Journal of Applied Analysis and Computation
Volume 12, Number 3, June 2022, 964–980
Website:http://www.jaac-online.com
DOI:10.11948/20220047

EXISTENCE OF INFINITELY MANY HOMOCLINIC SOLUTIONS OF DISCRETE SCHRÖDINGER EQUATIONS WITH LOCAL SUBLINEAR TERMS*

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Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract We obtain sufficient conditions on the existence of infinitely many homoclinic solutions for a class of discrete Schrödinger equations when the nonlinearities are assumed just to be sublinear near the origin. The problem we are going to study in this paper has two main difficulties, one is that the nonlinear terms are locally sublinear and the other is that the associated variational functional is indefinite. Some new techniques including cutoff methods and compact inclusions are applied here to overcome these two difficulties. Our results also improve some existing ones in the literature.

Keywords Discrete nonlinear Schrödinger equation, homoclinic solution, multiplicity, local sublinear term, critical point theory.

MSC(2010) 39A12, 39A70, 35Q51, 35Q55.

1. Introduction

Differential and difference equations (DDEs) have been widely used to model practical problems in various fields of natural sciences [1,3,10,11,13,15–18,20,26,35–37,41–45]. Serving as one class of the most basic DDEs, discrete nonlinear Schrödinger (DNLS) equations play a significant role in many nonlinear phenomena, such as Bose-Einstein condensates, nonlinear optics and biomolecular chains [10,14–16,18]. Since discrete breathers in the DNLS equations were widely observed in experiments [10,16,17], it has been a growing interest in proving the existence of discrete breathers of the DNLS equations.

Consider the discrete breathers of the following DNLS equation:

$$i \cdot \frac{d\psi_n}{dt} = -\Delta\psi_n + V_n\psi_n - f_n(\psi_n), \quad n = (n_1, n_2, \cdots, n_m) \in \mathbb{Z}^m. \tag{1.1}$$

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[*]This work was partially supported by National Natural Science Foundation of China (Nos. 12001127, 11971126), Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515010667), and Program for Changjiang Scholars and Innovative Research Team in University (No. IRT_16R16).

Here m is a fixed positive integer, i is the imaginary unit, $\psi_n = \psi_n(t)$,

$$\begin{aligned}\Delta\psi_n = &\psi_{(n_1+1,n_2,\cdots,n_m)} + \psi_{(n_1,n_2+1,\cdots,n_m)} + \cdots + \psi_{(n_1,n_2,\cdots,n_m+1)} \\ &+\psi_{(n_1-1,n_2,\cdots,n_m)} + \psi_{(n_1,n_2-1,\cdots,n_m)} + \cdots + \psi_{(n_1,n_2,\cdots,n_m-1)} \\ &-2m\psi_{(n_1,n_2,\cdots,n_m)}\end{aligned}$$

is the discrete Laplacian in m spatial dimensions, the (discrete) unbounded potential $V = \{V_n\}_{n\in\mathbb{Z}^m}$ is a real-valued sequence satisfying

(V) there exists $\nu_0 \in (0, 2)$ such that $\lim_{|n|\to+\infty} \frac{V_n}{|n|^{\nu_0}} = +\infty$,

where $|n| = |n_1|+|n_2|+...+|n_m|$ is the length of multi–index n, and the nonlinearity f_n is gauge invariant, i.e.,

$$f_n(e^{i\theta}u) = e^{i\theta} f_n(u), \quad \theta \in \mathbb{R},\ u \in \mathbb{R}.$$

Since discrete breathers are spatially localized time-periodic solutions and decay to zero at infinity, we assume that ψ_n of (1.1) has the form

$$\psi_n = u_n e^{-i\omega t},\ n \in \mathbb{Z}^m \quad \text{and} \quad \lim_{|n|\to+\infty} \psi_n = 0,$$

where $\{u_n\}$ is a real-valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.1) becomes

$$Lu_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}^m \tag{1.2}$$

and

$$\lim_{|n|\to+\infty} u_n = 0, \tag{1.3}$$

where $L := -\Delta + V$ is defined by

$$Lu_n = -\Delta u_n + V_n u_n, \quad n \in \mathbb{Z}^m. \tag{1.4}$$

If $f_n(0) \equiv 0$, then $u_n \equiv 0$ is a solution of (1.2), which is called the trivial solution. As usual, we say that a solution $u = \{u_n\}$ of (1.2) is homoclinic (to 0) if (1.3) holds. To find the discrete breathers of (1.1), we just need to seek the homoclinic solutions of (1.2).

In recent years, the existence and multiplicity results of homoclinic solutions for the DNLS equations with potentials and nonlinearities have been widely discussed by mainly using variational approaches [2,5,8,9,12,19,21–25,27–33,38–40,46–50]. To mention a few, one may refer to [9,29,30,33,38–40,46] for the superlinear nonlinearity, to [2,19,25,29,31,47–50] for the asymptotically linear nonlinearity, to [22–24] for the mixed nonlinearity, and to [4–8,12,21,32] for the sublinear nonlinearity. Among them, the results regarding unbounded potentials and superlinear or asymptotically-linear terms have received special attention [8,9,14,23,24,38,39,46]. However, there exist only a few results with unbounded potentials and sublinear terms [8, 21, 32]. Considering the importance of sublinear nonlinearity in inflation cosmology and supersymmetric field theories, quantum mechanics and nuclear physics [3, 11, 13], there needs a further exploration into the existence of homoclinic solutions for the DNLS equations with unbounded potentials and sublinear terms.

In this paper, we shall prove the existence of infinitely many homoclinic solutions of (1.2) with local sublinear terms f_n for $n \in \mathbb{Z}^m$ by using the classical dual fountain theorem [34]. To this end, we have the following assumptions on $f_n(u)$ for $u \in [-\epsilon, \epsilon]$ with some $\epsilon > 0$.

$(A1)$ $f_n(u)$ is continuous in u and $f_n(-u) = -f_n(u)$ on $[-\epsilon, \epsilon]$ for $n \in \mathbb{Z}^m$.

