Journal of Applied Analysis and Computation
Volume 12, Number 3, June 2022, 981–1006
Website:http://www.jaac-online.com
DOI:10.11948/20220093

STABILITY ANALYSIS OF A FRACTIONAL PREDATOR-PREY SYSTEM WITH TWO DELAYS AND INCOMMENSURATE ORDERS*

Yingxian Zhu[1], Shuangfei Li[1] and Yunxian Dai[1,†]

Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract In this paper, we consider a fractional predator-prey system with two delays and incommensurate orders. Firstly, the local stability of positive equilibrium of the system without delay is discussed. Secondly, we calculate the critical value of Hopf bifurcation by taking one delay as bifurcation parameter. Then, as two nonidentical delays change simultaneously, the stability switching curves, the directions of crossing and the existence of Hopf bifurcation are obtained. Finally, numerical simulations are presented to verify the given theoretical results.

Keywords Fractional predator-prey system, Hopf bifurcation, two delays, incommensurate orders, stability switching curves.

MSC(2010) 34C23, 34C60.

1. Introduction

Fractional calculus is a kind of generalization of integer-order calculus. Due to the effects of memory and hereditary properties, fractional calculus can more accurately describe the complex and rich dynamic behavior of the system rather than integer-order calculus [12, 15]. Because of the complexity of the calculation, fractional calculus has not been widely used for a long time. It was not until fractal theory was introduced in [18] that fractional calculus has progressively become one of the research hotspots. In recent years, fractional calculus has been successfully introduced into various fields, such as physics, chemistry, electricity, biology, economics, etc [3, 5, 7, 8, 21, 22].

It takes a certain amount of time to complete biological evolution and physical process, therefore delay is widely found in nature. The appearance of time delay means that the development of the system is not only dependent on the current state, but also related to the state in the previous period. On the other hand, time delay can cause Hopf bifurcation. When the parameter changes slightly near a critical value, the stability of the equilibrium of system changes, and there is the phenomenon of periodic solution near equilibrium. For fractional-order delay differential systems, Hopf bifurcation is also a common bifurcation phenomenon.

[†]The corresponding author. Email:dyxian1976@sina.com (Y. Dai)
[1]Department of System Science and Applied Mathematics, Kunming University of Science and Technology, Kunming 650500, China
[*]The authors were supported by National Natural Science Foundation of China (No. 11761040).

In [2, 17, 25, 30], the authors chose the delay as bifurcation parameter, and studied Hopf bifurcation of fractional-order delay differential systems.

The predator-prey model is an important model in population dynamics models [31]. Authors in [1] analyzed the dynamic behavior of the prey-predator system with time delays, and it was shown that the stability of the system can be changed with the change of harvest. In [26], authors studied a delayed predator-prey system with Beddington-DeAngelis functional response and got that the phenomenon of Hopf bifurcation is the main factor that switching the stability of the system to unstability with respect to delays. In [28], Wang and Tang took the Holling-III functional response, the complex diversity of biological environment and efforts into account, and discussed the Hopf bifurcation of a time-delayed predator-prey system which is as follows:

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x(t-\tau_1)}{K}) - \frac{\alpha(1-c)x^2y}{1+\alpha h(1-c)x^2}, \\ \frac{dy}{dt} = \frac{\beta\alpha(1-c)x^2(t-\tau_2)y(t-\tau_2)}{1+\alpha h(1-c)x^2(t-\tau_2)} - dy - qE_0y, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the densities of prey and predator, respectively. The growth rate and the environmental capacity of prey are represented by r and K, respectively. α denotes the attack coefficient; c $(0 < c < 1)$ represents a dimensionless parameter that measures habitat complexity; d is the death rate of the predator; q is the coefficient of catchability; E_0 denotes the harvesting effort; and $\frac{\alpha(1-c)x^2y}{1+\alpha h(1-c)x^2}$ is Holling-III functional response. All parameters of system (1.1) are positive.

For integer-order delay differential equations, a lot of researchers have paid too much attention to predator-prey systems with single delay, such as [4, 14]. Even though the integer-order predator-prey models with two delays have been discussed in [9, 29], the approaches are to make the two delays equal or fix one delay and choose the other one as bifurcation parameter. There is little work to consider differential equations with two delays varying simultaneously. In [10, 16], integer-order systems with two time delays varying simultaneously were discussed. By analyzing the characteristic equations of the following two forms

$$D(s;\tau_1,\tau_2) = U_0(s) + U_1(s)e^{-s\tau_1} + U_2(s)e^{-s\tau_2}$$

and

$$D(s;\tau_1,\tau_2) = U_0(s) + U_1(s)e^{-s\tau_1} + U_2(s)e^{-s\tau_2} + U_3(s)e^{-s(\tau_1+\tau_2)}$$

respectively in [10, 16], calculating the explicit expression of the stability switching curves, and giving the judgment method of the directions of change in stability, then the stability of the system was obtained. The method of [10, 16] has been applied to study the stability of the integer-order systems with two time delays varying simultaneously, such as [11, 20, 23].

For fractional-order delay differential equations, most of systems concerned by scholars are systems with one or two delays, but the ways to study the systems with two delays are either to make two delays equal or only fix one delay and choose the other delay as bifurcation parameter, such as [13, 25, 27, 32]. As far as we are aware, few authors have studied the stability of fractional-order systems when two delays change simultaneously. Therefore, it is significant to extend the method of [10, 16] to fractional-order systems with two delays.

Motived by the method of stability switching curves of [10,16] and based on the system of [28], we consider a fractional-order predator-prey system with two delays and incommensurate orders:

$$\begin{cases} D^{\gamma_1}x(t) = rx(1-\frac{x(t-\tau_1)}{K}) - \frac{\alpha(1-c)x^2y}{1+\alpha h(1-c)x^2}, \\ D^{\gamma_2}y(t) = \frac{\beta\alpha(1-c)x^2(t-\tau_2)y(t-\tau_2)}{1+\alpha h(1-c)x^2(t-\tau_2)} - dy - qE_0y, \end{cases} \tag{1.2}$$

where $\gamma_1,\ \gamma_2 \in (0,1],\ \gamma_1 \neq \gamma_2$. The other parameters have the same biological significance as system (1.1), and initial conditions are $x(t) > 0$ and $y(t) > 0$, $t \in [-\max\{\tau_i\}, 0]$ $(i = 1, 2)$. The highlights of this paper are generalized as follows:

(i) We generalize integer-order delayed predator-prey system to a new fractional predator-prey system with two delays and incommensurate orders.

(ii) The stability and the Hopf bifurcation of a fractional predator-prey system with two delays and incommensurate orders are obtained. It is shown that the delay and the order play an important role in the stability and the existence of Hopf bifurcation of the corresponding fractional-order system.

(iii) To the best of our knowledge, there are not many results on the stability of fractional-order system and the existence of Hopf bifurcation with two delays varying simultaneously. Discriminating from the general ways that making two delays equal or fixing one delay and choosing another delay as bifurcation parameter, this paper is the case that the stability of system and the existence of Hopf bifurcation are obtained by taking two delays as bifurcation parameters, and the stable region is a two-dimensional region about τ_1 and τ_2.

(iv) The method of determining the stability of the system by calculating the stability switching curves is first applied to the fractional differential equations with two delays. It is meaningful to study fractional-order differential equations with two delays.

The paper is organized as folllows. In Section 2, some basic knowledge and necessary lemmas on fractional calculus are presented. In Section 3, using the method in [10, 16], the stability and the existence of Hopf bifurcation of system (1.2) are obtained by calculating the stability switching curves and the directions of change in stability, taking the two delays as bifurcation parameters and considering the simultaneous change of the two delays. In Section 4, we perform numerical simulations to confirm the theoretical results. This paper ends with a conclusion.

2. Preliminaries

In fractional derivatives, the Grunwald-Letnikov(G-L) definition, the Riemann-Liouville(R-L) definition and the Caputo definition are commonly used. Then the Caputo definition can not only simplify the Laplace transform properly, but also allow that the initial conditions of the corresponding fractional-order equation can be expressed in integer order, which is more suitable for practical mathematical problems. In this paper, all the fractional derivatives are the Caputo definition. Next, we will introduce the Caputo definition and two fundamental lemmas for the following theoretical analysis.

Definition 2.1 ([24]). The Caputo fractional-order derivative is defined as follows:

$$D^{\gamma} f(t)=\frac{1}{\Gamma(n-\gamma)} \int_{t_0}^{t}(t-s)^{n-\gamma-1} f^{n}(s) ds,$$

where $f(t) \in C^{n}([t_0,\infty),\mathbb{R})$, $t>t_0$, $n-1 \leq \gamma < n$, $n \in \mathbb{Z}^{+}$, and $\Gamma(\cdot)$ is the Gamma function.

When $0<\gamma<1$, we have

$$D^{\gamma} f(t)=\frac{1}{\Gamma(n-\gamma)} \int_{t_0}^{t}(t-s)^{\gamma} f^{'}(s) ds.$$

Lemma 2.1 ([19]). *Consider the fractional-order system*

$$\begin{cases} D^{\gamma} x(t)=f(t,x(t)), \\ x(0)=x_0, \end{cases} \tag{2.1}$$

where $\gamma \in (0,1]$ and $f(t,x(t)) : \mathbb{R}^{+} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. The equilibria of system (2.1) *are locally asymptotically stable if all eigenvalues λ_i $(i=1,2,\ldots,n)$ of the Jacobian matrix $\frac{\partial f(t,x)}{\partial x}$ evaluated at the equilibria satisfy $|\arg(\lambda_i)| > \frac{\gamma\pi}{2}$.*

Lemma 2.2 ([6]). *For given n-dimensional linear fractional-order system:*

$$\begin{cases} D^{\gamma_1} x_1(t)=\alpha_{11} x_1(t-\tau_{11})+\alpha_{12} x_2(t-\tau_{12})+\cdots+\alpha_{1n} x_n(t-\tau_{1n}), \\ D^{\gamma_2} x_2(t)=\alpha_{21} x_1(t-\tau_{21})+\alpha_{22} x_2(t-\tau_{22})+\cdots+\alpha_{2n} x_n(t-\tau_{2n}), \\ \quad \vdots \\ D^{\gamma_n} x_n(t)=\alpha_{n1} x_1(t-\tau_{n1})+\alpha_{n2} x_2(t-\tau_{n2})+\cdots+\alpha_{nn} x_n(t-\tau_{nn}), \end{cases} \tag{2.2}$$

where $\gamma_i \in (0,1]$ $(i=1,2,\cdots,n)$, and the initial conditions $x_i(t) \in C[-\max\{\tau_{ij}\},0]$, $t \in [-\max\{\tau_{ij}\},0]$ $(i,j=1,2,\cdots,n)$.

