in this work could provide complementary information and better understanding of the mechanism of ionic flows through ion channels. Numerical simulations are performed to provide intuitive illustration of our analytical results.

Keywords I-V relations, individual fluxes, critical potentials, finite ion sizes.

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1. Introduction

Mathematical analysis plays unique and significant role in understanding the mechanics of phenomena arising from life science and discovering new features under the assumption that a more or less explicit solution of the associated mathematical model can be obtained. In this work, we analyze the qualitative properties of ionic flows through ion channels via Poisson-Nernst-Planck (PNP) systems with Bikerman's local hard-sphere potential ([9]). Of particular interest is to examine finite ion size effects on ionic flows from higher order expansions (in the volume of the cation) in terms of the total flow rate of charges (I-V relations) and individual fluxes.

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PNP system is a basic macroscopic model for electrodiffusion of charges through ion channels ([12,16–19,25,26,29,35,36,58,59,61,62], etc.). Under various reasonable conditions, one can derive the PNP system as a reduced model from molecular dynamics, Boltzmann equations, and variational principles ([3,32,33,66]).

The simplest PNP system is the classical PNP system, which treats ions as *point-charges*, and neglects ion-to-ion interaction. It has been simulated and analyzed to a great extent (see, e.g., [1, 4, 5, 8, 10–15, 21, 23, 25, 26, 28, 30, 31, 36–41, 44, 49–52, 55–57, 60, 63–65, 67–70, 72–79]). However, finite ion sizes perform significant roles in understanding mechanism of ionic flows through membrane channels. A lot of structural properties of ion channels, such as *selectivity*, rely on ion sizes critically. For instance, Na^+ (sodium) and K^+ (potassium), having the *same* valence, are mainly distinguished by their ionic sizes. This is extremely important when the PNP model involves two cations with the same valence but distinct ion sizes, which provides important information for the selectivity phenomena of ion channels ([6]). To examine ion size effects on ionic flows, one must consider ion-specific components of the electrochemical potential in the PNP models.

The PNP type models with ion sizes have been investigated computationally and analytically for ion channels and have shown great success ([2,6,7,20,22,24,26,27,32–34,42,43,45–47,53,54,71,80], etc.). Recently, the authors of [42] provided an analytical treatment of a one-dimensional PNP system with two oppositely charged ion species and Bikerman's local hard-sphere potential ([9]) to account for finite ion size effects. They treated the model as a singularly perturbed system and rigorously established the existence and uniqueness results of the boundary value problem for small ion sizes. Furthermore, in [42], treating ion sizes as small parameters, the authors derived approximations of the I-V relation and the individual fluxes of the following form (up to the first order in the volume of the cation)

$$\mathcal{I}(V) = \mathcal{I}_0(V) + \nu\mathcal{I}_1(V) + o(\nu), \quad \mathcal{J}_k(V) = \mathcal{J}_{k0}(V) + \nu\mathcal{J}_{k1}(V) + o(\nu),$$

where ν is the volume of the cation and $\mathcal{J}_k$ is the individual flux. Detailed analysis of the leading terms $\mathcal{I}_1(V)$ and $\mathcal{J}_{k1}$ containing ion size effects was provided. Several critical potentials that either balance the ion size effects or separate the relative ion size effects are identified, which plays critical roles in the study. Particularly, the authors observed that the first-order term $\mathcal{I}_1(V)$ approaches zero as the left boundary concentration is close enough to the right one for the same ion species (that is, either $L_1 \to R_1$ or $L_2 \to R_2$ for two ion species case, which is equivalent under electroneutrality conditions $z_1L_1 = -z_2L_2 := L$ and $z_1R_1 = -z_2R_2 := R$). To better understand ion size effects on ionic flows for both the cases with $L \neq R$ and $L \to R$, in this work, we focus on

- the second-order terms in ν, more precisely, $\mathcal{I}_2(V)$ and $\mathcal{J}_{k2}(V)$ for $k = 1, 2$;
- the interaction between the first-order and second-order terms (the effect from the combination);
- the characterization of ion size effects close to $L = R$.

We would like to comment that the study in this work is not just a technical extension of the previous analysis in [42], instead, it is based on the understanding of the model. Particularly, the discovery of the first integrals for the second order nonlinear limiting fast system (stated in Lemma 3.1) is critical for our study. Compared to the work done in [42], the analysis on higher order expansions, particularly the interaction between the first order and the second order expansions provides better understanding of internal dynamics of ionic flows through membrane channels. Most importantly, the study provides an efficient way to adjust boundary concentrations to reduce/enhance the finite ion size effects on ionic flows. On the other hand, the study of the higher order expansion provides more detailed and precise information of the effect on ionic flows from finite ion sizes, and this could provide better initial guessing for related numerical studies to observe rich dynamics of ionic flows.

Remark 1.1. To avoid confusion for readers, we point out that in [42], for the entire Section 4, V should be changed to $\mathcal{V}$ to be consistent with the set-ups. Noticing this, in current work, we make suitable changes in (2.2) and (2.6).

The rest of this paper is organized as follows. In Section 2, we describe the one-dimensional PNP model for ion flows, and the setup of the boundary value problem of the singularly perturbed PNP system. In Section 3, we focus on the asymptotic dynamics of the second order limiting PNP systems in the volume ν of the cation. Section 4 deals with the discussion on finite ion size effects, which consists of three parts. In Section 4.1, a number of critical potentials are identified and their roles in studying finite ion size effects on ionic flows are characterized in details. In Section 4.2, we discuss the essential ion size effects from the first-order and second-order terms. In Section 4.3, our interest lies in the case studies of ion size effects near $L = R$. In Section 4.4, numerical simulations are performed to further illustrate our results. Some remarks are provided in Section 5.

2. Problem Setup

2.1. A quasi-one-dimensional steady-state PNP type system

A quasi-one-dimensional *steady-state* PNP model for ion flows of n ion species through a single channel ([52, 57]) is

$$\begin{aligned}
&\frac{1}{A(X)}\frac{d}{dX}\left(\varepsilon_r(X)\varepsilon_0 A(X)\frac{d\Phi}{dX}\right) = -e\left(\sum_{k=1}^{n} z_k C_k(X) + Q(X)\right),\\
&\frac{d\mathcal{J}_k}{dX} = 0, \quad -\mathcal{J}_k = \frac{1}{k_BT}\mathcal{D}_i(X)A(X)C_i(X)\frac{d\mu_i}{dX}, \quad k = 1, 2, \cdots, n,
\end{aligned} \tag{2.1}$$

where e is the elementary charge, k_B is the Boltzmann constant, T is the absolute temperature; Φ is the electric potential, $Q(X)$ is the permanent charge of the channel, $\varepsilon_r(X)$ is the relative dielectric coefficient, ε_0 is the vacuum permittivity; $A(X)$ is the area of the cross-section of the channel over the point $X \in [0, l]$; for the k th ion species, C_k is the concentration, z_k is the valence, μ_k is the electrochemical potential, $\mathcal{J}_k$ is the flux density, and $\mathcal{D}_k(X)$ is the diffusion coefficient. The boundary conditions are, for $k = 1, 2, \cdots, n$,

$$\Phi(0) = V, \quad C_k(0) = \mathcal{L}_k > 0; \quad \Phi(l) = 0, \quad C_k(l) = \mathcal{R}_k > 0. \tag{2.2}$$

For a solution of the *steady-state* boundary value problem of (2.1)-(2.2), the *total flow rate of charge through a cross-section* or *current* $\mathcal{I}$ is

$$\mathcal{I} = \sum_{k=1}^{n} z_k\mathcal{J}_k. \tag{2.3}$$

For fixed boundary concentrations $\mathcal{L}_k$ and the $\mathcal{R}_k$, the current $\mathcal{I}$ only depends on the electric potential V, and this is the so-called *I-V relations* (current-voltage relations).

2.2. The BVP and assumptions

We will take essentially the same assumptions as those in [42]. To be specific,

(i) considering one cation with $z_1 > 0$ and one anion with $z_2 < 0$;

(ii) assuming the permanent charge $Q(X)$ to be zero over the whole interval;

(iii) for the electrochemical potential μ_k, in addition to the ideal component μ_k^{id} defined by

$$\mu_k^{id}(X) = z_k e\Phi(X) + k_B T \ln \frac{C_k(X)}{C_0}, \tag{2.4}$$

with some characteristic number density C_0, which can be taken as

$$C_0 = \max\{\mathcal{L}_k, \mathcal{R}_k : k = 1, 2\},$$

we also include the local hard-sphere potential μ_i^{Bik} defined by

$$\mu_k^{Bik}(X) = -k_B T \ln\Big(1 - \sum_{j=1}^{n} \nu_j C_j(X)\Big), \tag{2.5}$$

where ν_j is the volume of a single jth ion species;

(iv) assuming that the relative dielectric coefficient and the diffusion coefficient are constants, namely, $\varepsilon_r(X) = \varepsilon_r$ and $\mathcal{D}_k(X) = \mathcal{D}_k$.