$(A2)$ There exist $c_1 > 0$ and $\max\left\{\frac{3}{2}, \frac{2+\nu_0}{1+\nu_0}\right\} < \nu_1 < 2$ such that

$$|f_n(u)| \leq c_1|u|^{\nu_1-1}, \quad u \in [-\epsilon, \epsilon], \quad n \in \mathbb{Z}^m. \tag{1.5}$$

$(A3)$ $\lim_{u\to 0}(F_n(u)/u^2) = +\infty$ uniformly for $n \in \mathbb{Z}^m$, where $F_n(u) = \int_0^u f_n(s)ds$.

$(A4)$ $2F_n(u) - f_n(u)u > 0$ for $0 < |u| \leq \epsilon$ and $n \in \mathbb{Z}^m$.

We remark that the authors in [5, 12] studied the existence of homoclinic solutions of (1.2) with periodic potentials and nonlinear terms being sublinear both at the origin and infinity. This is different from ours, since we focus on the situation with unbounded potentials and local sublinear terms.

In addition, (1.2) with unbounded potentials was considered in [7, 8, 32]. In those works, however, $f_n(u)$ is sublinear in u both at the origin and infinity. It also needs to assume that the temporal frequency ω is less than λ_1 [8, 32]. Here λ_1 is the smallest eigenvalue of L in l^2, where

$$l^p := \left\{ u = \{u_n\}_{n\in\mathbb{Z}^m} : u_n \in \mathbb{R},\, n \in \mathbb{Z}^m,\, \|u\|_p = \left(\sum_{n\in\mathbb{Z}^m} |u_n|^p\right)^{\frac{1}{p}} < +\infty \right\}$$

with the following embedding between l^p spaces:

$$l^q \subset l^p,\ \|u\|_p \leq \|u\|_q,\ 1 \leq q \leq p \leq \infty.$$

In this case, it is relatively easy to obtain the homoclinic solutions of (1.2) since the corresponding variational problem is definite.

Recently in [21] we proved the multiplicity results of homoclinic solutions of (1.2) with unbounded potentials by mainly assuming $\omega < \lambda_1$ and

- there exist two constants $1 \leq \nu_1 < \nu_2 < 2$ and two positive-valued sequences $a_i = \{a_{i,n}\} \in l^{2/(2-\nu_i)}$ of $i = 1, 2$, such that $|f_n(u)| \leq a_{1,n}|u|^{\nu_1-1} + a_{2,n}|u|^{\nu_2-1}$ for $u \in [-\epsilon, \epsilon]$ and $n \in \mathbb{Z}^m$.

We note that the above condition generally proves to be valid only for the definite case of $\omega < \lambda_1$. In this work, we shall use a new assumption $(A2)$ in order to deal with the case of general temporal frequencies (the indefinite case of $\lambda_1 \leq \omega$).

One difficulty in this work is that it is not easy to establish the variational setting associated with (1.2) on the working space E. Usually, if the nonlinear term f is superlinear or asymptotically-linear both at the origin and infinity, then it is easy to write down the corresponding variational functional J of (1.2) and to verify that J is a well-defined C^1 functional on E. However, this step seems difficult to finish if f is just sublinear near the origin. Another difficulty is that the resulting functional J on E is indefinite and it is not easy to show the compactness of bounded Palais–Smale (PS) sequences of J. If f is superlinear or asymptotically-linear both at the origin and infinity, then one can prove the PS condition according to a well-known compact embedding theorem due to Zhang and Pankov [38, 39]. In contrast, the local sublinearity of f prevents us from directly using the compact-embedding results of [38, 39] to obtain the PS condition. In this paper, we shall introduce new tricks to overcome the difficulties. One of the key ingredients in our method is a new compact embedding lemma of E into l^p under the assumption of (V) (see Lemma 2.1), which extends that of [38, 39] from $p \geq 2$ to $p > \max\{1, 2/(1+\nu_0)\}$. This together with cutoff methods and a priori estimates ensures the well–definition of J and the compactness of bounded PS sequences of J (see Section 3).

To the best of our knowledge, this is the first attempt in the literature on the multiplicity of homoclinic solutions for the indefinite problem with local sublinear terms. Moreover, our result extends some existing ones [8, 21, 32].

Our main result reads as follows.

Theorem 1.1. *Assume that (V) and $(A1)$–$(A4)$ hold. Then (1.2) exists infinitely many homoclinic solutions $\{u^{(k)}\}$ satisfying $\max_{n\in\mathbb{Z}^m} |u_n^{(k)}| \to 0$ as $k \to +\infty$.*

Here we present an example to demonstrate our result. Let $s \in (1.5, 2)$ and

$$f_n(u) \equiv \begin{cases} |u|^{s-2}u, & |u| \leq 1, \\ e^{|u|-1}\sin(0.5\pi u), & |u| > 1, \end{cases} \qquad V_n = |n|^2, \quad n \in \mathbb{Z}^m. \tag{1.6}$$

Then it is easy to verify that V_n and f_n in (1.6) satisfy the conditions of Theorem 1.1. Thus, (1.2) admits infinitely many homoclinic solutions converging to zero. We emphasize that $f_n(u)$ in (1.6) is not sublinear at infinity. Moreover, it does not satisfy the following standard assumption:

- there exist $a > 0$ and $p > 2$ such that $|f_n(u)| \leq a(1 + |u|^{p-1})$ for $u \in \mathbb{R}$ and $n \in \mathbb{Z}^m$,

which plays an important role in proving the existence of homoclinic solutions of DNLS equations with unbounded potentials and superlinear terms by using variational approaches [14, 23, 38, 39, 46].

The remaining of this paper reads as follows. In Section 2, we first establish the variational setting associated with (1.2) and then present some key lemmas which are useful for proving Theorem 1.1. Finally, we present a rigorous proof of Theorem 1.1 in the last section.

2. Preliminaries

In the section, we make some preparations in order to confirm our main result.