Let

$$\Delta(s)=\begin{bmatrix} s^{\gamma_1}-\alpha_{11} e^{-s\tau_{11}} & -\alpha_{12} e^{-s\tau_{12}} & \cdots & -\alpha_{1n} e^{-s\tau_{1n}} \\ -\alpha_{21} e^{-s\tau_{21}} & s^{\gamma_2}-\alpha_{22} e^{-s\tau_{22}} & \cdots & -\alpha_{2n} e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1} e^{-s\tau_{n1}} & -\alpha_{n2} e^{-s\tau_{n2}} & \cdots & s^{\gamma_n}-\alpha_{nn} e^{-s\tau_{nn}} \end{bmatrix},$$

where $\Delta(s)$ is the characteristic matrix of system (2.2). *The zero solution of system* (2.2) *is locally asymptotically stable if all the roots of* $\det(\Delta(s))=0$ *possess negative real parts.*

3. Main results

In this section, we discuss the stability of system (1.2) and the existence of Hopf bifurcation. We first analyze the existence of positive equilibrium of system (1.2).

Next, we discuss the local stability of the positive equilibrium of the system (1.2) without delay by Routh-Hurwitz criterion. Then, we address the existence of Hopf bifurcation at the positive equilibrium of the system (1.2) with one delay by taking τ_1 and τ_2 as bifurcation parameter, respectively. Finally, applying the method of [10,16], we study the change in stability of the positive equilibrium and the existence of Hopf bifurcation of the system (1.2) when two delays change simultaneously.

3.1. Existence of positive equilibrium of system (1.2)

The equilibria J_0, J_K, J^* of system (1.2) can be obtained by the following equations:

$$\begin{cases} rx(1-\frac{x}{K}) - \frac{\alpha(1-c)x^2y}{1+\alpha h(1-c)x^2} = 0, \\ \frac{\beta\alpha(1-c)x^2y}{1+\alpha h(1-c)x^2} - dy - qE_0y = 0, \end{cases}$$

where

$$\begin{aligned} &J = (0,0),\ J_K = (K,0),\ J^* = (x^*, y^*), \\ &x^* = \sqrt{\frac{d+qE_0}{\alpha(1-c)(\beta-(d+qE_0)h)}},\ y^* = \frac{\beta r x^*}{d+qE_0}(1-\frac{x^*}{K}). \end{aligned}$$

When the following assumptions (H_1) and (H_2) hold:

$$\begin{aligned} &(H_1):\ \beta - (d+qE_0)h > 0, \\ &(H_2):\ 1 - \frac{x^*}{K} > 0, \end{aligned}$$

J^* is a unique positive equilibrium of system (1.2).

Using the transformation $P_1(t) = x(t) - x^*$, $P_2(t) = y(t) - y^*$, the system (1.2) can be written as

$$\begin{cases} D^{\gamma_1}P_1(t) = r(P_1(t)+x^*)(1-\frac{P_1(t-\tau_1)+x^*}{K}) - \frac{\alpha(1-c)(P_1(t)+x^*)^2(P_2(t)+y^*)}{1+\alpha h(1-c)(P_1(t)+x^*)^2}, \\ D^{\gamma_2}P_2(t) = \frac{\beta\alpha(1-c)(P_1(t-\tau_2)+x^*)^2(P_2(t-\tau_2)+y^*)}{1+\alpha h(1-c)(P_1(t-\tau_2)+x^*)^2} - (d+qE_0)(P_2(t)+y^*). \end{cases} \tag{3.1}$$

The linearization of the system (1.2) at J^* is

$$\begin{cases} D^{\gamma_1}P_1(t) = a_{11}P_1(t) + a_{12}P_2(t) + b_{11}P_1(t-\tau_1), \\ D^{\gamma_2}P_2(t) = a_{22}P_2(t) + c_{21}P_1(t-\tau_2) - a_{22}P_2(t-\tau_2), \end{cases} \tag{3.2}$$

where

$$\begin{aligned} a_{11} =& \frac{r(K-x^*)[\alpha h(1-c)x^{*2}-1]}{K[1+\alpha h(1-c)x^{*2}]}, \\ a_{12} =& -\frac{d+qE_0}{\beta} < 0, \\ a_{22} =& -(d+qE_0) < 0, \\ b_{11} =& -\frac{rx^*}{K} < 0, \end{aligned}$$

$$c_{21} = \frac{2\beta r(K - x^*)}{K[1 + \alpha h(1 - c)x^{*2}]} > 0.$$

The corresponding characteristic equation at J^* is

$$D(s;\tau_1,\tau_2) = U_0(s) + U_1(s)e^{-s\tau_1} + U_2(s)e^{-s\tau_2} + U_3(s)e^{-s(\tau_1+\tau_2)} = 0, \tag{3.3}$$

where

$$\begin{aligned}
U_0(s) =& s^{\gamma_1+\gamma_2} - a_{22}s^{\gamma_1} - a_{11}s^{\gamma_2} + a_{11}a_{22},\\
U_1(s) =& - b_{11}s^{\gamma_2} + a_{22}b_{11},\\
U_2(s) =& a_{22}s^{\gamma_1} - a_{11}a_{22} - a_{12}c_{21},\\
U_3(s) =& - a_{22}b_{11}.
\end{aligned}$$

3.2. Stability analysis of positive equilibrium J_* of system (1.2) without delay

When $\tau_1 = \tau_2 = 0$, Eq.(3.3) becomes:

$$s^{\gamma_1+\gamma_2} - (a_{11} + b_{11})s^{\gamma_2} - a_{12}c_{21} = 0. \tag{3.4}$$

To get the following result, the hypothesis (H_3) is given:

$$(H_3) : a_{11} + b_{11} < 0.$$

Theorem 3.1. *For $\tau_1 = \tau_2 = 0$, the positive equilibrium J^* of system* (1.2) *is locally asymptotically stable if $(H_1) - (H_3)$ are satisfied.*

Proof. The proof is done by contradiction.

Due to

$$U_0(0) + U_1(0) + U_2(0) + U_3(0) = -a_{12}c_{21} > 0,$$

$s = 0$ is not a root of Eq.(3.4).

Suppose Eq.(3.4) has roots, which are denoted as $\rho e^{\kappa i}$, and $\rho > 0$, $\kappa \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Substituting $s = \rho e^{\kappa i} = \rho(\cos\kappa + i\sin\kappa)$ into Eq.(3.4), and separating the real and imaginary parts, then the following equations can be obtained:

$$\rho^{\gamma_1+\gamma_2}\cos(\gamma_1+\gamma_2)\kappa - (a_{11}+b_{11})\rho^{\gamma_2}\cos\gamma_2\kappa - a_{12}c_{21} = 0, \tag{3.5}$$

$$\rho^{\gamma_1+\gamma_2}\sin(\gamma_1+\gamma_2)\kappa - (a_{11}+b_{11})\rho^{\gamma_2}\sin\gamma_2\kappa = 0. \tag{3.6}$$

For Eq.(3.6), we can get

$$\rho^{\gamma_1}(\sin\gamma_1\kappa\cos\gamma_2\kappa + \cos\gamma_1\kappa\sin\gamma_2\kappa) - (a_{11}+b_{11})\sin\gamma_2\kappa = 0.$$

Due to the definitions of a_{12} and c_{21}, the ranges of γ_1, γ_2 and κ, the properties of trigonometric functions and (H_3), then $a_{12}c_{21} < 0$, $a_{11}+b_{11} < 0$, $\gamma_1\kappa$, $\gamma_2\kappa \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos\gamma_1\kappa$, $\cos\gamma_2\kappa > 0$, and $\sin\gamma_1\kappa$ and $\sin\gamma_2\kappa$ have the same sign. Thus, $\kappa = 0$ is the only root to Eq.(3.6). From Eq.(3.5), we can know that $\kappa = 0$ is not a root to Eq.(3.5). Accordingly, there is no common root to Eq.(3.5) and Eq.(3.6). Therefore, the null hypothesis is not true if $\rho > 0$ and (H_3) are satisfied.

In conclusion, all common eigenvalues of Eq.(3.5) and Eq.(3.6) have negative real parts if $(H_1) - (H_3)$ hold. This completes the proof. □

3.3. Stability analysis of positive equilibrium J^* of system (1.2) with one delay

In this subsection, we analyze the stability of positive equilibrium J^* of system (1.2) with one delay by taking τ_1 and τ_2 as bifurcation parameter, respectively.