We first make a dimensionless rescaling ([23]), more precisely, we let

$$\begin{aligned}
&\varepsilon^2 = \frac{\varepsilon_r \varepsilon_0 k_B T}{e^2 l^2 C_0}, \quad x = \frac{X}{l}, \quad h(x) = \frac{A(X)}{l^2}, \quad D_k = l C_0 \mathcal{D}_k;\\
&\phi(x) = \frac{e}{k_B T}\Phi(X), \quad c_k(x) = \frac{C_k(X)}{C_0}, \quad J_k = \frac{\mathcal{J}_k}{D_k};\\
&\bar{V} = \frac{e}{k_B T} V, \quad L_k = \frac{\mathcal{L}_k}{C_0}, \quad R_k = \frac{\mathcal{R}_k}{C_0}.
\end{aligned} \tag{2.6}$$

Together with the expressions for μ_k^{id} in (2.4) and μ_k^{Bik} in (2.5), system (2.1) becomes

$$\begin{aligned}
&\frac{\varepsilon^2}{h(x)} \frac{d}{dx}\Big(h(x)\frac{d}{dx}\phi\Big) = -(z_1 c_1 + z_2 c_2), \quad \frac{dJ_k}{dx} = 0,\\
&\frac{dc_1}{dx} = -f_1(c_1, c_2; \nu_1, \nu_2)\frac{d\phi}{dx} - \frac{1}{h(x)} g_1(c_1, J_1, J_2; \nu_1, \nu_2),\\
&\frac{dc_2}{dx} = f_2(c_1, c_2; \nu_1, \nu_2)\frac{d\phi}{dx} - \frac{1}{h(x)} g_2(c_2, J_1, J_2; \nu_1, \nu_2),
\end{aligned} \tag{2.7}$$

with boundary conditions, for $k = 1, 2$,

$$\phi(0) = \bar{V}, \quad c_k(0) = L_k > 0; \quad \phi(1) = 0, \quad c_k(1) = R_k > 0, \tag{2.8}$$

where

$$f_1 = (z_1 - z_1\nu_1 c_1 - z_2\nu_2 c_2)c_1, \quad f_2 = -(z_2 - z_1\nu_1 c_1 - z_2\nu_2 c_2)c_2,$$
$$g_1 = J_1 - (\nu_1 J_1 + \nu_2 J_2)c_1, \quad g_2 = J_2 - (\nu_1 J_1 + \nu_2 J_2)c_2.$$

Upon introducing $u = \varepsilon\dot{\phi}$ and $\tau = x$, one gets the standard singularly perturbed system, the so-called *slow system* with state variables $(\phi, u, c_1, c_2, J_1, J_2, \tau) \in \mathbb{R}^7$

$$\begin{aligned}
&\varepsilon\dot{\phi} = u, \quad \varepsilon\dot{u} = -z_1c_1 - z_2c_2 - \varepsilon\frac{h_\tau(\tau)}{h(\tau)}u,\\
&\varepsilon\dot{c}_1 = -f_1(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_1(c_1, J_1, J_2; \nu_1, \nu_2),\\
&\varepsilon\dot{c}_2 = f_2(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_2(c_2, J_1, J_2; \nu_1, \nu_2),\\
&\dot{J}_1 = \dot{J}_2 = 0, \quad \dot{\tau} = 1.
\end{aligned} \tag{2.9}$$

For $\varepsilon > 0$, the rescaling $x = \varepsilon\xi$ yields the *fast system*

$$\begin{aligned}
&\phi' = u, \quad u' = -z_1c_1 - z_2c_2 - \varepsilon\frac{h_\tau(\tau)}{h(\tau)}u,\\
&c_1' = -f_1(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_1(c_1, J_1, J_2; \nu_1, \nu_2),\\
&c_2' = f_2(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_2(c_2, J_1, J_2; \nu_1, \nu_2),\\
&J_1' = J_2' = 0, \quad \tau' = \varepsilon,
\end{aligned} \tag{2.10}$$

where prime denotes the derivative with respect to the variable ξ.

3. Asymptotic dynamics of the limiting PNP system

We focus on the limiting fast and slow dynamics of the second order systems in the small parameter $\nu = \nu_1$. In particular, we obtain explicit expressions of J_{k2} and I_2, the second order individual fluxes and I-V relations in ν, which are significant for our study on finite ion size effects to be discussed in Section 4.

3.1. Limiting fast dynamics for the second order

Setting $\varepsilon = 0$ in (2.10) gives *limiting fast system*

$$\begin{aligned}
&\phi' = u, \quad u' = -z_1c_1 - z_2c_2, \quad c_1' = -f_1(c_1, c_2; \nu_1, \nu_2)u,\\
&c_2' = f_2(c_1, c_2; \nu_1, \nu_2)u, \quad J_1' = J_2' = 0, \quad \tau' = 0.
\end{aligned} \tag{3.1}$$

Recall that ν_1 and ν_2 are the volumes of the two ion species. For small $\nu_1 > 0$ and $\nu_2 > 0$, we treat (3.1) as a regular perturbation of that with $\nu_1 = \nu_2 = 0$. While ν_1 and ν_2 are small, their ratio is of order $O(1)$. Correspondingly, we let

$$\nu_1 = \nu \quad \text{and} \quad \nu_2 = \lambda\nu, \tag{3.2}$$

and seek solution $\gamma(\xi;\nu) = \big(\phi(\xi;\nu), u(\xi;\nu), c_1(\xi;\nu), c_2(\xi;\nu), J_1(\nu), J_2(\nu), \tau\big)$ of system (3.1) of the form

$$\begin{aligned}
&\phi(\xi;\nu) = \phi_0(\xi) + \phi_1(\xi)\nu + \phi_2(\xi)\nu^2 + o(\nu^2),\\
&u(\xi;\nu) = u_0(\xi) + u_1(\xi)\nu + u_2(\xi)\nu^2 + o(\nu^2),\\
&c_k(\xi;\nu) = c_{k0}(\xi) + c_{k1}(\xi)\nu + c_{k2}(\xi)\nu^2 + o(\nu^2),\\
&J_k(\nu) = J_{k0} + J_{k1}\nu + J_{k2}\nu^2 + o(\nu^2).
\end{aligned} \tag{3.3}$$

Substituting (3.3) into system (2.10), we obtain

- the zeroth order limiting fast system in ν,

$$\begin{aligned}
&\phi_0' = u_0, \quad u_0' = -z_1c_{10} - z_2c_{20}, \quad c_{10}' = -z_1c_{10}u_0, \quad c_{20}' = -z_2c_{20}u_0,\\
&J_{10}' = J_{20}' = 0, \quad \tau' = 0;
\end{aligned} \tag{3.4}$$

- the first order limiting fast system in ν,

$$\begin{aligned}
&\phi_1' = u_1, \quad u_1' = -z_1c_{11} - z_2c_{21},\\
&c_{11}' = -z_1u_0c_{11} - z_1u_1c_{10} + (z_1c_{10} + \lambda z_2c_{20})c_{10}u_0,\\
&c_{21}' = -z_2u_0c_{21} - z_2u_1c_{20} + (z_1c_{10} + \lambda z_2c_{20})c_{20}u_0,\\
&J_{11}' = J_{21}' = 0, \quad \tau' = 0;
\end{aligned} \tag{3.5}$$

- the second order limiting fast system in ν,

$$\begin{aligned}
\phi_2' =& u_2, \quad u_2' = -z_1c_{12} - z_2c_{22},\\
c_{12}' =& -z_1c_{10}u_2 - z_1c_{11}u_1 - z_1c_{12}u_0 + (z_1c_{10} + \lambda z_2c_{20})(c_{11}u_0 + c_{10}u_1)\\
&+ (z_1c_{11} + \lambda z_2c_{21})c_{10}u_0,\\
c_{22}' =& -z_2c_{20}u_2 - z_2c_{21}u_1 - z_2c_{22}u_0 + (z_1c_{10} + \lambda z_2c_{20})(c_{21}u_0 + c_{20}u_1)\\
&+ (z_1c_{11} + \lambda z_2c_{21})c_{20}u_0,\\
J_{12}' =& J_{22}' = 0, \quad \tau' = 0.
\end{aligned} \tag{3.6}$$

The zeroth order system and the first order system have been studied in [42]. For the second order system, one has the following results.

Lemma 3.1. *System (3.6) has the following nontrivial first integrals:*

$$\begin{aligned}
G_1 =& \frac{c_{12}}{c_{10}} + z_1\phi_2 + u_0u_1 - \frac{c_{11}^2}{2c_{10}^2} + (\lambda-1)c_{21} + \frac{(\lambda-1)z_1}{z_1+z_2}c_{10}c_{20} + \frac{\lambda(\lambda-1)}{2}c_{20}^2,\\
G_2 =& \frac{c_{22}}{c_{20}} + z_2\phi_2 + u_0u_1 - \frac{c_{21}^2}{2c_{20}^2} + (\lambda-1)c_{21} + \frac{(\lambda-1)z_1}{z_1+z_2}c_{10}c_{20} + \frac{\lambda(\lambda-1)}{2}c_{20}^2,\\
G_3 =& c_{12} + c_{22} - u_0u_2 - \frac{u_1^2}{2} + c_{10}c_{11} + \lambda c_{20}c_{21} + \frac{c_{10}^3}{3} + \frac{\lambda^2}{3}c_{20}^3 + \frac{z_1 - z_2\lambda}{z_2 - z_1}c_{10}c_{21}\\
&+ \frac{2\lambda z_2^2 - \lambda z_1z_2 - z_1^2}{z_1(z_2 - z_1)}c_{20}c_{11} + \frac{3\lambda z_2}{2z_1 + z_2}c_{10}^2c_{20} + \frac{\lambda(z_1^2 + 2\lambda z_2^2)}{z_1(z_1 + 2z_2)}c_{10}c_{20}^2.
\end{aligned}$$

Recalling the results for the zeroth and first order systems from [42], together with Lemma 3.1, one has

Proposition 3.1. *Assume that $\nu \geq 0$ is small. One has*

(i) *The stable manifold* $W^s(\mathcal{Z})$ *intersects* B_L *transversally at points* $\big(V, u_0^l + u_1^l\nu + u_2^l\nu^2 + o(\nu^2), L_k, J_k(\nu), 0\big)$ *for* $k=1,2$*, and the* $\omega-$*limit set of* $N^L = M^L \cap W^s(\mathcal{Z})$ *is*

$$\omega(N^L) = \Big\{\big(\phi_0^L + \phi_1^L\nu + \phi_2^L\nu^2 + o(\nu^2), 0, c_{k0}^L + c_{k1}^L\nu + c_{k2}^L\nu^2 + o(\nu^2), J_k(\nu), 0\big)\Big\},$$

where $J_k(\nu) = J_{k0} + J_{k1}\nu + J_{k2}\nu^2 + o(\nu^2)$*,* $k=1,2$*, can be arbitrary. The zeroth order and first order results are (recalled from [42])*

$$\begin{aligned}
\phi_0^L =&\bar{V} - \frac{1}{z_1 - z_2}\ln\frac{-z_2L_2}{z_1L_1}, \quad z_1c_{10}^L = -z_2c_{20}^L = \big(z_1L_1\big)^{\frac{-z_2}{z_1-z_2}}\big(-z_2L_2\big)^{\frac{z_1}{z_1-z_2}},\\
u_0^l =&sgn(z_1L_1 + z_2L_2)\sqrt{2\Big(L_1 + L_2 + \frac{z_1 - z_2}{z_1z_2}(z_1L_1)^{\frac{-z_2}{z_1-z_2}}(-z_2L_2)^{\frac{z_1}{z_1-z_2}}\Big)}
\end{aligned}$$

and

$$\begin{aligned}
\phi_1^L =&0, \qquad z_1c_{11}^L = -z_2c_{21}^L = z_1c_{10}^L\big(L_1 + \lambda L_2 - c_{10}^L - \lambda c_{20}^L\big),\\
u_1^l =&\frac{1}{u_0^l}\bigg(\frac{\lambda}{2}\big(L_2^2 - (c_{20}^L)^2\big) + \frac{1}{2}\big(L_1^2 - (c_{10}^L)^2\big) - \frac{z_2(1-\lambda)}{z_1+z_2}e^{(z_1+z_2)(V-\phi_0^L)}L_1L_2\\
&- c_{10}^Lc_{20}^L - c_{11}^L - c_{21}^L\bigg).
\end{aligned}$$