We first introduce the working space for the problem. It follows from (V) that $V = \{V_n\}$ is bounded from below. Without loss of generality, we assume that

$$V_n \geq 1 \quad \text{for } n \in \mathbb{Z}^m. \tag{2.1}$$

Thus, $L = -\Delta + V$ given in (1.4) is a positive unbounded self–adjoint operator in l^2. Denote

$$E = \{u \in l^2 : L^{1/2}u \in l^2\}.$$

Then E is a Hilbert space equipped with the norm

$$\|u\|_E = \|L^{1/2}u\|_2, \quad u \in E.$$

We remark that the following compact embedding plays an important role for our main result.

Lemma 2.1. *If V satisfies condition (V), then the embedding map from E into l^p is compact for $p \in (\max\{1, 2/(1+\nu_0)\}, +\infty]$.*

Proof. We first see from [39] that the lemma is true for $p \geq 2$. Hence, assume that $p \in (\max\{1, 2/(1+\nu_0)\}, 2)$. Let $q = \nu_0/(2-p)$. Obviously, $pq > 1$. For any positive integer m, denote

$$W_m = \inf_{|n|\geq m} \frac{V_n}{|n|^{\nu_0}}.$$

Then $W_m \to +\infty$ as $m \to +\infty$. Let $K \subset E$ be a bounded set. It follows from (2.1) that there exists $M > 0$ such that for any $u = \{u_n\} \in K$ and any positive integer m,

$$\begin{aligned}
\sum_{|n|\geq m} |u_n|^p &= \sum_{|n|\geq m,\, |n|^q|u_n|\leq 1} |u_n|^p + \sum_{|n|\geq m,\, |n|^q|u_n|>1} |u_n|^p \\
&\leq \sum_{|n|\geq m} |n|^{-pq} + \sum_{|n|\geq m,\, |n|^q|u_n|>1} (|n|^q|u_n|)^p |n|^{-pq} \\
&\leq \sum_{|n|\geq m} |n|^{-pq} + \sum_{|n|\geq m,\, |n|^q|u_n|>1} (|n|^q|u_n|)^2 |n|^{-pq} \\
&\leq \sum_{|n|\geq m} |n|^{-pq} + \sum_{|n|\geq m} |u_n|^2 |n|^{\nu_0} \\
&\leq \sum_{|n|\geq m} |n|^{-pq} + \frac{1}{W_m} \sum_{|n|\geq m} V_n |u_n|^2 \\
&\leq \sum_{|n|\geq m} |n|^{-pq} + \frac{M}{W_m}.
\end{aligned}$$

Thus, for any given $\varepsilon > 0$, there is a positive integer m_0 such that

$$\sum_{|n|\geq m_0} |u_n|^p \leq \varepsilon.$$

This implies that K is relatively compact in l^p. The proof is completed. □

Condition (V) implies that the spectrum $\sigma(L)$ of L is discrete and consists of finite-multiplicity eigenvalues accumulating to $+\infty$ (see [39]). We can assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$$

are all eigenvalues of L (counted in their multiplicities) and the corresponding set of eigenfunctions is $\{e^k\}_{k=1}^{+\infty}$, which forms an orthogonal basis in l^2. Let $E = E^- \oplus E^0 \oplus E^+$, where E^+, E^0 and E^- correspond to the positive, zero and negative part of the spectrum of $L - \omega$ in E, respectively.

Assume that there exist two positive integers m^* and m_* with $m^* > m_* \geq 1$ such that

$$\lambda_{m_*} < \omega = \lambda_{m_*+1} = \cdots = \lambda_{m^*} < \lambda_{m^*+1}. \tag{2.2}$$

Then it holds that

$$E^- = \operatorname{span}\{e_1, \cdots, e_{m_*}\} \quad \text{and} \quad E^0 = \operatorname{span}\{e_{m_*+1}, \cdots, e_{m^*}\}.$$

We also admit the cases of $m_* = 0$ and $m_* = m^* \geq 1$, in corresponding to $E^- = \{0\}$ and $E^0 = \{0\}$, respectively.

For any $u, v \in E = E^- \oplus E^0 \oplus E^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$, we define an inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on E by

$$(u,v) = ((L-\omega)u^+, v^+)_2 - ((L-\omega)u^-, v^-)_2 + (u^0, v^0)_2, \quad \|u\| = (u,u)^{\frac{1}{2}},$$

where $(\cdot,\cdot)_2$ is the inner product in l^2. Clearly, $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent, and the decomposition $E = E^- \oplus E^0 \oplus E^+$ is orthogonal with respect to both $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$.

We can see from Lemma 2.1 that for each $p \in (\max\{1, 2/(1+\nu_0)\}, +\infty]$, there exists $\rho_p > 0$ such that

$$\|u\|_p \leq \rho_p \|u\|, \quad u \in E. \tag{2.3}$$

For the reader's convenience, we end this section by giving some notations and definitions, and by citing a dual variant fountain theorem, in order to obtain the main result.

Definition 2.1 ([34]). Let $J \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$. The function J satisfies the $(PS)_c$ condition if any sequence $\{u^{(k)}\} \subset E$ such that

$$J(u^{(k)}) \to c, \quad J'(u^{(k)}) \to 0 \tag{2.4}$$

has a convergent subsequence.

As usual, the sequence $\{u^{(k)}\}$ satisfying (2.4) is called as a $(PS)_c$ sequence.

Assume that $E = \oplus_{j=1}^{+\infty} E_j$, where E_j is a finite dimensional subspace of E for each $j \geq 1$. Denote $Y_k := \oplus_{j=1}^{k} E_j$ and $Z_k := \overline{\oplus_{j=k}^{+\infty} E_j}$.

Definition 2.2 ([34]). Let $J \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$. The function J satisfies the $(PS)_c^*$ condition (with respect to $\{Y_k\}$) if any sequence $\{u^{(k)}\} \subset E$ such that

$$k \to +\infty, \quad u^{(k)} \in Y_k, \quad J(u^{(k)}) \to c, \quad J'|_{Y_k}(u^{(k)}) \to 0 \tag{2.5}$$

contains a subsequence converging to a critical point of J.