3.3.1. Stability analysis of positive equilibrium J^* of system (1.2) with $\tau_1 > 0$, $\tau_2 = 0$

When $\tau_1 > 0$, $\tau_2 = 0$, the characteristic equation (3.3) can be transformed into the following form:

$$s^{\gamma_1+\gamma_2} - a_{11}s^{\gamma_2} - a_{12}c_{21} - b_{11}s^{\gamma_2}e^{-s\tau_1} = 0. \tag{3.7}$$

Let $s = \omega_{10}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ ($\omega_{10} > 0$) be a root of Eq.(3.7), then by substituting s into Eq.(3.7) and separating the real and imaginary parts, we can obtain the following equations:

$$\begin{cases} \quad b_{11}\omega_{10}^{\gamma_2}\cos\frac{\gamma_2\pi}{2}\cos\omega_{10}\tau_1 + b_{11}\omega_{10}^{\gamma_2}\sin\frac{\gamma_2\pi}{2}\sin\omega_{10}\tau_1 \\ = \omega_{10}^{\gamma_1+\gamma_2}\cos\frac{(\gamma_1+\gamma_2)\pi}{2} - a_{11}\omega_{10}^{\gamma_2}\cos\frac{\gamma_2\pi}{2} - a_{12}c_{21}, \\ \quad b_{11}\omega_{10}^{\gamma_2}\sin\frac{\gamma_2\pi}{2}\cos\omega_{10}\tau_1 - b_{11}\omega_{10}^{\gamma_2}\cos\frac{\gamma_2\pi}{2}\sin\omega_{10}\tau_1 \\ = \omega_{10}^{\gamma_1+\gamma_2}\sin\frac{(\gamma_1+\gamma_2)\pi}{2} - a_{11}\omega_{10}^{\gamma_2}\sin\frac{\gamma_2\pi}{2}. \end{cases} \tag{3.8}$$

Then

$$\begin{cases} \cos\omega_{10}\tau_1 = \frac{-a_{11}\omega_{10}^{\gamma_2} + \omega_{10}^{\gamma_1+\gamma_2}\cos\frac{\gamma_1\pi}{2} - a_{12}c_{21}\cos\frac{\gamma_2\pi}{2}}{b_{11}\omega_{10}^{\gamma_2}}, \\ \sin\omega_{10}\tau_1 = \frac{-\omega_{10}^{\gamma_1+\gamma_2}\sin\frac{\gamma_1\pi}{2} - a_{12}c_{21}\sin\frac{\gamma_2\pi}{2}}{b_{11}\omega_{10}^{\gamma_2}}. \end{cases} \tag{3.9}$$

It follows from Eq.(3.9) and $\sin^2\omega_{10}\tau_1 + \cos^2\omega_{10}\tau_1 = 1$ that

$$\begin{aligned} &\omega_{10}^{2\gamma_1+2\gamma_2} - 2a_{12}c_{21}\omega_{10}^{\gamma_1+\gamma_2}\cos\frac{(\gamma_1+\gamma_2)\pi}{2} + (a_{11}^2 - b_{11}^2)\omega_{10}^{2\gamma_2} \\ &- 2a_{11}\omega_{10}^{\gamma_1+2\gamma_2}\cos\frac{\gamma_1\pi}{2} + 2a_{11}a_{12}c_{21}\omega_{10}^{\gamma_2}\cos\frac{\gamma_2\pi}{2} + a_{12}^2c_{21}^2 = 0. \end{aligned} \tag{3.10}$$

Suppose that ω_{10} is a positive root to Eq.(3.10) and by means of the first equation of Eq.(3.9) we can derive that

$$\tau_{1j} = \frac{1}{\omega_{10}}\left\{\arccos\left(\frac{-a_{11}\omega_{10}^{\gamma_2} + \omega_{10}^{\gamma_1+\gamma_2}\cos\frac{\gamma_1\pi}{2} - a_{12}c_{21}\cos\frac{\gamma_2\pi}{2}}{b_{11}\omega_{10}^{\gamma_2}}\right) + 2j\pi\right\}, \tag{3.11}$$

$j = 0, 1, 2, \cdots$.

Define the bifurcation point

$$\tau_{10} = \min\{\tau_{1j}\}, j = 0, 1, 2, \cdots,$$

where τ_{1j} is given by Eq.(3.11).

To further present our main results, the following assumption is needed:

$$(H_4): \ \frac{(\gamma_1+\gamma_2)\omega_{10}^{\gamma_1}\cos(\omega_{10}\tau_{10} + \frac{\gamma_1\pi}{2}) - \gamma_2 a_{11}\cos\omega_{10}\tau_{10} - \gamma_2 b_{11}}{b_{11}\omega_{10}^2} \neq 0.$$

Lemma 3.1. *Let $s(\tau_1)=\epsilon(\tau_1)+i\omega(\tau_1)$ be the root of Eq.(3.7), $\epsilon(\tau_{10})=0$ and $\omega(\tau_{10})=\omega_{10}$. If (H_4) holds, then*

$$\mathrm{Re}[\frac{ds}{d\tau_1}]^{-1}\Big|_{\omega=\omega_{10},\tau_1=\tau_{10}}\neq 0.$$

Proof. Taking the derivative of Eq.(3.7) with respect to τ_1, then it is deduced that

$$\begin{aligned}&[(\gamma_1+\gamma_2)s^{\gamma_1+\gamma_2-1}-\gamma_2 a_{11}s^{\gamma_2-1}-\gamma_2 b_{11}s^{\gamma_2-1}e^{-s\tau_1}+\tau_1 b_{11}s^{\gamma_2}e^{-s\tau_1}]\frac{ds}{d\tau_1}\\&+b_{11}s^{\gamma_2+1}e^{-s\tau_1}=0.\end{aligned}$$

Thus, we can get that

$$[\frac{ds}{d\tau_1}]^{-1}=\frac{-(\gamma_1+\gamma_2)s^{\gamma_1}e^{s\tau_1}+\gamma_2 a_{11}e^{s\tau_1}+\gamma_2 b_{11}}{b_{11}s^2}-\frac{\tau_1}{s}. \tag{3.12}$$

It acquires from Eq.(3.12) that

$$\begin{aligned}\mathrm{Re}[\frac{ds}{d\tau_1}]^{-1}\Big|_{\omega=\omega_{10},\tau_1=\tau_{10}}&=\frac{(\gamma_1+\gamma_2)\omega_{10}^{\gamma_1}\cos(\omega_{10}\tau_{10}+\frac{\gamma_1\pi}{2})-\gamma_2 a_{11}\cos\omega_{10}\tau_{10}-\gamma_2 b_{11}}{b_{11}\omega_{10}^2}\\&\neq 0.\end{aligned} \tag{3.13}$$

Hence, (H_4) implies that transversality condition holds. This ends the proof of Lemma 3.1. □

According to Lemma 2.1, Lemma 2.2 and Lemma 3.1, the following theorem can be concluded.

Theorem 3.2. *When $\tau_1>0$, $\tau_2=0$, the following results are obtained if $(H_1)-(H_4)$ hold:*

(i) The positive equilibrium J^* of system (1.2) is locally asymptotically stable for $\tau_1\in[0,\tau_{10})$.

(ii) Hopf bifurcation will happen around the positive equilibrium J^* of system (1.2) for $\tau_1=\tau_{10}$.

3.3.2. Stability analysis of positive equilibrium J^* of system (1.2) with $\tau_1=0$, $\tau_2>0$

When $\tau_1=0$, $\tau_2>0$, the characteristic equation (3.3) can be written as:

$$\begin{aligned}&s^{\gamma_1+\gamma_2}-a_{22}s^{\gamma_1}-(a_{11}+b_{11})s^{\gamma_2}+a_{22}(a_{11}+b_{11})\\&+(a_{22}s^{\gamma_1}-a_{11}a_{22}-a_{12}c_{21}-a_{22}b_{11})e^{-s\tau_2}=0.\end{aligned} \tag{3.14}$$

Let $s=\omega_{20}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$ $(w_{20}>0)$ be a root of Eq.(3.14), then it follows from Eq.(3.14) with separating the real and imaginary parts that

$$\begin{cases} (a_{22}\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}-a_{22}(a_{11}+b_{11})-a_{12}c_{21})\cos\omega_{20}\tau_2+a_{22}\omega_{20}^{\gamma_1}\sin\frac{\gamma_1\pi}{2}\sin\omega_{20}\tau_2 \\ =-\omega_{20}^{\gamma_1+\gamma_2}\cos\frac{(\gamma_1+\gamma_2)\pi}{2}+a_{22}\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}+(a_{11}+b_{11})(-a_{22}+\omega_{20}^{\gamma_2}\cos\frac{\gamma_2\pi}{2}), \\ a_{22}\omega_{20}^{\gamma_1}\sin\frac{\gamma_1\pi}{2}\cos\omega_{20}\tau_2-(a_{22}\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}-a_{22}(a_{11}+b_{11})-a_{12}c_{21})\sin\omega_{20}\tau_2 \\ =-\omega_{20}^{\gamma_1+\gamma_2}\sin\frac{(\gamma_1+\gamma_2)\pi}{2}+a_{22}\omega_{20}^{\gamma_1}\sin\frac{\gamma_1\pi}{2}+(a_{11}+b_{11})\omega_{20}^{\gamma_2}\sin\frac{\gamma_2\pi}{2}. \end{cases} \tag{3.15}$$

In view of Eq.(3.15), we have

$$\begin{cases} \cos\omega_{20}\tau_2 \\ =\frac{a_{22}^2\omega_{20}^{2\gamma_1}+a_{22}^2(a_{11}+b_{11})^2+a_{12}a_{22}c_{21}(a_{11}+b_{11})}{a_{22}^2\omega_{20}^{2\gamma_1}+(a_{22}(a_{11}+b_{11})+a_{12}c_{21})^2-2(a_{22}^2(a_{11}+b_{11})+a_{12}a_{22}c_{21})\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}} \\ \quad -\frac{(a_{12}a_{22}c_{21}+2a_{22}^2(a_{11}+b_{11}))\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}}{a_{22}^2\omega_{20}^{2\gamma_1}+(a_{22}(a_{11}+b_{11})+a_{12}c_{21})^2-2(a_{22}^2(a_{11}+b_{11})+a_{12}a_{22}c_{21})\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}} \\ \quad -\frac{(a_{22}\omega_{20}^{2\gamma_1}+a_{12}c_{21}(a_{11}+b_{11})+a_{22}(a_{11}+b_{11})^2)\omega_{20}^{\gamma_2}\cos\frac{\gamma_2\pi}{2}}{a_{22}^2\omega_{20}^{2\gamma_1}+(a_{22}(a_{11}+b_{11})+a_{12}c_{21})^2-2(a_{22}^2(a_{11}+vb_{11})+a_{12}a_{22}c_{21})\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}} \\ \quad +\frac{2a_{22}(a_{11}+b_{11})\omega_{20}^{\gamma_1+\gamma_2}\cos\frac{\gamma_1\pi}{2}\cos\frac{\gamma_2\pi}{2}+a_{12}c_{21}\omega_{20}^{\gamma_1+\gamma_2}\cos\frac{(\gamma_1+\gamma_2)\pi}{2}}{a_{22}^2\omega_{20}^{2\gamma_1}+(a_{22}(a_{11}+b_{11})+a_{12}c_{21})^2-2(a_{22}^2(a_{11}+b_{11})+a_{12}a_{22}c_{21})\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}} \\ \equiv G_1(\omega_{20}), \\ \sin\omega_{20}\tau_2 \\ =\frac{a_{12}a_{22}c_{21}\omega_{20}^{\gamma_1}\sin\frac{\gamma_1\pi}{2}}{a_{22}^2\omega_{20}^{2\gamma_1}+(a_{22}(a_{11}+b_{11})+va_{12}c_{21})^2-2(a_{22}^2(a_{11}+b_{11})+a_{12}a_{22}c_{21})\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}} \\ \quad +\frac{(a_{22}\omega_{20}^{2\gamma_1}+a_{12}c_{21}(a_{11}+b_{11})+a_{22}(a_{11}+b_{11})^2)\omega_{20}^{\gamma_2}\sin\frac{\gamma_2\pi}{2}}{a_{22}^2\omega_{20}^{2\gamma_1}+(a_{22}(a_{11}+b_{11})+a_{12}c_{21})^2-2(a_{22}^2(a_{11}+b_{11})+a_{12}a_{22}c_{21})\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}} \\ \quad +\frac{-2a_{22}(a_{11}+b_{11})\omega_{20}^{\gamma_1+\gamma_2}\sin\frac{\gamma_2\pi}{2}\cos\frac{\gamma_1\pi}{2}-a_{12}c_{21}\omega_{20}^{\gamma_1+\gamma_2}\sin\frac{(\gamma_1+\gamma_2)\pi}{2}}{a_{22}^2\omega_{20}^{2\gamma_1}+(a_{22}(a_{11}+b_{11})+a_{12}c_{21})^2-2(a_{22}^2(a_{11}+b_{11})+a_{12}a_{22}c_{21})\omega_{20}^{\gamma_1}\cos\frac{\gamma_1\pi}{2}} \\ \equiv G_2(\omega_{20}). \end{cases} \tag{3.16}$$