The result for the second order limiting fast system reads

$$\begin{aligned}
\phi_2^L =& 0,\\
z_1c_{12}^L =& -z_2c_{22}^L\\
=&z_1c_{10}^L\bigg(\frac{(\lambda-1)z_1}{z_1+z_2}L_1L_2 + \frac{L_1^2}{2} + \frac{\lambda^2}{2}L_2^2 - \frac{\lambda^2z_1^2 + z_2^2}{2z_2^2}(c_{10}^L)^2\\
&- \frac{z_1\lambda - z_2}{z_2}c_{11}^L + \frac{z_1z_2(2-\lambda) + z_1^2\lambda}{z_2(z_1+z_2)}(c_{10}^L)^2\bigg),\\
u_2^l =&\frac{L_1^3 - (c_{10}^L)^3}{3u_0^l} + \frac{\lambda^2}{3u_0^l}\left(L_2^3 - (c_{20}^L)^3\right) - \frac{1}{u_0^l}\Big(\frac{1}{2}(u_1^l)^2 + c_{12}^L + c_{22}^L\Big)\\
&+ \frac{(2\lambda-1)z_2^2 - \lambda z_1^2}{z_2^2u_0^l}c_{10}^Lc_{11}^L + \frac{3\lambda z_2}{(2z_1+z_2)u_0^l}\left(L_1^2L_2 - (c_{10}^L)^2c_{20}^L\right)\\
&+ \frac{\lambda z_1^2 + 2\lambda^2z_2^2}{z_1(z_1+2z_2)u_0^l}\left(L_1L_2^2 - c_{10}^L(c_{20}^L)^2\right).
\end{aligned} \tag{3.7}$$

(ii) *The unstable manifold* $W^u(\mathcal{Z})$ *intersects* B_R *transversally at points* $\big(0, u_0^r + u_1^r\nu + u_2^r\nu^2 + o(\nu^2), R_k, J_k(\nu), 1\big)$ *for* $k=1,2$*, and the* $\alpha-$*limit set of* $N^R = M^R \cap W^u(\mathcal{Z})$ *is*

$$\alpha(N^R) = \Big\{\big(\phi_0^R + \phi_1^R\nu + \phi_2^R\nu^2 + o(\nu^2), 0, c_{k0}^R + c_{k1}^R\nu + c_{k2}^R\nu^2 + o(\nu^2), J_k(\nu), 1\big)\Big\},$$

where $J_k(\nu) = J_{k0} + J_{k1}\nu + J_{k2}\nu^2 + o(\nu^2)$*,* $k=1,2$*, can be arbitrary. The zeroth order and first order results are (recalled from [42])*

$$\phi_0^R = -\frac{1}{z_1 - z_2}\ln\frac{-z_2R_2}{z_1R_1}, \quad z_1c_{10}^R = -z_2c_{20}^R = \big(z_1R_1\big)^{\frac{-z_2}{z_1-z_2}}\big(-z_2R_2\big)^{\frac{z_1}{z_1-z_2}},$$

$$u_0^r = sgn\big(z_1R_1 + z_2R_2\big)\sqrt{2\Big(R_1 + R_2 + \frac{z_1 - z_2}{z_1z_2}(z_1R_1)^{\frac{-z_2}{z_1-z_2}}(-z_2R_2)^{\frac{z_1}{z_1-z_2}}\Big)}$$

and

$$\begin{aligned}\phi_1^R =& 0, \qquad z_1c_{11}^R = -z_2c_{21}^R = z_1c_{10}^R\big(R_1 + \lambda R_2 - c_{10}^R - \lambda c_{20}^R\big),\\ u_1^r =& \frac{1}{u_0^r}\Bigg(\frac{\lambda}{2}\big(R_2^2 - (c_{20}^R)^2\big) + \frac{1}{2}\big(R_1^2 - (c_{10}^R)^2\big) - \frac{z_2(1-\lambda)}{z_1+z_2}e^{(z_1+z_2)\phi_0^R}R_1R_2\\ & - c_{10}^Rc_{20}^R - c_{11}^R - c_{21}^R\Bigg).\end{aligned}$$

The result for the second order limiting fast system reads

$$\begin{aligned}\phi_2^R =& 0,\\ z_1c_{12}^R =& -z_2c_{22}^R\\ =& z_1c_{10}^R\Bigg(\frac{(\lambda-1)z_1}{z_1+z_2}R_1R_2 + \frac{R_1^2}{2} + \frac{\lambda^2}{2}R_2^2 - \frac{\lambda^2z_1^2 + z_2^2}{2z_2^2}(c_{10}^R)^2\\ & - \frac{z_1\lambda - z_2}{z_2}c_{11}^R + \frac{z_1z_2(2-\lambda) + z_1^2\lambda}{z_2(z_1+z_2)}(c_{10}^R)^2\Bigg),\\ u_2^r =& \frac{R_1^3 - (c_{10}^R)^3}{3u_0^r} + \frac{\lambda^2}{3u_0^r}\big(R_2^3 - (c_{20}^R)^3\big) - \frac{1}{u_0^r}\Big(\frac{1}{2}(u_1^r)^2 + c_{12}^R + c_{22}^R\Big)\\ & + \frac{(2\lambda-1)z_2^2 - \lambda z_1^2}{z_2^2u_0^r}c_{10}^Rc_{11}^R + \frac{3\lambda z_2}{(2z_1+z_2)u_0^r}\left(R_1^2R_2 - (c_{10}^R)^2c_{20}^R\right)\\ & + \frac{\lambda z_1^2 + 2\lambda^2z_2^2}{z_1(z_1+2z_2)u_0^r}\left(R_1R_2^2 - c_{10}^R(c_{20}^R)^2\right).\end{aligned} \tag{3.8}$$

Remark 3.1. In the expressions $u_j^l, u_j^r, \phi_j^L, \phi_j^R, c_{ij}^L$ and c_{ij}^R for $i = 1, 2$ and $j = 0, 1, 2,$, the superscripts l and L indicate the left limits of the quantities at $x = 0$ while the superscript r and R indicate the right limits of the quantities at $x = 1$.

3.2. Limiting slow dynamics for the second order

Next we construct the regular layer on $\mathcal{Z}$ that connects $\omega(N_L)$ and $\alpha(N_R)$. After suitable treatment (see [42] for details), the limiting slow system reads

$$\begin{aligned}&\dot{\phi} = p, \quad \dot{c}_1 = -f_1(c_1, -\frac{z_1}{z_2}c_1; \nu, \lambda\nu)p - \frac{1}{h(\tau)}g_1(c_1, J_1, J_2; \nu, \lambda\nu),\\ &\dot{J}_1 = \dot{J}_2 = 0, \quad \dot{\tau} = 1,\end{aligned} \tag{3.9}$$

where

$$p = -\frac{z_1g_1(c_1, J_1, J_2; \nu, \lambda\nu) + z_2g_2(-\frac{z_1}{z_2}c_1, J_1, J_2; \nu, \lambda\nu)}{z_1(z_1 - z_2)h(\tau)c_1}.$$

As for the layer problem, we look for solution of (3.9) of the form

$$\begin{aligned}\phi(x) &= \phi_0(x) + \phi_1(x)\nu + \phi_2(x)\nu^2 + o(\nu^2),\\ c_1(x) &= c_{10}(x) + c_{11}(x)\nu + c_{12}(x)\nu^2 + o(\nu^2),\\ J_i &= J_{i0} + J_{i1}\nu + J_{i2}\nu^2 + o(\nu^2),\end{aligned} \tag{3.10}$$

to connect $\omega(N_L)$ and $\alpha(N_R)$ given in Proposition 3.1.

For simplicity, we introduce

$$T_k^m = J_{1k} + J_{2k},\ T_k^c = z_1 J_{1k} + z_2 J_{2k},\ k = 0,1,2 \text{ and } H(x) = \int_0^x \frac{1}{h(s)} ds.$$

From system (3.9), one has

- the zeroth order limiting slow system in ν,

$$\begin{aligned} &\dot{\phi}_0 = -\frac{T_0^c}{z_1(z_1 - z_2)h(\tau)c_{10}}, \quad \dot{c}_{10} = \frac{z_2 T_0^m}{(z_1 - z_2)h(\tau)}, \\ &\dot{J}_{k0} = 0, \quad \dot{\tau} = 1; \end{aligned} \tag{3.11}$$

- the first order limiting slow system in ν,

$$\begin{aligned} &\dot{\phi}_1 = -\frac{T_0^c c_{11}}{z_1(z_1 - z_2)h(\tau)c_{10}^2} - \frac{T_1^c}{z_1(z_1 - z_2)h(\tau)c_{10}}, \\ &\dot{c}_{11} = \frac{(\lambda z_1 - z_2)T_0^m c_{10}}{(z_1 - z_2)h(\tau)} + \frac{z_2 T_1^m}{(z_1 - z_2)h(\tau)}, \quad \dot{J}_{k1} = 0, \quad \dot{\tau} = 1; \end{aligned} \tag{3.12}$$

- the second order limiting slow system in ν,

$$\begin{aligned} \dot{\phi}_2 = & -\frac{T_0^c\big(c_{11}^2 - c_{10}c_{12}\big)}{z_1(z_1 - z_2)h(\tau)c_{10}^3} + \frac{T_1^c c_{11}}{z_1(z_1 - z_2)h(\tau)c_{10}^2} \\ & - \frac{T_2^c}{z_1(z_1 - z_2)h(\tau)c_{10}}, \\ \dot{c}_{12} = & \frac{(\lambda z_1 - z_2)T_0^m c_{11}}{(z_1 - z_2)h(\tau)} + \frac{(\lambda z_1 - z_2)T_1^m c_{10}}{(z_1 - z_2)h(\tau)} + \frac{z_2 T_2^m}{(z_1 - z_2)h(\tau)}, \\ \dot{J}_{k2} = & 0, \quad \dot{\tau} = 1. \end{aligned} \tag{3.13}$$