Here the sequence $\{u^{(k)}\}$ in (2.5) is called as a $(PS)_c^*$ sequence (with respect to $\{Y_k\}$).

Lemma 2.2 (Dual fountain theorem [34]). *Let $J \in C^1(E,\mathbb{R})$ be an invariant functional. If, for every $k \geq k_0$, there exist $\varrho_k > \gamma_k > 0$ such that*

(B_1) $a_k := \inf_{u \in Z_k, \|u\| = \varrho_k} J(u) \geq 0$,

(B_2) $b_k := \max_{u \in Y_k, \|u\| = \gamma_k} J(u) < 0$,

(B_3) $d_k := \inf_{u \in Z_k, \|u\| \leq \varrho_k} J(u) \to 0$ *as* $k \to +\infty$,

(B_4) *J satisfies the $(PS)_c^*$ condition for every $c \in [b_{k_0}, 0)$,*

then J has a sequence of negative critical values converging to 0.

3. Proof of the main result

We can see from $(A1)$ that the nonlinear terms $f_n(u)$ and $F_n(u)$ in our assumptions are locally defined near the origin 0. In order to establish the variational setting associated with (1.2) on the working space E, it needs to modify the global definitions of $f_n(u)$ and $F_n(u)$. To this end, we change $F_n(u)$ for u outside a neighborhood of 0 to get a new function on $\mathbb{R}$. For $d \in (0, (\lambda_{m^*+1} - \omega)/2)$ with λ_{m^*+1} defined in (2.2), we see from $(A3)$ that there exists $\delta \in (0, \epsilon/2)$ with ϵ given in $(A1)$, such that

$$F_n(u) \geq du^2, \quad n \in \mathbb{Z}^m, \quad |u| \leq 2\delta. \tag{3.1}$$

Set

$$H_n(u) := \zeta(|u|)F_n(u) + (1 - \zeta(|u|))du^2, \quad n \in \mathbb{Z}^m, \quad u \in \mathbb{R}, \tag{3.2}$$

where $\zeta \in C^1([0, +\infty), [0, +\infty))$ is a cut-off function satisfying $\zeta(s) = 1$ for $s \in [0, \delta]$, $\zeta(s) = 0$ for $s \in [2\delta, +\infty)$ and $\zeta'(s) < 0$ for $s \in (\delta, 2\delta)$. It follows that $H_n(u)$ is continuously differentiable in $u \in \mathbb{R}$ for each $n \in \mathbb{Z}^m$. By (3.2), a direct computation shows that

$$h_n(u) := H_n'(u) = \zeta(|u|)(f_n(u) - 2du) + \zeta'(|u|)\left(\frac{F_n(u)}{|u|} - d|u|\right)u + 2du \tag{3.3}$$

for $n \in \mathbb{Z}^m$ and $u \neq 0$. We also have $h_n(0) = 0$ for $n \in \mathbb{Z}^m$.

Lemma 3.1. *For H and h respectively given in (3.2) and (3.3), one has the following conclusions.*

(i) There exists a constant c such that

$$|h_n(u)| \leq c(|u| + |u|^{\nu_1 - 1}), \quad n \in \mathbb{Z}^m, \quad u \in \mathbb{R}. \tag{3.4}$$

(ii) $\widetilde{H}_n(u) := 2H_n(u) - h_n(u)u \geq 0$ for $u \in \mathbb{R}$ and $n \in \mathbb{Z}^m$, and $\widetilde{H}_n(u) = 0$ if and only if $u = 0$ or $|u| \geq 2\delta$ where δ is given in (3.1).

Proof. (i) It follows from (1.5) and $2\delta < \epsilon$ that

$$|f_n(u)| \leq c_1|u|^{\nu_1 - 1}, \quad n \in \mathbb{Z}^m, \quad |u| \leq 2\delta.$$

Note from (3.3) that for $n \in \mathbb{Z}^m$,

$$h_n(u) = f_n(u) \quad \text{for} \quad |u| < \delta \quad \text{and} \quad h_n(u) = 2du \quad \text{for} \quad |u| > 2\delta.$$

Moreover, for $n \in \mathbb{Z}^m$ and $|u| \in [\delta, 2\delta]$, we have

$$\begin{aligned} |h_n(u)| &\leq |f_n(u)| + 2d|u| + \max_{|u| \in [\delta, 2\delta]} \left|\zeta'(|u|)\left(\frac{F_n(u)}{|u|} - d|u|\right)\right| \cdot |u| + 2d|u| \\ &\leq c_1|u|^{\nu_1 - 1} + c_2|u| + 4d|u| \\ &\leq c(|u| + |u|^{\nu_1 - 1}), \end{aligned}$$

where $c_2 = \max_{|u| \in [\delta, 2\delta]} |\zeta'(|u|)\left(F_n(u)/|u| - d|u|\right)|$ and $c = c_1 + c_2 + 4d$. Thus, (3.4) is verified.

(ii) For $n \in \mathbb{Z}^m$, we see from (3.3) and the definition of $\widetilde{H}_n$ that

$$\widetilde{H}_n(u) = \zeta(|u|)(2F_n(u) - f_n(u)u) - \zeta'(|u|)\left(F_n(u) - du^2\right)|u|. \tag{3.5}$$

Noting that

$$\zeta(|u|) > 0 \geq \zeta'(|u|), \ |u| < 2\delta, \tag{3.6}$$

and $\zeta(|u|) = \zeta'(|u|) = 0$ for $|u| \geq 2\delta$, it follows from $(A4)$ and (3.1) that $\widetilde{H}_n(u) \geq 0$. By (3.5) and the definition of ζ, we know that $\widetilde{H}_n(u) = 0$ if either $u = 0$ or $|u| \geq 2\delta$. Conversely, assume by contradiction that there exist $n_0 \in \mathbb{Z}^m$ and $u_0 \in (-2\delta, 0) \cup (0, 2\delta)$ such that $\widetilde{H}_{n_0}(u_0) = 0$. Then we obtain from $(A4)$ and (3.6) that

$$\zeta(|u_0|)(2F_{n_0}(u_0) - f_{n_0}(u_0)u_0) = \zeta'(|u_0|)\left(F_{n_0}(u_0) - du_0^2\right)|u_0| \leq 0, \tag{3.7}$$

which contradicts the fact that the first term in (3.7) must be positive when $0 < |u| < 2\delta$. This finishes the proof. □