By Eq.(3.16) and $\sin^2\omega_{20}\tau_2+\cos^2\omega_{20}\tau_2=1$, we can get

$$G_1^2(\omega_{20})+G_2^2(\omega_{20})=1. \tag{3.17}$$

Suppose that ω_{20} is a positive root to Eq.(3.17) and we can derive that

$$\tau_{2j}=\frac{1}{\omega_{20}}\left\{\arccos(G_1(\omega_{20}))+2j\pi\right\}, j=0,1,2,\cdots. \tag{3.18}$$

Define the bifurcation point

$$\tau_{20}=\min\left\{\tau_{2j}\right\}, j=0,1,2,\cdots,$$

where τ_{2j} is defined by Eq.(3.18).

In order to present our main results, the additional hypothesis is useful and necessary:

$$(H_5): \ \frac{M_1}{M_2} \neq 0,$$

where M_1 and M_2 can be obtained by Eq.(3.21).

Lemma 3.2. *Let $s(\tau_2) = \sigma(\tau_2) + i\omega(\tau_2)$ be the root of Eq.(3.14), $\sigma(\tau_{20}) = 0$ and $\omega(\tau_{20}) = \omega_{20}$, then*

$$\mathrm{Re}[\frac{ds}{d\tau_2}]^{-1}\Big|_{\omega=\omega_{20},\tau_2=\tau_{20}} \neq 0.$$

Proof. Taking the derivative of Eq.(3.14) with respect to τ_2, it is calculated that

$$\begin{aligned}&[(\gamma_1+\gamma_2)s^{\gamma_1+\gamma_2-1} - \gamma_2(a_{11}+b_{11})s^{\gamma_2-1} + \gamma_1 a_{22}s^{\gamma_1-1}(e^{-s\tau_2}-1) - \tau_2 a_{22}s^{\gamma_1}e^{-s\tau_2}\\ &+ \tau_2(a_{12}c_{21}+a_{22}(a_{11}+b_{11}))e^{-s\tau_2}]\frac{ds}{d\tau_2} - s(a_{22}(s^{\gamma_1}-a_{11}-b_{11})-a_{12}c_{21})e^{-s\tau_2} = 0.\end{aligned}$$

Clearly, we can get that

$$\begin{aligned}[\frac{ds}{d\tau_2}]^{-1} =&\frac{((\gamma_1+\gamma_2)s^{\gamma_1+\gamma_2-1} - \gamma_1 a_{22}s^{\gamma_1-1} - \gamma_2(a_{11}+b_{11})s^{\gamma_2-1})e^{s\tau_2} + \gamma_1 a_{22}s^{\gamma_1-1}}{s(a_{22}s^{\gamma_1} - a_{22}(a_{11}+b_{11}) - a_{12}c_{21})}\\ &- \frac{\tau_2}{s}.\end{aligned} \tag{3.19}$$

By calculation, it deduces from Eq.(3.19) that

$$\mathrm{Re}[\frac{ds}{d\tau_2}]^{-1}\Big|_{\omega=\omega_{20},\tau_2=\tau_{20}} = \frac{M_1}{M_2}, \tag{3.20}$$

where

$$\begin{aligned}M_1 =&\gamma_1 a_{22}(a_{11}a_{22}+a_{22}b_{11}+a_{12}c_{21})\cos\frac{\gamma_1\pi}{2}\\ &- \gamma_1 a_{22}(a_{11}a_{22}+a_{22}b_{11}+a_{12}c_{21})\omega_{20}^{\gamma_1}\cos(\frac{\gamma_1\pi}{2}+\omega_{20}\tau_{20})\\ &- (\gamma_1+\gamma_2)a_{22}\omega_{20}^{2\gamma_1+\gamma_2}\cos(\frac{\gamma_2\pi}{2}+\omega_{20}\tau_{20})\\ &+ 2\gamma_2 a_{22}(a_{11}+b_{11})\omega_{20}^{\gamma_1+\gamma_2}\cos(\frac{\gamma_2\pi}{2}+\omega_{20}\tau_{20})\cos\frac{\gamma_1\pi}{2} + \gamma_1 a_{22}^2\omega_{20}^{2\gamma_1}\cos\omega_{20}\tau_{20}\\ &- \gamma_2(a_{11}+b_{11})(a_{11}a_{22}+a_{12}c_{21}+a_{22}b_{11})\omega_{20}^{\gamma_2}\cos(\frac{\gamma_2\pi}{2}+\omega_{20}\tau_{20}) - \gamma_1 a_{22}^2\omega_{20}^{2\gamma_1}\\ &+ \gamma_1(a_{11}a_{22}+a_{22}b_{11}+a_{12}c_{21})\omega_{20}^{\gamma_1+\gamma_2}\cos(\frac{(\gamma_1+\gamma_2)\pi}{2}+\omega_{20}\tau_{20}),\\ M_2 =&a_{22}^2\omega_{20}^{2\gamma_1+2} + (a_{22}^2(a_{11}+b_{11})^2 + a_{12}^2c_{21}^2 + 2a_{12}a_{22}c_{21}(a_{11}+b_{11}))\omega_{20}^2\\ &- 2a_{22}(a_{11}a_{22}+a_{22}b_{11}-a_{12}c_{21})\omega_{20}^{\gamma_1+2}\cos\frac{\gamma_1\pi}{2}.\end{aligned} \tag{3.21}$$

Therefore, (H_5) implies that transversality condition holds. This fulfills the proof of Lemma 3.2. □

According to Lemma 2.1, Lemma 2.2 and Lemma 3.2, we can get the following theorem.

Theorem 3.3. *When $\tau_1 = 0$, $\tau_2 > 0$, the following results can be determined if $(H_1) - (H_3)$ and (H_5) are satisfied:*

(i) The positive equilibrium J^* of system (1.2) *is locally asymptotically stable for $\tau_2 \in [0, \tau_{20})$.*

(ii) System (1.2) *has a branch of periodic solutions bifurcating from the positive equilibrium near $\tau_2 = \tau_{20}$.*

3.4. Stability analysis of system (1.2) with two delays τ_1, $\tau_2 > 0$ and $\tau_1 \neq \tau_2$

To analyze the stability of system (1.2) when two delays change at the same time, stability switching curves, directions of crossing and Hopf bifurcation are discussed in this subsection.

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ be the sets of real numbers, nonnegative real numbers, nonnegative integer numbers, integer numbers and complex numbers, respectively. Similary, $\mathbb{R}^2$ and $\mathbb{R}^2_+$ denotes the sets of 2-dimensional vectors with components in $\mathbb{R}$ and $\mathbb{R}_+$.

Based on [16], in order to ensure that the equation of the form of Eq.(3.3) is the characteristic equation of a time-delay system, the following four assumptions are necessary.

(H_6): There are finite number of characteristic roots on $\mathbb{C}_+ := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$, if

$$\deg(U_0(s)) \geq \max\{\deg(U_1(s)), \deg(U_2(s)), \deg(U_3(s))\}.$$

(H_7): No zero frequency: for any τ_1 and τ_2, the following equation

$$U_0(0) + U_1(0) + U_2(0) + U_3(0) \neq 0$$

is true.

(H_8): The polynomials $U_i(s)$ $(i = 0, 1, 2, 3)$ have no common factors, which means that they're coprime polynomials.

(H_9): The polynomials $U_i(s)$ $(i = 0, 1, 2, 3)$ satisfy the condition, which is

$$\lim_{s\to\infty}\left(\left|\frac{U_1(s)}{U_0(s)}\right| + \left|\frac{U_2(s)}{U_0(s)}\right| + \left|\frac{U_3(s)}{U_0(s)}\right|\right) < 1.$$

To make sure that the Eq.(3.3) is the characteristic equation of the system (1.2), next, we verify that the above four hypotheses $(H_6) - (H_9)$ for Eq.(3.3) are true:

(i) According to Eq.(3.3) and $\gamma_1, \gamma_2 \in (0, 1]$, we can know that

$$\deg(U_0(s)) = \gamma_1 + \gamma_2, \ \ \deg(U_1(s)) = \gamma_2, \ \ \deg(U_2(s)) = \gamma_1, \ \ \deg(U_3(s)) = 0.$$

Hence,

$$\deg(U_0(s)) \geq \max\{\deg(U_1(s)), \deg(U_2(s)), \deg(U_3(s))\}.$$

That means that (H_6) is true.

(ii) Obviously, $U_0(0)+U_1(0)+U_2(0)+U_3(0) = -a_{12}c_{21} \neq 0$, thus (H_7) is satisfied.

(iii) If (H_8) is not ture, then there is a common factor $c(s)$ of the highest degree such that $U_i(s) = c(s)h_i(s)$ $(i = 0, 1, 2, 3)$, then $h_i(s)$ are not reducible. Therefore, Eq.(3.3) can be written as

$$c(s)[h_0(s) + h_1(s)e^{-s\tau_1} + h_2(s)e^{-s\tau_2} + h_3(s)e^{-s(\tau_1+\tau_2)}] = 0,$$

which still satisfies (H_8). Therefore, (H_8) is naturally true.