For the second order limiting slow system, one has

Lemma 3.2. *There is a unique solution* $\big(\phi_2(x), c_{12}(x), J_{12}, J_{22}, \tau(x)\big)$ *of (3.13) such that* $\big(\phi_2(0), c_{12}(0), \tau(0)\big) = \big(0, c_{12}^L, 0\big)$ *and* $\big(\phi_2(1), c_{12}(1), \tau(1)\big) = \big(0, c_{12}^R, 1\big)$*, where* c_{12}^L *and* c_{12}^R *are defined in Proposition 3.1. It is given by*

$$\begin{aligned} &\phi_2(x) = \frac{T_0^c T_2^m - T_2^c T_0^m}{z_1 z_2 (T_0^m)^2}\big(\ln c_{10}(x) - \ln c_{10}^L\big) + \mathcal{P}_1(x), \\ &c_{12}(x) = c_{12}^L + \frac{T_2^m}{T_0^m}\big(c_{10}(x) - c_{10}^L\big) + \mathcal{P}_2(x), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \mathcal{P}_1(x) = & \frac{T_0^c}{2z_1 z_2 T_0^m}\left[\left(\frac{c_{11}(x)}{c_{10}(x)}\right)^2 - \left(\frac{c_{11}^L}{c_{10}^L}\right)^2 - 2\left(\frac{c_{12}(x)}{c_{10}(x)} - \frac{c_{12}^L}{c_{10}^L}\right)\right] \\ & + \frac{(z_1\lambda - z_2)T_1^c}{z_1 z_2^2 T_0^m}\big(c_{10}(x) - c_{10}^L\big) + \frac{T_0^c T_1^m - T_1^c T_0^m}{z_1 z_2 (T_0^m)^2}\left(\frac{c_{11}(x)}{c_{10}(x)} - \frac{c_{11}^L}{c_{10}^L}\right) \\ & + \frac{T_1^m\big(T_1^c T_0^m - T_0^c T_1^m\big)}{z_1 z_2 (T_0^m)^3}\left(\ln c_{10}(x) - \ln c_{10}^L\right), \end{aligned}$$

$$\mathcal{P}_2(x) = \frac{\lambda z_1 - z_2}{z_2}\left(c_{11}(x)c_{10}(x) - c_{11}^L c_{10}^L\right) - \frac{1}{3}\left(\lambda\frac{z_1}{z_2} - 1\right)^2\left(c_{10}^3(x) - (c_{10}^L)^3\right).$$

Here, from [42],

$$\begin{aligned} T_0^c &= \frac{z_1(z_1 - z_2)(c_{10}^R - c_{10}^L)}{H(1)(\ln c_{10}^R - \ln c_{10}^L)}\left(\phi_0^L - \phi_0^R\right), \quad T_0^m = \frac{(z_1 - z_2)(c_{10}^R - c_{10}^L)}{z_2 H(1)}, \\ T_1^c &= \frac{z_1(z_1 - z_2)\left(c_{10}^L - c_{10}^R\right)}{H(1)\left(\ln c_{10}^L - \ln c_{10}^R\right)}N, \quad T_1^m = \frac{z_1 - z_2}{z_2 H(1)}M \end{aligned}$$

with

$$\begin{aligned} M &= c_{11}^R - c_{11}^L + \frac{z_1\lambda - z_2}{2z_2}(c_{10}^L + c_{10}^R)(c_{10}^L - c_{10}^R), \\ N &= \frac{\phi_0^L - \phi_0^R}{\ln c_{10}^L - \ln c_{10}^R}\left(\frac{c_{11}^R}{c_{10}^R} - \frac{c_{11}^L}{c_{10}^L} + \frac{z_1\lambda - z_2}{2z_2}(c_{10}^L - c_{10}^R)\right) - \frac{\phi_0^L - \phi_0^R}{c_{10}^L - c_{10}^R}M. \end{aligned}$$

In particular,

$$\begin{aligned} J_{12} &= \frac{\left(z_2 T_0^m - T_0^c\right)\left(c_{12}^R - c_{12}^L - \mathcal{P}_2(1)\right)}{(z_2 - z_1)(c_{10}^R - c_{10}^L)} - \frac{z_1 z_2 T_0^m}{(z_2 - z_1)(\ln c_{10}^R - \ln c_{10}^L)}\mathcal{P}_1(1), \\ J_{22} &= \frac{\left(T_0^c - z_1 T_0^m\right)\left(c_{12}^R - c_{12}^L - \mathcal{P}_2(1)\right)}{(z_2 - z_1)(c_{10}^R - c_{10}^L)} + \frac{z_1 z_2 T_0^m}{(z_2 - z_1)(\ln c_{10}^R - \ln c_{10}^L)}\mathcal{P}_1(1). \end{aligned}$$

Proof. Taking the integral from 0 to x for the first two equations in (3.13), respectively, together with $c_{12}(0) = c_{12}^L$ and $\phi_2(0) = 0$, one has

$$\begin{aligned} \phi_2(x) =& -\frac{T_0^c}{z_1(z_1 - z_2)}\left(\int_0^x \frac{c_{11}^2(s)}{h(s)c_{10}^3(s)}ds - \int_0^x \frac{c_{12}(s)}{h(s)c_{10}^2(s)}ds\right) \\ &+ \frac{T_1^c}{z_1(z_1 - z_2)}\int_0^x \frac{c_{11}(s)}{h(s)c_{10}^2(s)}ds - \frac{T_2^c}{z_1(z_1 - z_2)}\int_0^x \frac{1}{h(s)c_{10}(s)}ds, \\ c_{12}(x) =& c_{12}^L + \frac{(z_1\lambda - z_2)T_0^m}{z_1 - z_2}\int_0^x \frac{c_{11}(s)}{h(s)}ds + \frac{(z_1\lambda - z_2)T_1^m}{z_1 - z_2}\int_0^x \frac{c_{10}(s)}{h(s)}ds \\ &+ \frac{T_2^m}{T_0^m}\left(c_{10}(x) - c_{10}^L\right). \end{aligned} \tag{3.15}$$

From (3.11) and (3.12), by careful computations, one has

$$\begin{aligned} &\int_0^x \frac{c_{11}(s)}{h(s)c_{10}(s)}ds = \frac{z_1 - z_2}{z_2 T_0^m}\int_0^x \frac{c_{11}(s)\dot{c}_{10}(s)}{c_{10}(s)}ds = \frac{z_1 - z_2}{z_2 T_0^m}\int_0^x c_{11}(s)d\ln c_{10}(s) \\ =& \frac{z_1 - z_2}{z_2 T_0^m}\left(c_{11}(x)\ln c_{10}(x) - c_{11}^L \ln c_{10}^L - \int_0^x \dot{c}_{11}(s)\ln c_{10}(s)ds\right) \\ =& \frac{z_1 - z_2}{z_2 T_0^m}\left(c_{11}(x)\ln c_{10}(x) - c_{11}^L \ln c_{10}^L\right) - \frac{\lambda z_1 - z_2}{z_2}\int_0^x \frac{c_{10}(s)\ln c_{10}(s)}{h(s)}ds \\ &- \frac{T_1^m}{T_0^m}\int_0^x \frac{\ln c_{10}(s)}{h(s)}ds \\ =& \frac{z_1 - z_2}{z_2 T_0^m}\left(c_{11}(x)\ln c_{10}(x) - c_{11}^L \ln c_{10}^L\right) - \frac{(z_1 - z_2)(\lambda z_1 - z_2)}{2z_2^2 T_0^m}\int_0^x \ln c_{10}(s)dc_{10}^2(s) \end{aligned}$$

$$
\begin{aligned}
&- \frac{(z_1-z_2)T_1^m}{z_2(T_0^m)^2}\int_0^x \ln c_{10}(s)dc_{10}(s)\\
&= \frac{z_1-z_2}{z_2T_0^m}\big(c_{11}(x)\ln c_{10}(x) - c_{11}^L\ln c_{10}^L\big) - \frac{(z_1-z_2)(z_1\lambda - z_2)}{2z_2^2T_0^m}\Big(c_{10}^2(x)\ln c_{10}(x)\\
&\quad - (c_{10}^L)^2\ln c_{10}^L - \frac{1}{2}\big(c_{10}^2(x) - (c_{10}^L)^2\big)\Big) - \frac{(z_1-z_2)T_1^m}{z_2(T_0^m)^2}\big(c_{10}(x)\ln c_{10}(x) - c_{10}^L\ln c_{10}^L\\
&\quad - c_{10}(x) + c_{10}^L\big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_0^x \frac{c_{10}(s)}{h(s)}ds &= \frac{z_1-z_2}{2z_2T_0^m}\big(c_{10}^2(x) - (c_{10}^L)^2\big),\\
\int_0^x \frac{c_{11}(s)}{h(s)}ds &= \frac{z_1-z_2}{z_2T_0^m}\Big(c_{11}(x)c_{10}(x) - c_{11}^Lc_{10}^L - \frac{z_1\lambda - z_2}{3z_2}\big(c_{10}^3(x) - (c_{10}^L)^3\big)\Big)\\
&\quad - \frac{(z_1-z_2)T_1^m}{2z_2(T_0^m)^2}\big(c_{10}^2(x) - (c_{10}^L)^2\big),\\
\int_0^x \frac{c_{11}(s)}{h(s)c_{10}^2(s)}ds &= \frac{z_1-z_2}{z_2T_0^m}\left(\frac{c_{11}^L}{c_{10}^L} - \frac{c_{11}(x)}{c_{10}(x)}\right) + \frac{(z_1-z_2)(z_1\lambda - z_2)}{z_2^2T_0^m}\big(c_{10}(x) - c_{10}^L\big)\\
&\quad + \frac{(z_1-z_2)T_1^m}{z_2(T_0^m)^2}\big(\ln c_{10}(x) - \ln c_{10}^L\big),\\
\int_0^x \frac{c_{11}^2(s)}{h(s)c_{10}^3(s)}ds &= \frac{z_1-z_2}{z_2T_0^m}\left(\frac{(c_{11}^L)^2}{2(c_{10}^L)^2} - \frac{c_{11}^2(x)}{2c_{10}^2(x)}\right) + \frac{z_1\lambda - z_2}{z_2}\int_0^x \frac{c_{11}(s)}{h(s)c_{10}(s)}ds\\
&\quad + \frac{T_1^m}{T_0^m}\int_0^x \frac{c_{11}(s)}{h(s)c_{10}^2(s)}ds,\\
\int_0^x \frac{c_{12}(s)}{h(s)c_{10}^2(s)}ds &= \frac{z_1-z_2}{z_2T_0^m}\left(\frac{c_{12}^L}{c_{10}^L} - \frac{c_{12}(x)}{c_{10}(x)}\right) + \frac{z_1\lambda - z_2}{z_2}\int_0^x \frac{c_{11}(s)}{h(s)c_{10}(s)}ds\\
&\quad + \frac{(z_1\lambda - z_2)T_1^m}{z_2T_0^m}H(x) + \frac{T_2^m}{T_0^m}\int_0^x \frac{1}{h(s)c_{10}(s)}ds,\\
\int_0^x \frac{1}{h(s)c_{10}(s)}ds &= \frac{z_1-z_2}{z_2T_0^m}\big(\ln c_{10}(x) - \ln c_{10}^L\big).
\end{aligned}
$$