Lemma 3.2. *If $u^{(k)} \rightharpoonup u$ in E, then $w^{(k)} \to w$ in l^2, where $w^{(k)} = \left\{h_n(u_n^{(k)})\right\}_{n\in\mathbb{Z}^m}$ and $w = \{h_n(u_n)\}_{n\in\mathbb{Z}^m}$.*

Proof. Since $u^{(k)} \rightharpoonup u$ in E, we see that $u^{(k)} \to u$ in l^2 and $l^{2\nu_1-2}$, respectively. Thus $\{\|u^{(k)}\|_2\}$ and $\{\|u^{(k)}\|_{2\nu_1-2}\}$ are bounded. It is true for $n \in \mathbb{Z}^m$ that

$$\begin{aligned}
&|h_n(u_n^{(k)}) - h_n(u_n)|^2 \leq 2(|h_n(u_n^{(k)})|^2 + |h_n(u_n)|^2)\\
&\leq 2c^2((|u_n^{(k)}| + |u_n^{(k)}|^{\nu_1-1})^2 + (|u_n| + |u_n|^{\nu_1-1})^2)\\
&\leq 6c^2(|u_n^{(k)}|^2 + |u_n^{(k)}|^{2\nu_1-2} + |u_n|^2 + |u_n|^{2\nu_1-2}),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^m} |h_n(u_n^{(k)}) - h_n(u_n)|^2\\
&\leq 6c^2 \sum_{n\in\mathbb{Z}^m} (|u_n^{(k)}|^2 + |u_n^{(k)}|^{2\nu_1-2} + |u_n|^2 + |u_n|^{2\nu_1-2})\\
&= 6c^2(\|u^{(k)}\|_2^2 + \|u^{(k)}\|_{2\nu_1-2}^{2\nu_1-2} + \|u\|_2^2 + \|u\|_{2\nu_1-2}^{2\nu_1-2})\\
&< +\infty.
\end{aligned} \tag{3.8}$$

By using the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to+\infty} \sum_{n\in\mathbb{Z}^m} |h_n(u_n^{(k)}) - h_n(u_n)|^2 = \sum_{n\in\mathbb{Z}^m} \lim_{k\to+\infty} |h_n(u_n^{(k)}) - h_n(u_n)|^2 = 0.$$

The lemma is proved. □

To confirm Theorem 1.1, we consider the following auxiliary equation associated with (1.2):

$$Lu_n - \omega u_n = h_n(u_n), \quad n \in \mathbb{Z}^m \tag{3.9}$$

with the boundary condition (1.3), where h_n is given in (3.3). It suffices to show that (3.9) has a sequence $\{u^{(k)}\}$ in E with all $u^{(k)} \neq 0$ such that the l^∞-norm of $u^{(k)}$ converges to zero. Take k large enough such that the l^∞-norm of $u^{(k)}$ is less than δ, where δ is given in (3.1). Then $h_n(u_n^{(k)}) = f_n(u_n^{(k)})$ for $n \in \mathbb{Z}^m$ and (3.9) is reduced to (1.2). Thus Theorem 1.1 follows.

Consider the functional J defined on E by

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{n\in\mathbb{Z}^m} H_n(u_n), \tag{3.10}$$

where $H_n(u)$ is given in (3.2). We claim that J is a well-defined C^1 functional on E, and for the derivative J' of J,

$$(J'(u), v) = ((L-\omega)u, v)_2 - \sum_{n\in\mathbb{Z}^m} h_n(u_n)v_n, \quad u, v \in E. \tag{3.11}$$

In fact, it follows from (3.4) that

$$|H_n(u)| \le d_0(|u|^2 + |u|^{\nu_1}), \quad n \in \mathbb{Z}^m, \quad u \in \mathbb{R},$$

for some $d_0 > 0$. This gives us that

$$\begin{aligned}\sum_{n\in\mathbb{Z}^m} H_n(u_n) &\le \sum_{n\in\mathbb{Z}^m} d_0(|u|^2 + |u|^{\nu_1}) \\ &\le d_0(\|u\|_2^2 + \|u\|_{\nu_1}^{\nu_1}).\end{aligned}$$

We see from (2.3) that J is well defined on E. For any $u, v \in E$ and any sequence $\{\theta_n\}$ with $\theta_n \in (0,1)$ for $n \in \mathbb{Z}^m$, similarly to the proof of (3.8), it is true that

$$\sum_{n\in\mathbb{Z}^m} \max_{t\in(0,1)} |h_n(u_n + t\theta_n v_n)v_n| < +\infty.$$

Combining (3.10) and the above inequality, we have

$$\begin{aligned}(J'(u), v) &= \lim_{t\to 0^+} \frac{J(u+tv) - J(u)}{t} \\ &= \lim_{t\to 0^+} \frac{1}{2t}[((L-\omega)(u+tv), u+tv)_2 - ((L-\omega)u, u)_2] \\ &\quad - \lim_{t\to 0^+} \frac{1}{t} \sum_{n\in\mathbb{Z}^m} [H_n(u_n + tv_n) - H_n(u_n)] \\ &= ((L-\omega)u, v)_2 - \lim_{t\to 0^+} \left[\sum_{n\in\mathbb{Z}^m} h_n(u_n + t\theta_n v_n)v_n\right] \\ &= ((L-\omega)u, v)_2 - \sum_{n\in\mathbb{Z}^m} h_n(u_n)v_n.\end{aligned}$$