(iv) For (H_9), the hypothesis is naturally satisfied if the delay equation is of retarded type. In fact, (H_9) is the exclusion of large oscillations, that is, the exclusion of $i\omega$ being the root to Eq.(3.3), if ω is arbitrarily large. Hence, ω is bounded.

3.4.1. Stability switching curves

Applying the method of [10, 16], the feasible region of the system (1.2) is found in this subsubsection. Then all pairs of points (τ_1, τ_2) in the feasible region such that the characteristic equation (3.3) at least has one pair of pure imaginary roots, which constitute the stability switching curves T.

We need to find a series of points $(\tau_1, \tau_2) \in \mathbb{R}^2_+$, such that the charisteristic equation (3.3) has a root $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\omega > 0)$. It's obvious by (H_7) that $s \neq 0$. Therefore, the characteristic equation (3.3) can be written as

$$D(i\omega; \tau_1, \tau_2) = U_0(i\omega) + U_1(i\omega)e^{-i\omega\tau_1} + (U_2(i\omega) + U_3(i\omega)e^{-i\omega\tau_1})e^{-i\omega\tau_2} = 0. \tag{3.22}$$

Since $|e^{-i\omega\tau_2}| = 1$, we can get

$$|U_0 + U_1e^{-i\omega\tau_1}| = |U_2 + U_3e^{-i\omega\tau_1}|. \tag{3.23}$$

Square both sides of Eq.(3.23), and it could be equal to

$$(U_0 + U_1e^{-i\omega\tau_1})(\overline{U}_0 + \overline{U}_1e^{i\omega\tau_1}) = (U_2 + U_3e^{-i\omega\tau_1})(\overline{U}_2 + \overline{U}_3e^{i\omega\tau_1}).$$

By calculation, we can obtain that

$$|U_0|^2 + |U_1|^2 - |U_2|^2 - |U_3|^2 = 2E_1(\omega)\cos(\omega\tau_1) - 2E_2(\omega)\sin(\omega\tau_1), \tag{3.24}$$

where

$$\begin{aligned}
E_1(\omega) =&\mathrm{Re}(U_2\overline{U}_3) - \mathrm{Re}(U_0\overline{U}_1)\\
=&a_{12}a_{22}b_{11}c_{21} - a_{11}b_{11}\omega^{2\gamma_2} + b_{11}\omega^{\gamma_1+2\gamma_2}\cos\frac{\gamma_1\pi}{2}\\
&+ 2a_{11}a_{22}b_{11}\omega^{\gamma_2}\cos\frac{\gamma_2\pi}{2} - 2a_{22}b_{11}\omega^{\gamma_1+\gamma_2}\cos\frac{\gamma_1\pi}{2}\cos\frac{\gamma_2\pi}{2},\\
E_2(\omega) =&\mathrm{Im}(U_2\overline{U}_3) - \mathrm{Im}(U_0\overline{U}_1)\\
=&b_{11}\omega^{\gamma_1+2\gamma_2}\sin\frac{\gamma_1\pi}{2} - 2a_{22}b_{11}\omega^{\gamma_1+\gamma_2}\sin\frac{\gamma_1\pi}{2}\cos\frac{\gamma_2\pi}{2}.
\end{aligned}$$

There is a continnous function such that

$$\theta_1(\omega) = \arg\left\{U_2\overline{U}_3 - U_0\overline{U}_1\right\},$$

which means that

$$E_1(\omega) = \sqrt{E_1(\omega)^2 + E_2(\omega)^2}\cos(\theta_1(\omega)),$$

$$E_2(\omega) = \sqrt{E_1(\omega)^2 + E_2(\omega)^2}\sin(\theta_1(\omega)).$$

Therefore, Eq.(3.24) can be equivalently written as

$$|U_0|^2 + |U_1|^2 - |U_2|^2 - |U_3|^2 = 2\sqrt{E_1(\omega)^2 + E_2(\omega)^2}\cos(\theta_1(\omega) + \omega\tau_1). \qquad (3.25)$$

Since $|\cos(\theta_1(\omega)+\omega\tau_1)| \leq 1$, then for any $\tau_1 \in \mathbb{R}_+$, there's a necessary condition that satisfies the above equation:

$$F(\omega) = (|U_0|^2 + |U_1|^2 - |U_2|^2 - |U_3|^2)^2 - 4(E_1(\omega)^2 + E_2(\omega)^2) \leq 0. \qquad (3.26)$$

Based on (H_7), we can get that $F(0) \neq 0$. When $\omega \to +\infty$, then $F(\omega) \to +\infty$ by (H_6) and (H_9). Therefore, $F(\omega)$ has a finite number of roots on $\mathbb{R}_+$. The range of ω satisfying Eq.(3.26) is denoted by Ω, which is

$$\Omega = \{\omega | F(\omega) \leq 0\}. \qquad (3.27)$$

From Eq.(3.25), let

$$\cos(\phi_1(\omega)) = \frac{|U_0|^2 + |U_1|^2 - |U_2|^2 - |U_3|^2}{2\sqrt{E_1(\omega)^2 + E_2(\omega)^2}}, \quad \phi_1 \in [0, \pi].$$

Then

$$\tau^{\pm}_{1,n_1}(\omega) = \frac{\pm\phi_1(\omega) - \theta_1(\omega) + 2n_1\pi}{\omega}, \ n_1 \in \mathbb{Z}. \qquad (3.28)$$

Substituting Eq.(3.28) into Eq.(3.22), we can get

$$\tau^{\pm}_{2,n_2}(\omega) = \frac{1}{\omega}\arg\left\{-\frac{U_2 + U_3 e^{-i\omega\tau_1\pm}}{U_0 + U_1 e^{-i\omega\tau_1\pm}}\right\} + 2n_2\pi, \quad n_2 \in \mathbb{Z}. \qquad (3.29)$$

Another way to calculate τ_2 is to analyze τ_2 in the same way that we analyzed for τ_1, which gives

$$D(i\omega; \tau_1, \tau_2) = U_0(i\omega) + U_2(i\omega)e^{-i\omega\tau_2} + (U_1(i\omega) + U_3(i\omega)e^{-i\omega\tau_2})e^{-i\omega\tau_1} = 0, \ (3.30)$$

then, we have

$$\tau^{\pm}_{2,n_2}(\omega) = \frac{\pm\phi_2(\omega) - \theta_2(\omega) + 2n_2\pi}{\omega}, \ n_2 \in \mathbb{Z}, \qquad (3.31)$$

where

$$\begin{aligned}
\cos(\phi_2(\omega)) &= \frac{|U_0|^2 - |U_1|^2 + |U_2|^2 - |U_3|^2}{2\sqrt{E_3(\omega)^2 + E_4(\omega)^2}}, \phi_2 \in [0, \pi], \\
E_3(\omega) &= \sqrt{E_3(\omega)^2 + E_4(\omega)^2}\cos(\theta_2(\omega)), \\
E_4(\omega) &= \sqrt{E_3(\omega)^2 + E_4(\omega)^2}\sin(\theta_2(\omega)),
\end{aligned}$$

and

$$\begin{aligned}
E_3(\omega) =& \mathrm{Re}(U_1\overline{U}_3) - \mathrm{Re}(U_0\overline{U}_2) \\
=& a_{22}^2\omega^{2\gamma_1} + 2a_{11}a_{22}\omega^{\gamma_1+\gamma_2}\cos\frac{\gamma_1\pi}{2}\cos\frac{\gamma_2\pi}{2} + a_{12}c_{21}\omega^{\gamma_1+\gamma_2}\cos\frac{(\gamma_1+\gamma_2)\pi}{2} \\
& - (a_{22}\omega^{2\gamma_1} - a_{22}b_{11}^2 + a_{11}^2a_{22} + a_{11}a_{12}c_{21})\omega^{\gamma_2}\cos\frac{\gamma_2\pi}{2}
\end{aligned}$$

$$
\begin{aligned}
&- a_{22}(2a_{11}a_{22} + a_{12}c_{21})\omega^{\gamma_1}\cos\frac{\gamma_1\pi}{2} + a_{11}a_{12}a_{22}c_{21} + (a_{11}^2 - b_{11}^2)a_{22}^2,\\
E_4(\omega) =&\mathrm{Im}(U_1\overline{U}_3) - \mathrm{Im}(U_0\overline{U}_2)\\
=&(a_{22}(-a_{11}^2 + b_{11}^2 - \omega^{2\gamma_1}) - a_{11}a_{12}c_{21})\omega^{\gamma_2}\sin\frac{\gamma_2\pi}{2} - a_{12}a_{22}c_{21}\omega^{\gamma_1}\sin\frac{\gamma_1\pi}{2}\\
&+ a_{12}c_{21}\omega^{\gamma_1+\gamma_2}\sin\frac{(\gamma_1+\gamma_2)\pi}{2} + 2a_{11}a_{22}\omega^{\gamma_1+\gamma_2}\cos\frac{\gamma_1\pi}{2}\sin\frac{\gamma_2\pi}{2}.
\end{aligned}
$$

Similarly to Eq.(3.26), we have

$$F_1(\omega) = (|U_0|^2 + |U_2|^2 - |U_1|^2 - |U_3|^2)^2 - 4(E_3(\omega)^2 + E_4(\omega)^2)^2 \leq 0. \tag{3.32}$$

By comparing Eq.(3.26) and Eq.(3.32), it is not difficult to find that they are equivalent. The range of ω that satisfies Eq.(3.26) is the same as the range of ω that satisfies Eq.(3.32), and they are denoted as Ω. Ω is also known as the feasible region and it is consistent with the range of all frequencies corresponding to all points on the stability switching curves.

From the above derivation, we can get that the stability switching curves are

$$T := \left\{(\tau_{1,n_1}^{\pm}(\omega), \tau_{2,n_2}^{\pm}(\omega)) \in \mathbb{R}_+^2 : \omega \in \Omega,\ n_1,\ n_2 \in \mathbb{Z}\right\}. \tag{3.33}$$

3.4.2. Directions of crossing

With the previous discussion, in this subsubsection, we will focus on the directions of change in the stability of system (1.2).