Substituting these integrals into (3.15) and regrouping some terms, one obtain the ϕ_2 and c_{12} equations in (3.14). Evaluating ϕ_2 and c_{12} in (3.14) at $x = 1$, one can uniquely solve the two resulting algebraic equations in J_{12} and J_{22}, and obtain their expressions. This completes the proof. □

4. Effects on ionic flows from finite ion sizes

We now examine the finite ion size effect on the I-V relation $\mathcal{I} = z_1D_1J_1 + z_2D_2J_2$ and the individual fluxes $\mathcal{J}_k = D_kJ_k$, $k = 1, 2$ based on the explicit approximations obtained from the solutions to the limiting PNP systems. Of particular interest are i) the ion size effect from the second order terms $\mathcal{I}_2 = z_1D_1J_{12} + z_2D_2J_{22}$ and $\mathcal{J}_{k2} = D_kJ_{k2}$; ii) the interplay between the first order and second order terms (the effect from the combination); and iii) the characterization of ion size effects close to $L = R$.

For our following discussions, we assume electroneutrality boundary conditions

$$z_1L_1 = -z_2L_2 = L, \qquad z_1R_1 = -z_2R_2 = R. \tag{4.1}$$

Lemma 4.1. *Under conditions (4.1), one has*

$$\begin{aligned}&\phi_0^L = V,\ z_1c_{10}^L = -z_2c_{20}^L = L,\ z_1c_{10}^R = -z_2c_{20}^R = R,\\ &\phi_0^R = \phi_1^L = c_{11}^L = c_{21}^L = \phi_1^R = c_{11}^R = c_{21}^R = 0,\ \phi_2^L = c_{12}^L = c_{22}^L = \phi_2^R = c_{12}^R = c_{22}^R = 0.\end{aligned}$$

From Lemmas 3.2 and 4.1, we have (the expressions for J_{k0} and J_{k1} are from [42])

Lemma 4.2. *Assume $L \neq R$. Under conditions (4.1), one has*

$$\begin{aligned}
J_{10} =& \frac{L-R}{z_1H(1)}\Big(1+\frac{e}{k_BT}\frac{z_1V}{\ln L-\ln R}\Big), \quad J_{20} = -\frac{L-R}{z_2H(1)}\Big(1+\frac{e}{k_BT}\frac{z_2V}{\ln L-\ln R}\Big),\\
J_{11} =& \frac{\lambda z_1-z_2}{z_1z_2H(1)}f_0(L,R)f_1(L,R)\frac{e}{k_BT}V - \frac{\lambda z_1-z_2}{2z_1^2z_2H(1)}\big(L^2-R^2\big),\\
J_{21} =& -\frac{\lambda z_1-z_2}{z_1z_2H(1)}f_0(L,R)f_1(L,R)\frac{e}{k_BT}V + \frac{\lambda z_1-z_2}{2z_1z_2^2H(1)}\big(L^2-R^2\big),\\
J_{12} =& \frac{(\lambda z_1-z_2)^2}{z_1^2z_2^2H(1)}f_0(L,R)\Big(f_2(L,R)+\frac{1}{2}f_0^2(L,R)\Big)\frac{e}{k_BT}V + \frac{(\lambda z_1-z_2)^2}{3z_1^3z_2^2H(1)}\big(L^3-R^3\big),\\
J_{22} =& -\frac{(\lambda z_1-z_2)^2}{z_1^2z_2^2H(1)}f_0(L,R)\Big(f_2(L,R)+\frac{1}{2}f_0^2(L,R)\Big)\frac{e}{k_BT}V - \frac{(\lambda z_1-z_2)^2}{3z_1^2z_2^3H(1)}\big(L^3-R^3\big),
\end{aligned}$$

where

$$\begin{aligned}
f_0(L,R) =& \frac{L-R}{\ln L-\ln R}, \quad f_1(L,R) = f_0(L,R) - \frac{L+R}{2},\\
f_2(L,R) =& \frac{1}{3}\big(R^2+RL+L^2\big) - \frac{3(L+R)}{4}f_0(L,R).
\end{aligned}$$

In particular,

$$\begin{aligned}
I_0(V;0) =& \frac{e(z_1D_1-z_2D_2)}{k_BTH(1)}f_0(L,R)V + \frac{D_1-D_2}{H(1)}\big(L-R\big),\\
I_1(V;\lambda,0) =& \frac{e(\lambda z_1-z_2)(z_1D_1-z_2D_2)}{z_1z_2k_BTH(1)}f_0(L,R)f_1(L,R)V\\
&- \frac{(\lambda z_1-z_2)(D_1-D_2)}{2z_1z_2H(1)}\big(L^2-R^2\big),\\
I_2(V;\lambda,0) =& \frac{e(z_1D_1-z_2D_2)(\lambda z_1-z_2)^2}{z_1^2z_2^2k_BTH(1)}f_0(L,R)\Big(f_2(L,R)+\frac{1}{2}f_0^2(L,R)\Big)V\\
&+ \frac{(D_1-D_2)(\lambda z_1-z_2)^2}{3z_1^2z_2^2H(1)}\big(L^3-R^3\big).
\end{aligned} \tag{4.2}$$

To end this section, we introduce the following result, which will be used in our later discussion.

Lemma 4.3. *Assume $L > R$. One has*

(i) $f_0(L,R) > 0,\ f_1(L,R) < 0$;

(ii) *For $f_2(L,R)$, for simplicity, we rewrite it as $f_2(x)$ with $x = \frac{L}{R} > 1$. Then, there exists an $x^* > 1$ such that $f_2(x) < 0$ for $1 < x < x^*$, and $f_2(x) > 0$ for $x > x^*$.*

Furthermore, for fixed $R > 0$,

$$\lim_{L\to R} f_0(L,R) = R,\ \lim_{L\to R} f_1(L,R) = 0 \text{ and } \lim_{L\to R} f_2(L,R) = -\frac{R^2}{2}.$$

4.1. Critical potentials and their role descriptions

In this subsection, our main concern is identifying the critical potentials and the roles they play in the study of finite ion size effects on ionic flows.

Definition 4.1. We define nine potentials $V_k^c,\ V_{1k}^c$ and V_{2k}^c, $k = 0,1,2$ by

$$I_k(V_k^c;\lambda,0) = 0,\ J_{1k}(V_{1k}^c;\lambda,0) = 0,\ J_{2k}(V_{2k}^c;\lambda,0) = 0.$$

Remark 4.1. Actually, in Definition 4.1, $I_0,\ J_{10}$ and J_{20} are independent of the parameter λ.

Directly from Lemma 4.2 and Definition 4.1, one has

Lemma 4.4. *Suppose $L \neq R$. Then,*

$$\begin{aligned}
&z_1V_{10}^c = z_2V_{20}^c = -\frac{k_BT}{e}(\ln L - \ln R), \quad V_0^c = \frac{z_1(D_1-D_2)}{z_1D_1 - z_2D_2}V_{10}^c,\\
&z_1V_{11}^c = z_2V_{21}^c = \frac{k_BT}{2e}\ \frac{L^2-R^2}{f_0(L,R)f_1(L,R)}, \quad V_1^c = \frac{z_1(D_1-D_2)}{z_1D_1 - z_2D_2}V_{11}^c,\\
&z_1V_{12}^c = z_2V_{22}^c = -\frac{k_BT}{3e}\frac{L^3-R^3}{f_0(L,R)\big(\frac{1}{2}f_0^2(L,R) + f_2(L,R)\big)}, \quad V_2^c = \frac{z_1(D_1-D_2)}{z_1D_1 - z_2D_2}V_{12}^c.
\end{aligned}$$

In particular, one has

$$V_0^c = \frac{z_1D_1V_{10}^c - z_2D_2V_{20}^c}{z_1D_1 - z_2D_2}, \quad V_1^c = \frac{z_1D_1V_{11}^c - z_2D_2V_{21}^c}{z_1D_1 - z_2D_2}, \quad V_2^c = \frac{z_1D_1V_{12}^c - z_2D_2V_{22}^c}{z_1D_1 - z_2D_2}.$$

From Lemma 4.2, direct calculations yield

Lemma 4.5. *Assume $L \neq R$. Under conditions (4.1), one has $\partial_V I_0 > 0,\ \partial_V I_1 > 0,\ \partial_V I_2 > 0$; $\partial_V J_{10} > 0,\ \partial_V J_{11} > 0,\ \partial_V J_{12} > 0$; and $\partial_V J_{20} < 0,\ \partial_V J_{21} < 0,\ \partial_V J_{22} < 0$.*

It follows from Lemma 4.5 that

Proposition 4.1. *Assume $L \neq R$. Regarding I_k, J_{1k} and J_{2k}, $k = 0,1,2$ as functions of V, one has*

(i) *I_0, I_1 and I_2 are all increasing in V. Furthermore, $I_0 > 0$ (resp. $I_0 < 0$) if $V > V_0^c$ (resp. $V < V_0^c$); $I_1 > 0$ (resp. $I_1 < 0$) if $V > V_1^c$ (resp. $V < V_1^c$); and $I_2 > 0$ (resp. $I_2 < 0$) if $V > V_2^c$ (resp. $V < V_2^c$).*

(ii) *J_{10}, J_{11} and J_{12} are all increasing in V. Furthermore, $J_{10} > 0$ (resp. $J_{10} < 0$) if $V > V_{10}^c$ (resp. $V < V_{10}^c$); $J_{11} > 0$ (resp. $J_{11} < 0$) if $V > V_{11}^c$ (resp. $V < V_{11}^c$); and $J_{12} > 0$ (resp. $J_{12} < 0$) if $V > V_{12}^c$ (resp. $V < V_{12}^c$).*