This verifies (3.11). Next, we show that the derivative J' is continuous. Let $u^{(k)} \to u \in E$ as $k \to +\infty$. For any $v \in E$, we have

$$\begin{aligned}&|(J'(u^{(k)}) - J'(u), v)| \\ &\le \left|\Big((L-\omega)(u^{(k)} - u), v\Big)_2\right| + \sum_{n\in\mathbb{Z}^m} \left|h_n(u_n^{(k)}) - h_n(u_n)\right| |v_n| \\ &\le o_k(1) + \|v\|_2 \left[\sum_{n\in\mathbb{Z}^m} \left|h_n(u_n^{(k)}) - h_n(u_n)\right|^2\right]^{\frac{1}{2}},\end{aligned}$$

where $o_k(1)$ means $o_k(1) \to 0$ as $k \to +\infty$. It follows from Lemma 3.2 that

$$(J'(u^{(k)}) - J'(u), v) \to 0 \quad \text{as} \quad k \to +\infty,$$

which implies $J \in C^1(E, \mathbb{R})$. The claim is confirmed.

We see from (3.11) that (3.9) is the corresponding Euler-Lagrange equation of J. To find nontrivial homoclinic solutions of (3.9), we need only to look for nonzero critical points of J in E.

Before seeking nonzero critical points of J, we check critical points at the zero energy level.

Lemma 3.3. *Assume that (A1)–(A4) are satisfied. Then 0 is the unique critical point of J such that $J(u) = 0$.*

Proof. It is easy to see that 0 is a critical point of J with $J(u) = 0$. Let $u \in E$ be a critical point of J with $J(u) = 0$. Then we have

$$0 = 2J(u) - (J'(u), u) = \sum_{n\in\mathbb{Z}^m} \widetilde{H}_n(u_n).$$

It follows from Lemma 3.1 that either $u_n = 0$ or $|u_n| \geq 2\delta$ for each $n \in \mathbb{Z}^m$. We claim that $u_n \equiv 0$. If it is not true, then we have $A_\delta := \{n \in \mathbb{Z}^m : |u_n| \geq 2\delta\} \neq \emptyset$ and

$$\begin{aligned}
0 &= ((L-\omega)u, v)_2 - \sum_{n\in\mathbb{Z}^m} h_n(u_n)v_n \\
&= ((L-\omega)u, v)_2 - \sum_{n\in A_\delta} h_n(u_n)v_n \\
&= ((L-\omega)u, v)_2 - 2d \sum_{n\in A_\delta} u_n v_n \\
&= (Lu, v)_2 - (2d+\omega)\cdot(u, v)_2.
\end{aligned}$$

This is impossible since $d \in (0, (\lambda_{m^*+1} - \omega)/2)$. The proof is completed. $\square$

In what follows, we verify the conditions in Lemma 2.2 to find nonzero critical points of J in E.

Lemma 3.4. *There exist a positive integer k_0 and two sequences $\{\varrho_k\}$ and $\{\gamma_k\}$ with $0 < \gamma_k < \varrho_k$ for $k \geq k_0$ and $\varrho_k \to 0$ as $k \to +\infty$ such that*

$$a_k := \inf_{u\in Z_k,\, \|u\|=\varrho_k} J(u) \geq 0, \tag{3.12}$$

$$b_k := \max_{u\in Y_k,\, \|u\|=\gamma_k} J(u) < 0, \quad \textit{and} \tag{3.13}$$

$$d_k := \inf_{u\in Z_k,\, \|u\|\leq\varrho_k} J(u) \to 0, \quad \textit{as} \quad k \to +\infty, \tag{3.14}$$

where $Y_k = span\{e_1, \cdots, e_k\}$ and $Z_k = \overline{span\{e_k, e_{k+1}, \cdots\}}$.

Proof. Note that $Z_k \subset E^+$ for $k \geq m^* + 1$, where m^* is specified in (2.2). In this case, we get for $u \in Z_k$ that

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}^m} H_n(u_n) \\
&\geq \frac{1}{2}\|u\|^2 - c\sum_{n\in\mathbb{Z}^m} (|u_n|^2 + |u_n|^{\nu_1}) \\
&= \frac{1}{2}\|u\|^2 - c(\|u\|_2^2 + \|u\|_{\nu_1}^{\nu_1})
\end{aligned}$$

$$\geq \frac{1}{2}\|u\|^2 - c\beta_k^2\|u\|^2 - c\zeta_k^{\nu_1}\|u\|^{\nu_1}, \tag{3.15}$$

where $\beta_k := \sup_{A_k} \|u\|_2$, $\zeta_k := \sup_{A_k} \|u\|_{\nu_1}$ and $A_k := \{u \in Z_k : \|u\| = 1\}$. We claim that $\beta_k \to 0$ as $k \to +\infty$. In fact, it is clear that $0 < \beta_{k+1} \leq \beta_k$. Thus, there exists $\beta_0 \geq 0$ such that $\beta_k \to \beta_0$ as $k \to +\infty$. For each $k \geq m^* + 1$, there exists $u^{(k)} \in Z_k$ such that $\|u^{(k)}\| = 1$ and $\|u^{(k)}\|_2 > \beta_k/2$. By the definition of Z_k, $u^{(k)} \rightharpoonup 0$ in E, and so $u^{(k)} \to 0$ in l^2. Thus we have proved that $\beta_0 = 0$. Similarly, we can show that $\zeta_k \to 0$ as $k \to +\infty$. Thus, there exists a positive integer $k_0 \geq m^* + 1$ such that

$$c\beta_k^2 \leq \frac{1}{4} \quad \text{and} \quad \varrho_k := (8c\zeta_k^{\nu_1})^{\frac{1}{2-\nu_1}} < 1$$

for $k \geq k_0$. Obviously, $\varrho_k \to 0$ as $k \to +\infty$. For $u \in Z_k$ with $\|u\| = \varrho_k$, it follows from (3.15) that

$$J(u) \geq \frac{1}{4}\|u\|^2 - c\zeta_k^{\nu_1}\|u\|^{\nu_1} = \frac{1}{8}\varrho_k^2.$$

This proves (3.12).