Suppose $(\tau_1^*, \tau_2^*) \in T$, and there exists $\omega^* > 0$ such that $(i\omega^*; \tau_1^*, \tau_2^*)$ is a root of the characteristic equation (3.3). If $\frac{\partial D}{\partial s}(i\omega^*; \tau_1^*, \tau_2^*) \neq 0$, then let $s(\tau_1^*, \tau_2^*) = \eta(\tau_1^*, \tau_2^*) + i\omega(\tau_1^*, \tau_2^*)$ be a simple root of Eq.(3.3). In the neighborhood of (τ_1^*, τ_2^*), $\eta(\tau_1^*, \tau_2^*) = 0$ and $\omega(\tau_1^*, \tau_2^*) = \omega^*$ are satisfied. We call the increasing direction of $\omega \in \Omega$ as the positive direction of the stability switching curves T, and as moving along the positive direction of the curves T, the left-hand (right-hand) side is called as the region on the left (right).

Due to the tangent vector of T along the positive direction is $\mathbf{m} = (\frac{\partial \tau_1^*}{\partial \omega^*}, \frac{\partial \tau_2^*}{\partial \omega^*})$, the normal vector of T pointing to the right region is $\mathbf{n} = (\frac{\partial \tau_2^*}{\partial \omega^*}, -\frac{\partial \tau_1^*}{\partial \omega^*})$ and (τ_1^*, τ_2^*) moves along the direction $\mathbf{q} = (\frac{\partial \tau_1^*}{\partial \eta}, \frac{\partial \tau_2^*}{\partial \eta})$. Also, as η increases from negative to positive through 0, the direction of a pair of pure imaginary roots of characteristic equation (3.3) across the imaginary axis to the right on the complex pane is determined by the sign of the inner product of $\mathbf{m}$ and $\mathbf{n}$, which is

$$\delta(\omega^*) := \mathbf{n}\cdot\mathbf{q} = \left(\frac{\partial \tau_2^*}{\partial \omega^*}, -\frac{\partial \tau_1^*}{\partial \omega^*}\right)\left(\frac{\partial \tau_1^*}{\partial \eta}, \frac{\partial \tau_2^*}{\partial \eta}\right) = \frac{\partial \tau_1^*}{\partial \eta}\frac{\partial \tau_2^*}{\partial \omega^*} - \frac{\partial \tau_1^*}{\partial \omega^*}\frac{\partial \tau_2^*}{\partial \eta} = \begin{vmatrix} \frac{\partial \tau_1^*}{\partial \eta} & \frac{\partial \tau_1^*}{\partial \omega^*} \\ \frac{\partial \tau_2^*}{\partial \eta} & \frac{\partial \tau_2^*}{\partial \omega^*} \end{vmatrix}. \tag{3.34}$$

If $\delta(\omega^*) > 0$ ($\delta(\omega^*) < 0$), then the region on the right (left) has characteristic roots with positive real parts when moving along the positive direction of stability switching curves T.

Since $D(s; \tau_1^*, \tau_2^*)$ is an analytical function of s, τ_1^* and τ_2^*, if

$$\det\begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix} = R_1 I_2 - R_2 I_1 \neq 0,$$

then

$$\Delta(\omega^*) = \begin{pmatrix} \frac{\partial \tau_1^*}{\partial \eta} & \frac{\partial \tau_1^*}{\partial \omega^*} \\ \frac{\partial \tau_2^*}{\partial \eta} & \frac{\partial \tau_2^*}{\partial \omega^*} \end{pmatrix} \Bigg|_{\sigma=0,\omega^*\in\Omega} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 & -I_0 \\ I_0 & R_0 \end{pmatrix}, \tag{3.35}$$

where

$$\begin{aligned}
R_0 =& \mathrm{Re}\left\{ \frac{\partial D(s;\tau_1^*,\tau_2^*)}{\partial \eta}\Big|_{s=i\omega^*} \right\} \\
=& (\gamma_1+\gamma_2){\omega^*}^{\gamma_1+\gamma_2} \sin\frac{(\gamma_1+\gamma_2)\pi}{2} - \gamma_1 a_{22}{\omega^*}^{\gamma_1-1}\sin\frac{\gamma_1\pi}{2} \\
& - \gamma_2 a_{11}{\omega^*}^{\gamma_2-1}\sin\frac{\gamma_2\pi}{2} - \tau_1^* a_{22}b_{11}\cos\omega^*\tau_1^* \\
& - \gamma_2 b_{11}{\omega^*}^{\gamma_2-1}\sin(\frac{\gamma_2\pi}{2}-\omega^*\tau_1^*) + \tau_1^* b_{11}{\omega^*}^{\gamma_2}\cos(\frac{\gamma_2\pi}{2}-\omega^*\tau_1^*) \\
& + \gamma_1 a_{22}{\omega^*}^{\gamma_1-1}\sin(\frac{\gamma_1\pi}{2}-\omega^*\tau_2^*) - \tau_2^* a_{22}{\omega^*}^{\gamma_1}\cos(\frac{\gamma_1\pi}{2}-\omega^*\tau_2^*) \\
& + \tau_2^*(a_{11}a_{22}+a_{12}c_{21})\cos\omega^*\tau_2^* + (\tau_1^*+\tau_2^*)a_{22}b_{11}\cos\omega^*(\tau_1^*+\tau_2^*), \\
I_0 =& \mathrm{Im}\left\{ \frac{\partial D(s;\tau_1^*,\tau_2^*)}{\partial \eta}\Big|_{s=i\omega^*} \right\} \\
=& -(\gamma_1+\gamma_2){\omega^*}^{\gamma_1+\gamma_2} \cos\frac{(\gamma_1+\gamma_2)\pi}{2} + \gamma_1 a_{22}{\omega^*}^{\gamma_1-1}\cos\frac{\gamma_1\pi}{2} \\
& + \gamma_2 a_{11}{\omega^*}^{\gamma_2-1}\cos\frac{\gamma_2\pi}{2} + \tau_1^* a_{22}b_{11}\sin\omega^*\tau_1^* \\
& + \gamma_2 b_{11}{\omega^*}^{\gamma_2-1}\cos(\frac{\gamma_2\pi}{2}-\omega^*\tau_1^*) + \tau_1^* b_{11}{\omega^*}^{\gamma_2}\sin(\frac{\gamma_2\pi}{2}-\omega^*\tau_1^*) \\
& - \gamma_1 a_{22}{\omega^*}^{\gamma_1-1}\cos(\frac{\gamma_1\pi}{2}-\omega^*\tau_2^*) - \tau_2^* a_{22}{\omega^*}^{\gamma_1}\sin(\frac{\gamma_1\pi}{2}-\omega^*\tau_2^*) \\
& - \tau_2^*(a_{11}a_{22}+a_{12}c_{21})\sin\omega^*\tau_2^* - (\tau_1^*+\tau_2^*)a_{22}b_{11}\sin\omega^*(\tau_1^*+\tau_2^*), \\
R_1 =& \mathrm{Re}\left\{ -i\omega^*(U_1(i\omega^*)e^{-i\omega^*\tau_1^*} + U_3(i\omega^*)e^{-i\omega^*(\tau_1^*+\tau_2^*)}) \right\} \\
=& a_{22}b_{11}\omega^*\sin\omega^*(\tau_1^*+\tau_2^*) - b_{11}{\omega^*}^{\gamma_2+1}\sin(\frac{\gamma_2\pi}{2}-\omega^*\tau_1^*) - a_{22}b_{11}\omega^*\sin\omega^*\tau_1^*, \\
I_1 =& \mathrm{Im}\left\{ -i\omega^*(U_1(i\omega^*)e^{-i\omega^*\tau_1^*} + U_3(i\omega^*)e^{-i\omega^*(\tau_1^*+\tau_2^*)}) \right\} \\
=& a_{22}b_{11}\omega^*\cos\omega^*(\tau_1^*+\tau_2^*) + b_{11}{\omega^*}^{\gamma_2+1}\cos(\frac{\gamma_2\pi}{2}-\omega^*\tau_1^*) - a_{22}b_{11}\omega^*\cos\omega^*\tau_1^*, \\
R_2 =& \mathrm{Re}\left\{ -i\omega^*(U_2(i\omega^*)e^{-i\omega^*\tau_2^*} + U_3(i\omega^*)e^{-i\omega^*(\tau_1^*+\tau_2^*)}) \right\} \\
=& a_{22}{\omega^*}^{\gamma_1+1}\sin(\frac{\gamma_1\pi}{2}-\omega^*\tau_2^*) + (a_{11}a_{22}+a_{12}c_{21})\omega^*\sin\omega^*\tau_2^* \\
& + a_{22}b_{11}\omega^*\sin\omega^*(\tau_1^*+\tau_2^*), \\
I_2 =& \mathrm{Im}\left\{ -i\omega^*(U_2(i\omega^*)e^{-i\omega^*\tau_2^*} + U_3(i\omega^*)e^{-i\omega^*(\tau_1^*+\tau_2^*)}) \right\} \\
=& -a_{22}{\omega^*}^{\gamma_1+1}\cos(\frac{\gamma_1\pi}{2}-\omega^*\tau_2^*) + (a_{11}a_{22}+a_{12}c_{21})\omega^*\cos\omega^*\tau_2^* \\
& + a_{22}b_{11}\omega^*\cos\omega^*(\tau_1^*+\tau_2^*).
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}\operatorname{Re}\left\{\frac{\partial D(s;\tau_1^*,\tau_2^*)}{\partial\omega^*}\bigg|_{s=i\omega^*}\right\}=-I_0,\\ \operatorname{Im}\left\{\frac{\partial D(s;\tau_1^*,\tau_2^*)}{\partial\omega^*}\bigg|_{s=i\omega^*}\right\}=R_0.\end{aligned}\tag{3.36}$$

We have

$$\delta(\omega^*)=\det(\Delta(\omega^*))=\begin{vmatrix}R_1 & R_2\\ I_1 & I_2\end{vmatrix}^{-1}\begin{vmatrix}R_0 & -I_0\\ I_0 & R_0\end{vmatrix}.$$

Since

$$\begin{vmatrix}R_0 & -I_0\\ I_0 & R_0\end{vmatrix}=R_0^2+I_0^2\geq 0,$$

then $\delta(\omega^*)$ and $R_1I_2-R_2I_1$ have the same sign, which is

$$sign\left\{\delta(\omega^*)\right\}=sign\left\{R_1I_2-R_2I_1\right\}.\tag{3.37}$$