(iii) *J_{20}, J_{21} and J_{22} are all decreasing in V. Furthermore, $J_{20} < 0$ (resp. $J_{20} > 0$) if $V > V_{20}^c$ (resp. $V < V_{20}^c$); $J_{21} < 0$ (resp. $J_{21} > 0$) if $V > V_{21}^c$ (resp. $V < V_{21}^c$); and $J_{22} < 0$ (resp. $J_{22} > 0$) if $V > V_{22}^c$ (resp. $V < V_{22}^c$).*

The scaling laws for $I_0, I_1, J_{k0}, J_{k1}, (k = 1, 2)$ and their critical potentials have been discussed in [47]. For I_2, J_{k2} and their critical potentials defined in the Definition 4.1, one has

Proposition 4.2. *Viewing I_2, J_{k2}, V_2^c and V_{k2}^c as functions of (L, R) for $k = 1, 2$, one has*

(i) *I_2, J_{12} and J_{22} are homogeneous of degree three in (L, R), that is, for any $s > 0$, $I_2(V; sL, sR) = s^3 I_2(V; L, R)$, $J_{12}(V; sL, sR) = s^3 J_{12}(V; L, R)$ and $J_{22}(V; sL, sR) = s^3 J_{22}(V; L, R)$;*

(ii) *V_2^c and V_{12}^c, V_{22}^c are homogeneous of degree zero in (L, R), that is, taking V_2^c for example, for any $s > 0$, $V_2^c(sL, sR) = V_2^c(L, R)$.*

Proof. Note that $f_0(sL, sR) = sf_0(L, R)$ and $f_2(sL, sR) = s^2 f_2(L, R)$ for any $s > 0$, where $f_0(L, R)$ and $f_2(L, R)$ are defined in Lemma 4.2. The statements can be verified directly. ☐

In terms of the parameters (D_1, D_2), (L, R) and λ, we can provide a partial order for the critical potentials identified in the Definition 4.1, which is crucial for us to further discuss the finite ion size effects on ionic flows.

Lemma 4.6. *Assume $L > R$, $D_2 > D_1$ and $\lambda > 1$. One has*

(i) *$V_{11}^c < V_{12}^c < V_{10}^c$, $V_{20}^c < V_{22}^c < V_{21}^c$ and $V_0^c < V_2^c < V_1^c$ if $1 < x < x^*$,*

(ii) *$V_{12}^c < V_{11}^c < V_{10}^c$, $V_{20}^c < V_{21}^c < V_{22}^c$ and $V_0^c < V_1^c < V_2^c$ if $x > x^*$,*

where, with $x = L/R > 1$, x^* is the unique zero of the function

$$p(x) = -9(x+1)^2 \ln x + 6(x^2 - 1) + 8(x^2 + x + 1) \ln x.$$

Proof. We provide a detailed proof for the order of V_{10}^c, V_{11}^c and V_{12}^c for both $1 < x < x^*$ and $x > x^*$. From Lemma 4.4, direct calculation gives

$$V_{10}^c - V_{11}^c = -\frac{k_B T}{e} \frac{\ln L - \ln R}{z_1} \frac{f_0(L, R)}{f_1(L, R)}.$$

From Lemma 4.3, one has $f_0(L, R) > 0$ and $f_1(L, R) < 0$ for $L > R$. It follows that $V_{10}^c - V_{11}^c > 0$, that is, $V_{10}^c > V_{11}^c$. Similarly,

$$V_{11}^c - V_{12}^c = \frac{k_B T}{e} \frac{1}{6z_1} \frac{L - R}{f_1(L, R)(\frac{1}{2} f_0^2(L, R) + f_2(L, R))} q(L, R),$$

where $q(L, R) = 3(L + R)\left(\frac{1}{2} f_0^2(L, R) f_2(L, R)\right) + 2(L^2 + LR + R^2) f_1(L, R)$. With $x = L/R$, $q(L, R)$ can be written as

$$q(x) = \frac{R^3 (x - 1)}{4(\ln x)^2} p(x),$$

where $p(x) = -9(x+1)^2 \ln x + 6(x^2 - 1) + 8(x^2 + x + 1) \ln x$. Careful calculation yields

$$p'(x) = -18(x+1) \ln x - \frac{9(x^2 + 2x + 1)}{x} + 12x + 8(2x + 1) \ln x + \frac{8(x^2 + x + 1)}{x},$$

$$p''(x) = -2\ln x + \frac{1}{x^2} - \frac{10}{x} + 9, \quad p'''(x) = \frac{-2(x^2 - 5x + 1)}{x^3}.$$

It is easy to check that $p(1) = p'(1) = p''(1) = 0$ and $p'''(x) > 0$ (resp. $p'''(x) < 0$) if $1 < x < x_1$ (resp. $x > x_1$), where x_1 (actually equals $\frac{5+\sqrt{21}}{2}$) is the unique zero of $p'''(x) = 0$. This indicates that $p''(x)$ is increasing over $(1, x_1)$ and decreasing for $x > x_1$. Together with $p''(1) = 0$ and $\lim\limits_{x\to+\infty} p''(x) = -\infty$, we get $p''(x) > 0$ (resp. $p''(x) < 0$) over $(1, x_2)$ (resp. $x > x_2$), where $x_2 > x_1$ is the unique root of $p''(x) = 0$ for $x > 1$. Similar argument shows that

- $p'(x) > 0$ (resp. $p'(x) < 0$) over $(1, x_3)$ (resp. $x > x_3$), where $x_3 > x_2$ is the unique root of $p'(x) = 0$ for $x > 1$;
- $p(x) > 0$ (resp. $p(x) < 0$) over $(1, x^*)$ (resp. $x > x^*$), where $x^* > x_3$ is the unique root of $p(x) = 0$ for $x > 1$.

It follows that $q(x) > 0$ (resp. $q(x) < 0$) for $x \in (1, x^*)$ (resp. $x > x^*$). Therefore, $V_{11}^c - V_{12}^c < 0$ (resp. $V_{11}^c - V_{12}^c > 0$) for $x \in (1, x^*)$ (resp. $x > x^*$).

Together with $V_{10}^c > V_{11}^c$, one has $V_{12}^c < V_{11}^c < V_{10}^c$ for $x > x^*$. On the other hand, for $x \in (1, x^*)$, we have $V_{11}^c < V_{12}^c$ and $V_{11}^c < V_{10}^c$. To get the total order of V_{10}^c, V_{11}^c and V_{12}^c, we further compare V_{10}^c with V_{12}^c. Similar argument leads to $V_{10}^c > V_{12}^c$ for $x > 1$, and hence $V_{11}^c < V_{12}^c < V_{10}^c$ for $1 < x < x^*$. This completes the proof. □

From Proposition 4.1, Lemma 4.5 and Lemma 4.6, we obtain

Theorem 4.1. *Assume $L > R$, $D_2 > D_1$ and $\lambda > 1$. For the individual fluxes $J_k(V), k = 1, 2$ and the total flux $I(V)$, with $|J_1(V)|$, $|J_2(V)|$ and $|I(V)|$ denoting the magnitude of $J_1(V)$, $J_2(V)$ and $I(V)$, respectively. One has, for $1 < x < x^*$, where x^* is identified in Lemma 4.6,*

(A) *For the individual flux $J_1(V)$,*

  (A1) *if $V < V_{11}^c$, $J_{10}(V) < 0$, $J_{11}(V) < 0$ and $J_{12}(V) < 0$, that is, the ion size effects from $J_{11}(V)$ and $J_{12}(V)$ both reduce $J_1(V)$ while both enhance $|J_1(V)|$;*

  (A2) *if $V_{11}^c < V < V_{12}^c$, $J_{10}(V) < 0$, $J_{11}(V) > 0$ and $J_{12}(V) < 0$, that is, the ion size effect from $J_{11}(V)$ enhances $J_1(V)$, while the one from $J_{12}(V)$ reduces $J_1(V)$. Furthermore, $J_{11}(V)$ reduces $|J_1(V)|$ while $J_{12}(V)$ enhances $|J_1(V)|$;*

  (A3) *if $V_{12}^c < V < V_{10}^c$, $J_{10}(V) < 0$, $J_{11}(V) > 0$ and $J_{12}(V) > 0$, that is, the ion size effects from $J_{11}(V)$ and $J_{12}(V)$ both enhance $J_1(V)$ while both reduce $|J_1(V)|$;*

  (A4) *if $V > V_{10}^c$, $J_{10}(V) > 0$, $J_{11}(V) > 0$ and $J_{12}(V) > 0$, that is, the ion size effects from $J_{11}(V)$ and $J_{12}(V)$ both enhance $J_1(V)$ and $|J_1(V)|$.*

(B) *For the individual flux $J_2(V)$,*

  (B1) *if $V < V_{20}^c$, $J_{20}(V) > 0$, $J_{21}(V) > 0$ and $J_{22}(V) > 0$, that is, the ion size effects from $J_{21}(V)$ and $J_{22}(V)$ both enhance $J_2(V)$ and $|J_2(V)|$;*

  (B2) *if $V_{20}^c < V < V_{22}^c$, $J_{20}(V) < 0$, $J_{21}(V) > 0$ and $J_{22}(V) > 0$, that is, the ion size effects from $J_{21}(V)$ and $J_{22}(V)$ enhance $J_2(V)$. Furthermore, $J_{21}(V)$ and $J_{22}(V)$ reduce $|J_2(V)|$;*

(B3) *if* $V_{22}^c < V < V_{21}^c$, $J_{20}(V) < 0$, $J_{21}(V) > 0$ *and* $J_{22}(V) < 0$, *that is, the ion size effect from* $J_{21}(V)$ *enhances* $J_2(V)$ *while the one from* $J_{22}(V)$ *reduces* $J_2(V)$. *Furthermore,* $J_{22}(V)$ *enhances* $|J_2(V)|$ *while* $J_{21}(V)$ *reduces* $|J_2(V)|$;

(B4) *if* $V > V_{21}^c$, $J_{20}(V) < 0$, $J_{21}(V) < 0$ *and* $J_{22}(V) < 0$, *that is, the ion size effects from* $J_{21}(V)$ *and* $J_{22}(V)$ *both reduce* $J_2(V)$ *while both enhances* $|J_2(V)|$.