Next, we prove (3.13). By the definition of Y_k, for each $k \geq k_0$, $\dim Y_k < +\infty$. Thus there exists $T_k > 0$ such that $\|u\|^2 \leq T_k\|u\|_2^2$ for $u \in Y_k$. By $(A3)$ and (3.2), there exists $s_k \in (0, \epsilon/2)$ such that

$$H_n(u) \geq T_k|u|^2, \quad n \in \mathbb{Z}^m, \quad |u| < s_k.$$

For $u \in Y_k$ with $\|u\| = \gamma_k := \min\{s_k/(2\rho_\infty), \varrho_k/2\}$ where ρ_∞ and ϱ_k are respectively given in (2.3) and (3.12), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{n\in\mathbb{Z}^m} H_n(u_n) \\ &\leq \frac{1}{2}\|u^+\|^2 - T_k \sum_{n\in\mathbb{Z}^m} |u_n|^2 \\ &= \frac{1}{2}\|u^+\|^2 - T_k\|u\|_2^2 \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 \\ &= -\frac{1}{2}\|u\|^2. \end{aligned}$$

In other words, it holds that

$$\max_{u\in Y_k,\, \|u\|=\gamma_k} J(u) \leq -\frac{1}{2}\gamma_k^2,$$

and so (3.13) is proved.

Since $\varrho_k \to 0$ as $k \to +\infty$, we see that

$$d_k = \inf_{u\in Z_k,\, \|u\|\leq\varrho_k} J(u) \to J(0) = 0$$

as $k \to +\infty$, which verifies (3.14). The proof is completed. $\square$

Lemma 3.5. *J satisfies the $(PS)^*_\alpha$ condition for $\alpha \in \mathbb{R}$.*

Proof. Assume that $\{u^{(k)}\}$ is a $(PS)^*_\alpha$ sequence in E for $\alpha \in \mathbb{R}$. In order to show that $\{u^{(k)}\}$ has a convergent subsequence in E, we first claim that $\{\|u^{(k)}\|\}$ is bounded. Assume by contradiction that, up to a subsequence, $\|u^{(k)}\| \to +\infty$ as $k \to +\infty$. Let

$$w^{(k)} = \frac{u^{(k)}}{\|u^{(k)}\|}, \quad w^{(k)+} = \frac{u^{(k)+}}{\|u^{(k)}\|}, \quad w^{(k)-} = \frac{u^{(k)-}}{\|u^{(k)}\|}, \quad w^{(k)0} = \frac{u^{(k)0}}{\|u^{(k)}\|}.$$

Then $w^{(k)} = w^{(k)+} + w^{(k)-} + w^{(k)0}$ and $\|w^{(k)}\| = 1$. Taking a subsequence if necessary, there exists $w \in E$ such that

$$w^{(k)} \rightharpoonup w, \quad w^{(k)+} \rightharpoonup w^+, \quad w^{(k)-} \to w^-, \quad w^{(k)0} \to w^0, \quad \text{in} \quad E.$$

Thus, we have

$$w^{(k)} \to w, \quad w^{(k)+} \to w^+, \quad w^{(k)-} \to w^-, \quad w^{(k)0} \to w^0, \quad \text{in} \quad l^2.$$

Case 1: $w = 0$**.** It holds that

$$w^{(k)-} \to 0, \quad w^{(k)0} \to 0, \quad \text{in} \quad E,$$

which implies that

$$\|w^{(k)-}\|^2 \to 0, \quad \|w^{(k)0}\|^2 \to 0, \quad \text{as} \quad k \to +\infty. \tag{3.16}$$

Since $\{u^{(k)}\}$ is a $(PS)^*_\alpha$ sequence in E, we have

$$\begin{aligned} o_k(1) &= \frac{1}{\|u^{(k)}\|^2}(J'(u^{(k)}), u^{(k)+}) \\ &= \frac{1}{\|u^{(k)}\|^2}\left[(u^{(k)+}, u^{(k)+}) - \sum_{n\in\mathbb{Z}^m} h_n(u_n^{(k)})u_n^{(k)+}\right] \\ &= \|w^{(k)+}\|^2 - \frac{1}{\|u^{(k)}\|^2}\sum_{n\in\mathbb{Z}^m} h_n(u_n^{(k)})u_n^{(k)+}. \end{aligned} \tag{3.17}$$

In addition, we see from $(A2)$, (2.3) and (3.4) that

$$\begin{aligned} &\frac{1}{\|u^{(k)}\|^2}\sum_{n\in\mathbb{Z}^m} |h_n(u_n^{(k)})u_n^{(k)+}| \\ &\le \frac{c\sum_{n\in\mathbb{Z}^m}(|u_n^{(k)}| + |u_n^{(k)}|^{\nu_1-1})|u_n^{(k)+}|}{\|u^{(k)}\|^2} \\ &\le \frac{c\|u^{(k)+}\|_2(\|u^{(k)}\|_2 + \|u^{(k)}\|_{2\nu_1-2}^{\nu_1-1})}{\|u^{(k)}\|^2} \\ &\le c\|w^{(k)+}\|_2\left(\|w^{(k)}\|_2 + \frac{\rho_{2\nu_1-2}^{\nu_1-1}\|u^{(k)}\|^{\nu_1-1}}{\|u^{(k)}\|}\right) \to 0 \end{aligned} \tag{3.18}$$

as $k \to +\infty$. Combining (3.17) and (3.18) gives us that

$$\|w^{(k)+}\|^2 \to 0 \quad \text{as} \quad k \to +\infty. \tag{3.19}$$

Thus, by using (3.16) and (3.19), we obtain

$$1 = \|w^{(k)}\|^2 = \|w^{(k)+}\|^2 + \|w^{(k)-}\|^2 + \|w^{(k)0}\|^2 \to 0$$

as $k \to +\infty$. This is a contradiction.