For $({\tau_1}^*,{\tau_2}^*)\in T$, we have

$$U_2(i\omega^*)e^{i\omega^*(\tau_1^*-\tau_2^*)}=-U_0(i\omega^*)e^{i\omega^*\tau_1^*}-U_1(i\omega^*)-U_3(i\omega^*)e^{-i\omega^*\tau_2^*}.\tag{3.38}$$

We can verify that

$$\begin{aligned}&R_1I_2-R_2I_1\\ =&\operatorname{Im}\left\{(R_1-I_1i)(R_2+I_2i)\right\}\\ =&\operatorname{Im}\left\{\overline{(R_1+I_1i)}(R_2+I_2i)\right\}\\ =&\operatorname{Im}\left\{\overline{(-i\omega^*)(U_1e^{-i\omega^*\tau_1^*}+U_3e^{-i\omega^*(\tau_1^*+\tau_2^*)})}(-i\omega^*)(U_2e^{-i\omega^*\tau_2^*}+U_3e^{-i\omega^*(\tau_1^*+\tau_2^*)})\right\}\\ =&{\omega^*}^2\operatorname{Im}\left\{\overline{U}_1U_2e^{i\omega^*(\tau_1^*-\tau_2^*)}+\overline{U}_1U_3e^{-i\omega^*\tau_2^*}+U_2\overline{U}_3e^{i\omega^*\tau_1^*}\right\}\\ =&{\omega^*}^2\operatorname{Im}\left\{\overline{U}_1(-U_0e^{i\omega^*\tau_1^*}-U_1-U_3e^{-i\omega^*\tau_2^*})+\overline{U}_1U_3e^{-i\omega^*\tau_2^*}+U_2\overline{U}_3e^{i\omega^*\tau_1^*}\right\}\\ =&{\omega^*}^2\operatorname{Im}\left\{(U_2\overline{U}_3-U_0\overline{U}_1)e^{i\omega^*\tau_1^*}\right\}\\ =&{\omega^*}^2\operatorname{Im}\left\{|U_2\overline{U}_3-U_0\overline{U}_1|e^{i\omega^*\tau_1^*}e^{\theta_1}\right\}\\ =&\pm\,{\omega^*}^2|U_2\overline{U}_3-U_0\overline{U}_1|\sin\phi_1.\end{aligned}$$

Hence,

$$\delta(\omega^*\in\Omega)=\pm sign\left\{{\omega^*}^2|U_2\overline{U}_3-U_0\overline{U}_1|\sin\phi_1\right\}=\pm 1.\tag{3.39}$$

Theorem 3.4. *For $\frac{\partial D}{\partial s}(i\omega^*;\tau_1^*,\tau_2^*)\neq 0$ and $(\tau_1^*,\tau_2^*)\in T$, the direction of characteristic roots s crossing the imaginary axis from left to right, as (τ_1^*,τ_2^*) passes through the stability switching curves to the region on the right (left) if $sign\left\{\delta(\omega^*)\right\}>0$ ($sign\left\{\delta(\omega^*)\right\}<0$).*

3.4.3. Hopf bifurcation

Taking the derivstive of τ_1 of Eq.(3.3), we can get that

$$\begin{aligned}&[U_0'(s)+U_1'(s)e^{-s\tau_1}-\tau_1U_1(s)e^{-s\tau_1}+U_2'(s)e^{-s\tau_2}-\tau_2U_2(s)e^{-s\tau_2}\\&+U_3'(s)e^{-s(\tau_1+\tau_2)}-(\tau_1+\tau_2)U_3(s)e^{-s(\tau_1+\tau_2)}]\frac{ds}{d\tau_1}-[sU_2(s)e^{-s\tau_2}\\&+sU_3(s)e^{-s(\tau_1+\tau_2)}]\frac{d\tau_2}{d\tau_1}-sU_1(s)e^{-s\tau_1}-sU_3(s)e^{-s(\tau_1+\tau_2)}=0,\end{aligned}\tag{3.40}$$

where $U_i'(s)$ is the derivaties of $U_i(s)$ $(i=0,1,2,3)$.

Eq.(3.22) can be written equivalently as follows:

$$\tau_2=\frac{\ln(-\frac{U_2e^{s\tau_1}+U_3}{U_0e^{s\tau_1}+U_1})}{s}.\tag{3.41}$$

By Eq.(3.40) and Eq.(3.41), it can get that

$$[\frac{ds}{d\tau_1}]^{-1}=\frac{V}{D}+\frac{U_1'}{sU_1}-\frac{\tau_1}{s},\tag{3.42}$$

where

$$\begin{aligned}V=&U_0'U_1U_3+U_0'U_3^2e^{-s\tau_2}+(U_0'U_1U_2+U_0'U_0U_3)e^{s\tau_1}+U_0'U_0U_2e^{2s\tau_1}\\&+U_0'U_2e^{s(2\tau_1-\tau_2)}+2U_0'U_2U_3e^{s(\tau_1-\tau_2)},\\D=&sU_1^2U_3e^{-s\tau_1}+2sU_1U_2U_3e^{-s\tau_2}+sU_1U_3^2e^{-s(\tau_1+\tau_2)}\\&+sU_0U_1U_2e^{s\tau_1}+sU_1U_2^2e^{s(\tau_1-\tau_2)}+sU_1^2U_2+sU_0U_1U_3.\end{aligned}$$

Let $U_i'^{r}$ and $U_i'^{i}$ be the real and imaginary parts of the derivative of U_i' $(i=0,1,2,3)$, respectively, and the real and imaginary parts of $U_i(s)$ $(i=0,1,2,3)$ are denoted by A_i and B_i, respectively. If $s=\omega_0(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$ is a root of Eq.(3.3), and $\tau_1=\overline{\tau}_1$ is a bifurcation point, then the following equation can be deduced by calculation:

$$\mathrm{Re}[\frac{ds}{d\tau_1}]^{-1}\Big|_{\omega=\omega_0,\tau_1=\overline{\tau}_1}=\frac{V_1D_1+V_2D_2}{D_1^2+D_2^2}+\frac{-\omega_0B_1U_1'^{r}+\omega_0A_1U_1'^{i}}{\omega_0^2A_1^2+\omega_0^2B_1^2},\tag{3.43}$$

where

$$\begin{aligned}V_1=&U_0'^{r}A_1A_3-U_0'^{i}B_1A_3+U_0'^{r}A_3^2\cos\omega_0\tau_2+U_0'^{i}A_3^2\sin\omega_0\tau_2\\&+(U_0'^{r}A_1A_2-U_0'^{i}A_2B_1-U_0'^{r}B_1B_2-U_0'^{i}A_1B_2+U_0'^{r}A_1A_3-U_0'^{i}B_1A_3)\cos\omega_0\overline{\tau}_1\\&-(U_0'^{r}A_1B_2-U_0'^{i}B_1B_2+U_0'^{r}A_2B_1+U_0'^{i}A_1A_2+U_0'^{r}B_1A_3+U_0'^{i}A_1A_3)\sin\omega_0\overline{\tau}_1\\&+(U_0'^{r}A_0A_2-U_0'^{i}B_0A_2-U_0'^{r}B_0B_2-U_0'^{i}A_0B_2)\cos2\omega_0\overline{\tau}_1\\&-(U_0'^{r}A_0B_2-U_0'^{i}B_0B_2+U_0'^{r}B_0A_2+U_0'^{i}A_0A_2)\sin2\omega_0\overline{\tau}_1\\&+(U_0'^{r}A_2-U_0'^{i}B_2)\cos(2\omega_0\overline{\tau}_1-\omega_0\tau_2)-(U_0'^{i}A_2+U_0'^{r}B_2)\sin(2\omega_0\overline{\tau}_1-\omega_0\tau_2)\\&+2(U_0'^{r}A_2A_3-U_0'^{i}B_2A_3)\cos(\omega_0\overline{\tau}_1-\omega_0\tau_2)\\&-2(U_0'^{r}B_2A_3+U_0'^{i}A_2A_3)\sin(\omega_0\overline{\tau}_1-\omega_0\tau_2),\end{aligned}$$