(C) *For the total flow rate of charge* I,

(C1) *if* $V < V_0^c$, *then,* $I_0(V) < 0, I_1(V) < 0$ *and* $I_2(V) < 0$, *that is, the ion size effects from* $I_1(V)$ *and* $I_2(V)$ *both reduce* $I(V)$. *Furthermore,* $I_1(V)$ *and* $I_2(V)$ *both enhance* $|I(V)|$;

(C2) *if* $V_0^c < V < V_2^c$, *then,* $I_0(V) > 0, I_1(V) < 0$ *and* $I_2(V) < 0$, *that is, the ion size effects from* $I_1(V)$ *and* $I_2(V)$ *both reduce* $I(V)$ *and* $|I(V)|$;

(C3) *if* $V_2^c < V < V_1^c$, *then,* $I_0(V) > 0, I_1(V) < 0$ *and* $I_2(V) > 0$, *that is, the ion size effect from* $I_1(V)$ *reduces* $I(V)$ *while* $I_2(V)$ *enhances* $I(V)$. *Furthermore,* $I_1(V)$ *reduces* $|I(V)|$, *but* $I_2(V)$*enhances* $|I(V)|$;

(C4) *if* $V > V_1^c$, *then,* $I_0(V) > 0, I_1(V) > 0$ *and* $I_2(V) > 0$, *that is, the ion size effects from* $I_1(V)$ *and* $I_2(V)$ *both enhance* $I(V)$ *and* $|I(V)|$.

Remark 4.2. For the case with $x > x^*$, similar results as those stated in Theorem 4.1 can be obtained.

4.2. Essential effects from finite ion sizes

To better understand the finite ion size effects on ionic flows, we introduce another three critical potentials V_1^b, V_2^b and V^b, which characterizes the essential effects on ionic flows from finite ion sizes, that is, the combining effects from the first and the second order terms.

For convenience in our following discussion, we define three functions $I^b(V;\lambda)$, $J_1^b(V;\lambda)$ and $J_2^b(V;\lambda)$ by

$$\begin{aligned}&I^b(V;\lambda) = I_1(V;\lambda) + \nu I_2(V;\lambda), \quad J_1^b(V;\lambda) = J_{11}(V;\lambda) + \nu J_{12}(V;\lambda),\\ &J_2^b(V;\lambda) = J_{21}(V;\lambda) + \nu J_{22}(V;\lambda).\end{aligned}$$

Definition 4.2. We define three critical potentials V_1^b, V_2^b and V^b by

$$I^b(V^b;\lambda) = 0, \quad J_1^b(V_1^b;\lambda) = 0 \quad \text{and} \quad J_2^b(V_2^b;\lambda) = 0.$$

Lemma 4.7. *Assume* $L \neq R$ *and* $\lambda > 1$, *one has*

$$z_1 V_1^b = z_2 V_2^b = -\frac{k_B T}{e}\frac{\frac{z_1\lambda - z_2}{3z_1z_2}(L^2+RL+R^2)\nu - \frac{1}{2}(L+R)}{\frac{z_1\lambda - z_2}{z_1z_2}\Big(\frac{1}{2}f_0^2(L,R) + f_2(L,R)\Big)\nu + f_1(L,R)}(\ln L - \ln R),$$

$$V^b = \frac{z_1(D_1 - D_2)}{z_1D_1 - z_2D_2}V_1^b.$$

We comment that the critical potentials defined in Definition 4.2 balance the ion size effects on the total flux $I(V)$, and the individual fluxes $J_1(V)$ and $J_2(V)$, respectively. Also, V^b and V_k^b as functions of (L,R) don't share the scaling laws as other critical potentials defined in Definition 4.1, which is not a surprise since it reflects the mixed ion size effects from both the first and the second order corrections.

Lemma 4.8. *Assume* $L > R,\ D_2 > D_1$ *and* $\lambda > 1$. *One has* $V_0^c < V^b$, $V_{10}^c > V_1^b$ *and* $V_{20}^c < V_2^b$.

Notice that $\partial_V J_1^b = -\partial_V J_2^b = \frac{\partial_V I^b}{z_1D_1 - z_2D_2} = \frac{e}{k_BT}\frac{z_1\lambda - z_2}{z_1z_2}\frac{f_0(L,R)}{H(1)}g(L,R)$, where

$$g(L,R) = f_1(L,R) + \nu\frac{z_1\lambda - z_2}{z_1z_2}\left(f_2(L,R) + \frac{1}{2}f_0^2(L,R)\right).$$

From Lemma 4.3, together with the valence $z_2 < 0$, for the function $g(L,R)$, one has

Lemma 4.9. *Assume* $L > R$. *One has* $g(L,R) < 0$.

Together with Lemma 4.8, we obtain

Theorem 4.2. *Assume* $L > R$ *and* $\lambda > 1$. *For* $\nu > 0$ *small, one has*

(i) $I^b(V)$ *is increasing in the potential* V. *Hence,* $I^b(V) > 0$ *(resp.* $I^b(V) < 0$*) if* $V > V^b$ *(resp.* $V < V^b$*); that is, the ion size eventually enhances (resp. reduces) the total flux* $I(V)$ *if* $V > V^b$ *(resp.* $V < V^b$*). Furthermore, the ion size eventually enhances (resp. reduces)* $|I(V)|$ *if* $V > V^b$ *or* $V < V_0^c$ *(resp.* $V_0^c < V < V^b$*).*

(ii) $J_1^b(V)$ *is increasing in the potential* V. *Hence,* $J_1^b(V) > 0$ *(resp.* $J_1^b(V) < 0$*) if* $V > V_1^b$ *(resp.* $V < V_1^b$*), that is, the ion size eventually enhances (resp. reduces) the individual flux* $J_1(V)$ *if* $V > V_1^b$ *(resp.* $V < V_1^b$*). Furthermore, the ion size eventually enhances (resp. reduces)* $|J_1(V)|$ *if* $V < V_1^b$ *or* $V > V_{10}^c$ *(resp.* $V_1^b < V < V_{10}^c$*).*

(iii) $J_2^b(V)$ *is decreasing in the potential* V. *Hence,* $J_2^b(V) > 0$ *(resp.* $J_2^b(V) < 0$*) if* $V < V_2^b$ *(resp.* $V > V_2^b$*), that is, the ion size eventually enhances (resp. reduces) the individual flux* $J_2(V)$ *if* $V < V_2^b$ *(resp.* $V > V_2^b$*). Furthermore, the ion size eventually enhances (resp. reduces)* $|J_2(V)|$ *if* $V > V_2^b$ *or* $V < V_{20}^c$ *(resp.* $V_{20}^c < V < V_2^b$*).*

We now provide a result of the total order of the critical potentials V_k^b and V^b, $k = 1, 2$.

Lemma 4.10. *Assume* $L > R,\ D_2 > D_1$ *and* $\lambda > 1$. *One has* $V_1^b < V^b < V_2^b$.

Correspondingly, the following result can be established.

Theorem 4.3. *Assume* $L > R$, $D_2 > D_1$ *and* $\lambda > 1$. *For* $\nu > 0$ *small, one has*

(i) *If* $V < V_1^b$, *then, the ion size effect eventually reduces both* $J_1(V)$ *and* $I(V)$ *while enhances* $J_2(V)$*;*

(ii) *If* $V_1^b < V < V^b$, *then, the ion size effect eventually enhances both* $J_1(V)$ *and* $J_2(V)$ *while reduces* $I(V)$*;*

(iii) *If* $V^b < V < V_2^b$, *then, the ion size effect eventually enhances* $J_1(V)$, $J_2(V)$ *and* $I(V)$*;*

(vi) *If* $V > V_2^b$, *then, the ion size eventually enhances* $J_1(V)$ *and* $I(V)$ *while reduces* $J_2(V)$.

4.3. Ion size effects near L=R

As $L \to R$, $I_1(V;\lambda,0) \to 0$ and $J_{k1}(V;\lambda,0) \to 0$. In other words, as $L \to R$, the leading terms $I_1(V;\lambda,0), J_{11}(V;\lambda,0)$ and $J_{21}(V;\lambda,0)$ cannot provide information for the effects from finite ion size, and higher order terms need to be considered. While this is the motivation of the work, we surprisingly found that the second order terms $I_2,\ J_{12}$ and J_{22} are also approach zero as $L \to R$. To summarize, one has

Lemma 4.11. *Fixed $R > 0$. Then, for $k = 1, 2$,*

$$\lim_{L\to R} J_{10}(V;0) = -\lim_{L\to R} J_{20}(V;0) = \frac{R}{H(1)}\frac{e}{k_BT}V, \quad \lim_{L\to R} J_{k1}(V;\lambda,0) = 0,$$

$$\lim_{L\to R} J_{k2}(V;\lambda,0) = 0,\ \lim_{L\to R} I_0(V;0) = \frac{(z_1D_1 - z_2D_2)R}{H(1)}\frac{e}{k_BT}V,\ \lim_{L\to R} I_k(V;\lambda,0) = 0.$$

Proposition 4.3. *As $L \to R$, one has*

(i) $V_{10}^c = V_{20}^c = V_0^c = 0$*;* $V_{11}^c \to -\infty,\ V_{21}^c \to \infty,\ V_1^c \to -\infty$ *if* $D_1 > D_2$*, and* $V_1^c \to \infty$ *if* $D_1 < D_2$*;* $V_{12}^c \to -\infty,\ V_{22}^c \to \infty,\ V_2^c \to -\infty$ *if* $D_1 > D_2$*, and* $V_2^c \to \infty$ *if* $D_1 < D_2$*.*

(ii) $I_k(V;\lambda),\ J_{1k}(V;\lambda)$ *and* $J_{2k}(V;\lambda)$ *all approach zero for* $k = 1, 2$*; that is, the ion size effect does not affect the total flux* $I(V)$ *and the individual fluxes* $J_k(V)$*,* $k = 1, 2$*; and hence does not affact* $|I(V)|$ *and* $|J_k(V)|$ *for* $k = 1, 2$*.*

4.4. Numerical illustrations

To provide more intuitive illustration of the effects on ionic flows from finite ion sizes, we perform the following numerical simulations to the approximations obtained in Lemma 4.2. To be specific, we consider the cation to be Na$^+$ and the anion to be Cl$^-$, and λ is the ratio of the volume of Na$^+$ to Cl$^-$. We may take ([48,53])

$$D_{Na} = 1.334 \times 10^{-9} m^2/s,\ D_{Cl} = 2.032 \times 10^{-9} m^2/s,\ k_B = 1.38 \times 10^{-23} JK^-,$$
$$T = 298.15K,\ e = 1.602 \times 10^{-19} C,\ z_1 = -z_2 = 1 \text{ and } \lambda = 1.885.$$

Our main interest is to identify the critical potentials $V_k^c,\ V_{1k}^c$ and V_{2k}^c identified in Definition 4.1 (see Figure 1 for $L \neq R$ and Figure 3 for L close to R), and the critical potential V_k^b and V^b defined in Definition 4.2 (see Figure 2) for $L \neq R$; and observe the monotonicity of $I_k,\ J_{1k},\ J_{2k},\ I^b$ and J_k^b viewed as functions of the potential V. This provides more intuitive understandings of our analytical results (such as Theorems 4.1 and 4.2).