Case 2: $w \neq 0$**.** In this case, it is true that

$$o_k(1) = (u^{(k)+}, \phi^{(k)}) - (u^{(k)-}, \phi^{(k)}) - \sum_{n\in\mathbb{Z}^m} h_n(u_n^{(k)})\phi_n^{(k)}$$

where $\phi^{(k)} = \phi\,|_{Y_k}$ for $\phi = \sum_{i=1}^{+\infty} t_i e_i \in E$ and $Y_k = \operatorname{span}\{e_1, \cdots, e_k\}$. Then we can obtain that

$$\begin{aligned} o_k(1) &= \frac{(u^{(k)+}, \phi^{(k)})}{\|u^{(k)}\|} - \frac{(u^{(k)-}, \phi^{(k)})}{\|u^{(k)}\|} - \frac{\sum_{n\in\mathbb{Z}^m} 2du_n^{(k)}\phi_n^{(k)}}{\|u^{(k)}\|} \\ &\quad - \frac{\sum_{n\in\mathbb{Z}^m}(h_n(u_n^{(k)}) - 2du_n^{(k)})\phi_n^{(k)}}{\|u^{(k)}\|} \\ &= (w^{(k)+}, \phi^{(k)}) - (w^{(k)-}, \phi^{(k)}) - 2d\cdot(w^{(k)}, \phi^{(k)})_2 \\ &\quad - \frac{\sum_{n\in\mathbb{Z}^m}(h_n(u_n^{(k)}) - 2du_n^{(k)})\phi_n^{(k)}}{\|u^{(k)}\|}. \end{aligned} \tag{3.20}$$

Similarly to the proof of (3.4), we can see that

$$|h_n(u) - 2du| \leq C|u|^{\nu_1 - 1}, \quad n \in \mathbb{Z}^m, \quad u \in \mathbb{R},$$

for some $C > 0$. It follows from $(A2)$ and (2.3) that

$$\begin{aligned} &\frac{1}{\|u^{(k)}\|}\sum_{n\in\mathbb{Z}^m} |(h_n(u_n^{(k)}) - 2du_n^{(k)})\phi_n^{(k)}| \\ &\leq \frac{C}{\|u^{(k)}\|}\sum_{n\in\mathbb{Z}^m} |u_n^{(k)}|^{\nu_1-1}|\phi_n^{(k)}| \\ &\leq \frac{C\|u^{(k)}\|_{2\nu_1-2}^{\nu_1-1}\|\phi^{(k)}\|_2}{\|u^{(k)}\|} \\ &\leq \frac{C\rho_{2\nu_1-2}^{\nu_1-1}\|u^{(k)}\|^{\nu_1-1}\|\phi^{(k)}\|_2}{\|u^{(k)}\|}. \end{aligned} \tag{3.21}$$

Letting $k \to +\infty$, (3.20) together with (3.21) shows that

$$(w^+, \phi) - (w^-, \phi) - 2d\cdot(w, \phi)_2 = 0, \quad w \in E.$$

This is impossible since $d \in (0, (\lambda_{m^*+1} - \omega)/2)$.

In summary of the above two cases, we have shown that $\{u^{(k)}\}$ is bounded in E.

Next, we verify that $\{u^{(k)}\}$ admits a convergent subsequence in E. By the boundedness of $\{u^{(k)}\}$ in E, we may assume that

$$u^{(k)} \rightharpoonup u \quad \text{in} \quad E.$$

Then we have

$$u^{(k)+} \rightharpoonup u^+ \text{ in } E \quad \text{and} \quad u^{(k)+} \to u^+ \text{ in } l^2.$$

Since $(J'(u^{(k)}) - J'(u), u^{(k)+} - u^{+}) \to 0$ as $k \to +\infty$, we see from (3.11) that

$$\|u^{(k)+} - u^{+}\|^2 = o_k(1) + \sum_{n\in\mathbb{Z}^m} (h_n(u_n^{(k)}) - h_n(u_n))(u_n^{(k)+} - u_n^{+}).$$

Note from Lemma 3.2 that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^m} (h_n(u_n^{(k)}) - h_n(u_n))(u_n^{(k)+} - u_n^{+})\\ \le& \left[\sum_{n\in\mathbb{Z}^m} (h_n(u_n^{(k)}) - h_n(u_n))^2\right]^{\frac{1}{2}} \|u^{(k)+} - u^{+}\|_2 \to 0\end{aligned}$$

as $k \to +\infty$. This further implies that $u^{(k)+} \to u^{+}$ in E. Since $\dim(E^{-} \oplus E^{0}) < +\infty$, we obtain that both $u^{(k)-} \to u^{-}$ and $u^{(k)0} \to u^{0}$ in E. As a result, $u^{(k)} \to u$ in E. The proof is finished. □

Proof of Theorem 1.1. We have verified the conditions in Lemma 2.2. It follows that J has a sequence of nontrivial critical points $\{u^{(k)}\}$ in E satisfying $J(u^{(k)}) \to 0$ as $k \to +\infty$. Thus, $\{u^{(k)}\}$ is a sequence of nontrivial homoclinic solutions of (3.9). In addition, $\{u^{(k)}\}$ is also a $(PS)_0$ sequence in E. Similarly to the proof of Lemma 3.5, we can show that J satisfies the $(PS)_0$ condition. Up to a subsequence, we assume that $u^{(k)} \to u$ in E. Since 0 is a critical point of J with $J(0) = 0$, we see from Lemma 3.3 that $u = 0$. Thus, $u^{(k)} \to 0$ in E. According to (2.3), we find $u^{(k)} \to 0$ in l^{∞}. Let k be large enough such that $\|u^{(k)}\|_{\infty} < \delta$ with δ given in (3.1). Then $h_n(u_n^{(k)}) = f_n(u_n^{(k)})$ for $n \in \mathbb{Z}^m$ and (3.9) becomes (1.2). The desired result follows and the proof is completed.

Acknowledgements. The authors wish to thank the anonymous referees for their careful reading and useful comments of the paper.

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