5. Concluding remarks

In this work, we further study the effects on ionic flows from finite ion sizes via the method of asymptotic expansions up to the second order due to the observation that the first-order terms approach zero, in other words, the finite ion size effects on ionic flows disappear, when the left and right boundary concentrations are close for the same ion species. On the other hand, considering higher order terms may help us perceive the properties of the expansion and generalize it for any size, not just

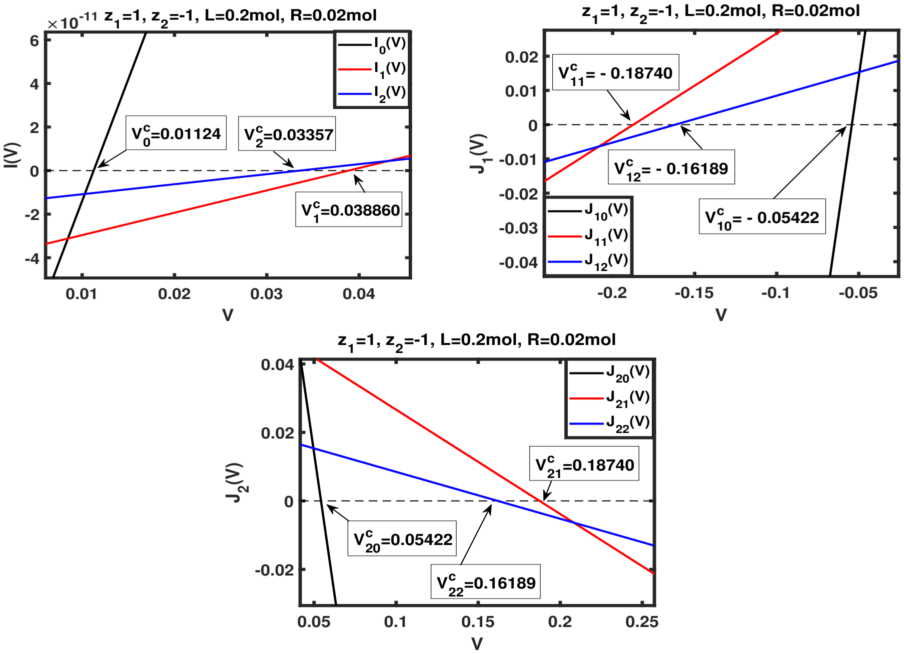

Figure 1. Numerical identifications of the critical potentials V_k^c, V_{1k}^c and V_{2k}^c for $k = 0, 1, 2$. The monotonicity of the functions $I_k(V)$, $J_{1k}(V)$ and $J_{2k}(V)$ for $k = 0, 1, 2$ is shown clearly in each figure. This provides intuitive illustration of Theorem 4.1 with $1 < x < x^* = 339.75691$.

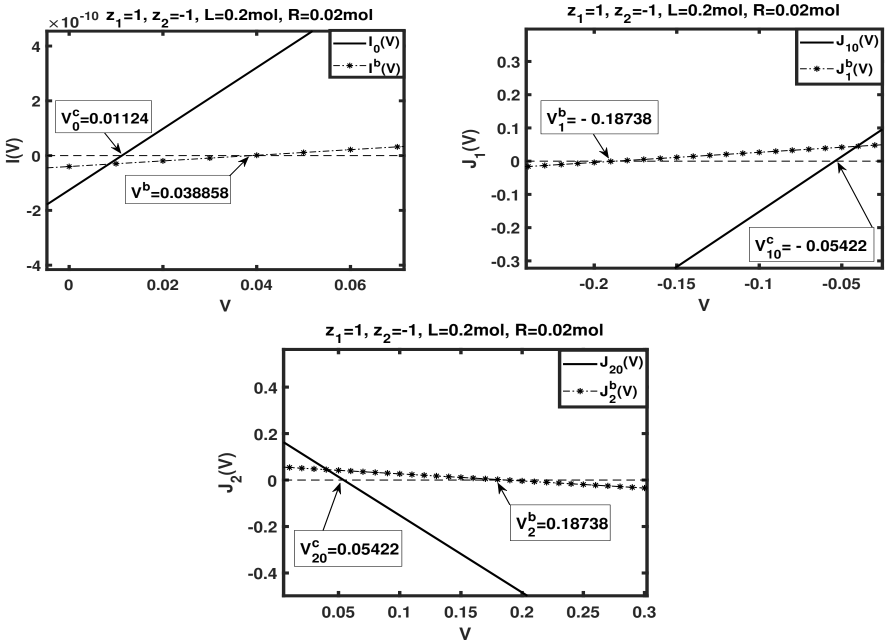

Figure 2. Numerical identifications of the critical potentials V_0^c, V^b, V_{10}^c, V_1^b, V_{20}^c and V_2^b with small ν. The monotonicity of the functions $I_0(V)$, $J_{10}(V)$ and $J_{20}(V)$ (solid line in each figure) and the functions $I^b(V)$, $J_1^b(V)$ and $J_2^b(V)$ (dashed line with star in each figure) can be observed clearly. This provides intuitive illustration of Theorem 4.2.

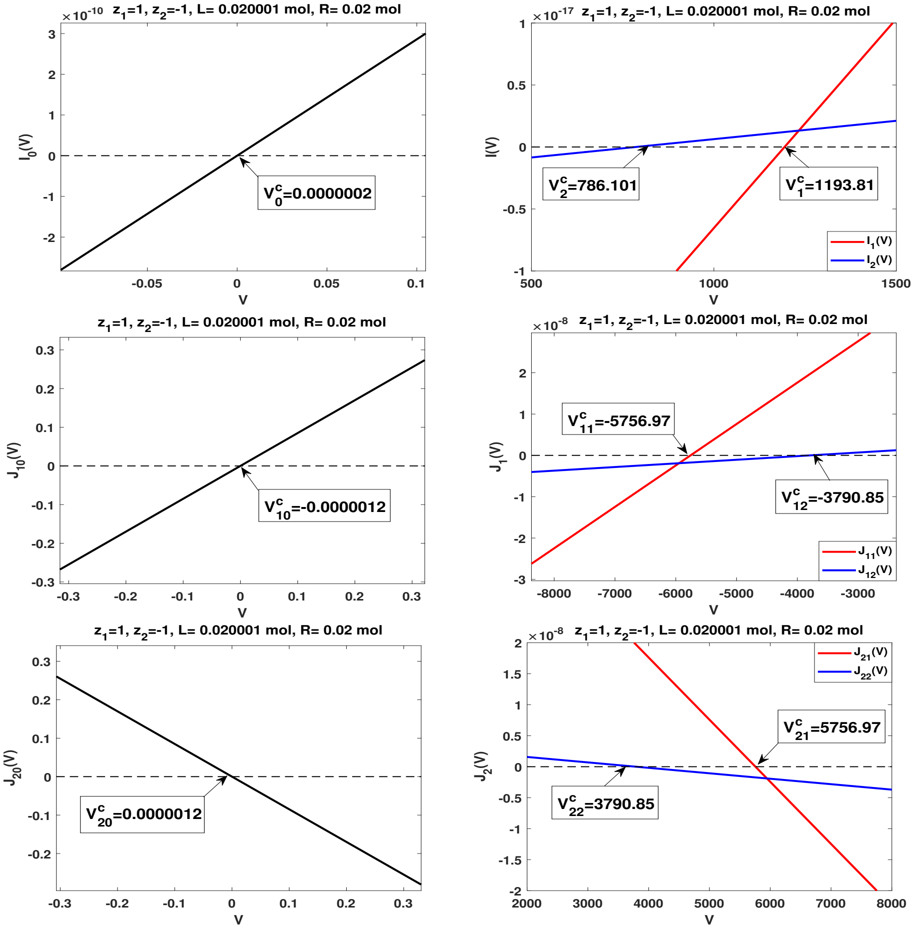

Figure 3. Numerical identifications of the critical potentials V_k^c, V_{1k}^c and V_{2k}^c for $k = 0, 1, 2$. The monotonicity of the functions $I_k(V)$, $J_{1k}(V)$ and $J_{2k}(V)$ for $k = 0, 1, 2$ is shown clearly in each figure for the case with L close to R. It is clear that, for $k = 1, 2$, the values of V_0^c and V_{k0}^c become very small, the values of V_k^c V_{1k}^c and V_{2k}^c for $k = 1, 2$ become very large, and the curves I_k, J_{1k} and J_{2k} become very small as L close to R. The observation is consistent with our analytical results stated in Section 4.3.

the small sizes of ions. For $L \neq R$, the interactions between the first-order and the second-order terms are also described to better understand the ionic flow properties. Moreover, critical potentials are identified to help us monitor the dynamics of ionic flows. In particular, some critical potentials such as V^b, V_1^b and V_2^b identified in the Definition 4.2, can be estimated experimentally. Take the potential V^b for example, one can take an experimental I-V relation as $I(V; \lambda, \nu)$ and numerically (or analytically) compute $I_0(V)$ for ideal case that allows one to get an estimate of V^b. However, as $L \to R$, we surprisingly found that the second order terms are also approaching 0. We would like to propose the following conjecture for the specific setup in this work: all higher order terms in the finite ion size will approach 0 as $L \to R$. The analysis in this work, particularly for the case $L \neq R$, will provide complementary information and better understanding of the mechanism of ionic flows through membrane channels. Numerical simulations performed in this work provide intuitive illustration of our main analytical results.

We finally comment that, for this simple setup, complicated nonlinear interactions among physical parameters, particularly, the diffusion constants (D_1, D_2) and the boundary concentrations (L, R) are characterized, which are not intuitive, and provide some deep insights into the internal dynamics of ionic flows through membrane channels. This could be very helpful for the future studies along this direction, not only mathematically or numerically, but experimentally since the internal dynamics of ion channels cannot be measured with present technology.

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