$$
\begin{aligned}
V_2 =& U_0^{'r} B_1 A_3 + U_0^{'i} A_1 A_3 - U_0^{'r} A_3^2 \sin \omega_0 \tau_2 + U_0^{'i} A_3^2 \cos \omega_0 \tau_2 \\
&+ (U_0^{'r} A_1 A_2 - U_0^{'i} A_2 B_1 - U_0^{'r} B_1 B_2 - U_0^{'i} A_1 B_2 + U_0^{'r} A_1 A_3 - U_0^{'i} B_1 A_3) \sin \omega_0 \overline{\tau}_1 \\
&+ (U_0^{'r} A_1 B_2 - U_0^{'i} B_1 B_2 + U_0^{'r} A_2 B_1 + U_0^{'i} A_1 A_2 + U_0^{'r} B_1 A_3 + U_0^{'i} A_1 A_3) \cos \omega_0 \overline{\tau}_1 \\
&+ (U_0^{'r} A_0 A_2 - U_0^{'i} B_0 A_2 - U_0^{'r} B_0 B_2 - U_0^{'i} A_0 B_2) \sin 2\omega_0 \overline{\tau}_1 \\
&+ (U_0^{'r} A_0 B_2 - U_0^{'i} B_0 B_2 + U_0^{'r} B_0 A_2 + U_0^{'i} A_0 A_2) \cos 2\omega_0 \overline{\tau}_1 \\
&+ (U_0^{'r} A_2 - U_0^{'i} B_2) \sin(2\omega_0 \overline{\tau}_1 - \omega_0 \tau_2) + (U_0^{'i} A_2 + U_0^{'r} B_2) \cos(2\omega_0 \overline{\tau}_1 - \omega_0 \tau_2) \\
&+ 2(U_0^{'r} A_2 A_3 - U_0^{'i} B_2 A_3) \sin(\omega_0 \overline{\tau}_1 - \omega_0 \tau_2) \\
&+ 2(U_0^{'r} B_2 A_3 + U_0^{'i} A_2 A_3) \cos(\omega_0 \overline{\tau}_1 - \omega_0 \tau_2), \\
D_1 =& - \omega_0 (2A_1 A_2 B_1 + A_1^2 B_2 - B_1^2 B_2 + A_0 A_3 B_1 + A_1 A_3 B_0) \\
&+ \omega_0 (-A_0 A_1 B_2 + B_0 B_1 B_2 - A_0 A_2 B_1 - A_1 A_2 B_0 + 2A_1 A_3 B_1) \cos \omega_0 \overline{\tau}_1 \\
&- \omega_0 (A_0 A_1 A_2 - B_0 B_1 A_2 - A_0 B_1 B_2 - A_1 B_0 B_2 - A_1^2 A_3 + A_3 B_1^2) \sin \omega_0 \overline{\tau}_1 \\
&- 2\omega_0 (A_2 A_3 B_1 + A_1 A_3 B_2) \cos \omega_0 \tau_2 + 2\omega_0 (A_1 A_2 A_3 - A_3 B_1 B_2) \sin \omega_0 \tau_2 \\
&+ \omega_0 (-2A_1 A_2 B_2 - A_2^2 B_1 + B_1 B_2^2) \cos \omega_0 (\overline{\tau}_1 - \tau_2) \\
&- \omega_0 (-2A_2 B_1 B_2 + A_1 A_2^2 - A_1 B_2^2) \sin \omega_0 (\overline{\tau}_1 - \tau_2) \\
&- \omega_0 B_1 A_3^2 \cos \omega_0 (\overline{\tau}_1 + \tau_2) + \omega_0 A_1 A_3^2 \sin \omega_0 (\overline{\tau}_1 + \tau_2), \\
D_2 =& \omega_0 (-2A_1 B_1 B_2 + A_1^2 A_2 - B_1^2 A_2 + A_0 A_1 A_3 - A_3 B_0 B_1) \\
&+ \omega_0 (-A_0 A_1 B_2 + B_0 B_1 B_2 - A_0 A_2 B_1 - A_1 A_2 B_0 - 2A_1 A_3 B_1) \sin \omega_0 \overline{\tau}_1 \\
&+ \omega_0 (A_0 A_1 A_2 - B_0 B_1 A_2 - A_0 B_1 B_2 - A_1 B_0 B_2 + A_1^2 A_3 - A_3 B_1^2) \cos \omega_0 \overline{\tau}_1 \\
&+ 2\omega_0 (A_2 A_3 B_1 + A_1 A_3 B_2) \sin \omega_0 \tau_2 + 2\omega_0 (A_1 A_2 A_3 - A_3 B_1 B_2) \cos \omega_0 \tau_2 \\
&+ \omega_0 (-2A_1 A_2 B_2 - A_2^2 B_1 + B_1 B_2^2) \sin \omega_0 (\overline{\tau}_1 - \tau_2) \\
&+ \omega_0 (-2A_2 B_1 B_2 + A_1 A_2^2 - A_1 B_2^2) \cos \omega_0 (\overline{\tau}_1 - \tau_2) \\
&+ \omega_0 B_1 A_3^2 \sin \omega_0 (\overline{\tau}_1 + \tau_2) + \omega_0 A_1 A_3^2 \cos \omega_0 (\overline{\tau}_1 + \tau_2), \\
A_0 =& \omega_0^{\gamma_1+\gamma_2} \cos \frac{(\gamma_1+\gamma_2)\pi}{2} - a_{22} \omega_0^{\gamma_1} \cos \frac{\gamma_1 \pi}{2} - a_{11} \omega_0^{\gamma_2} \cos \frac{\gamma_2 \pi}{2} + a_{11} a_{22}, \\
B_0 =& \omega_0^{\gamma_1+\gamma_2} \sin \frac{(\gamma_1+\gamma_2)\pi}{2} - a_{22} \omega_0^{\gamma_1} \sin \frac{\gamma_1 \pi}{2} - a_{11} \omega_0^{\gamma_2} \sin \frac{\gamma_2 \pi}{2}, \\
A_1 =& - b_{11} \omega_0^{\gamma_2} \cos \frac{\gamma_2 \pi}{2} + a_{22} b_{11}, \\
B_1 =& - b_{11} \omega_0^{\gamma_2} \sin \frac{\gamma_2 \pi}{2}, \\
A_2 =& a_{22} \omega_0^{\gamma_1} \cos \frac{\gamma_1 \pi}{2} - a_{11} a_{22} - a_{12} c_{21}, \\
B_2 =& a_{22} \omega_0^{\gamma_1} \sin \frac{\gamma_1 \pi}{2}, \\
A_3 =& - a_{22} b_{11}, \\
B_3 =& 0.
\end{aligned}
$$

If the following condition holds:

$$
(H_{10}) : \mathrm{Re}[\frac{ds}{d\tau_1}]^{-1}\Big|_{(\omega=\omega_0, \tau_1=\overline{\tau}_1)} = \frac{V_1 D_1 + V_2 D_2}{D_1^2 + D_2^2} + \frac{-\omega_0 B_1 U_1^{'r} + \omega_0 A_1 U_1^{'i}}{\omega_0^2 A_1^2 + \omega_0^2 B_1^2} \neq 0,
$$

then the transversal condition is true.

Clearly, we can obtain the following theorem.

Theorem 3.5. *When $\tau_1 > 0, \tau_2 > 0$, and $\tau_1 \neq \tau_2$, the system* (1.2) *appears Hopf bifurcation if $(\tau_1, \tau_2) \in T$, and $(H_1) - (H_3)$ and (H_{10}) are true.*

4. Numerical simulations

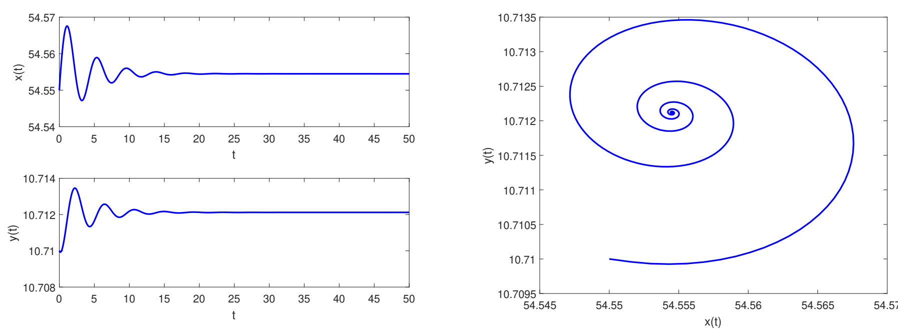
In this section, we consider the following system:

$$\begin{cases} D^{\gamma_1}x(t) = 2.5x(1 - \frac{x(t-\tau_1)}{300}) - \frac{0.03(1-0.72)x^2y}{1+0.03\times0.056(1-0.72)x^2}, \\ D^{\gamma_2}y(t) = \frac{0.12\times0.03(1-0.72)x^2(t-\tau_2)y(t-\tau_2)}{1+0.03\times0.056(1-0.72)x^2(t-\tau_2)} - 1.2y - 0.1\times0.5y, \end{cases} \tag{4.1}$$

where $\gamma_1 = 0.9$, $\gamma_2 = 0.95$.

For system (4.1), $\beta - (d + qE_0)h = 0.05 > 0$ and $1 - \frac{x^*}{K} = 0.8182 > 0$, then (H_1) and (H_2) are true. There is a unique positive equilibrium $J^* = (54.55, 10.71)$ in system (4.1).

We can calculate $a_{11} + b_{11} = 0.3409 - 0.4546 = -0.1137 < 0$, then (H_3) is true. From Theorem 3.1, the positive equilibrium J^* is locally asymptotically stable with $\tau_1 = \tau_2 = 0$ (see Fig.1).



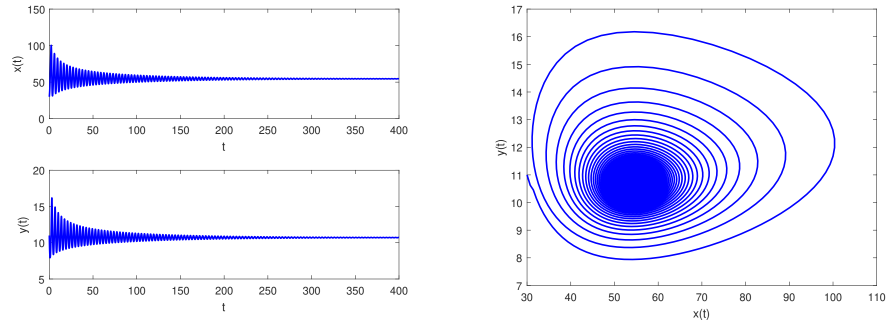
(a) (b)

Figure 1. The positive equilibrium J^* of system (4.1) is locally asymptotically stable when $\tau_1 = 0$ and $\tau_2 = 0$ with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.

When $\tau_1 > 0$ and $\tau_2 = 0$, we can calculate that $\omega_{10} = 1.7953$, $\tau_{10} = 0.8993$ and $\mathrm{Re}[\frac{ds}{d\tau_1}]^{-1}|_{\omega=\omega_{10},\tau_1=\tau_{10}} = 1.8197 \neq 0$. When $\tau_1 = 0.8 < 0.8993$, the positive equilibrium is locally asymptocically stable (see Fig.2). When $\tau_1 = 1 > 0.8993$, the system (4.1) occurs Hopf bifurcation (see Fig.3).

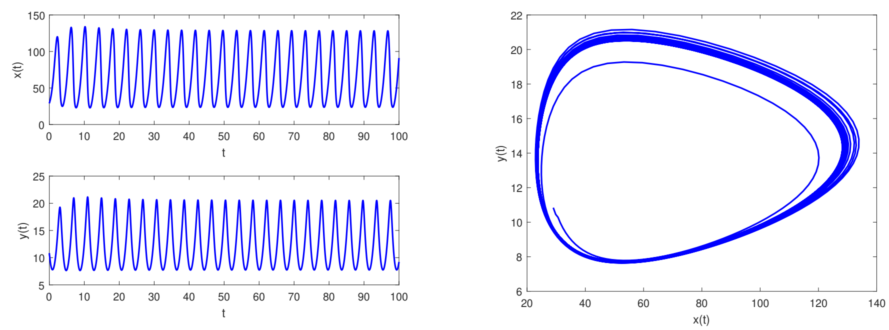
When $\tau_1 = 0$ and $\tau_2 > 0$, we can easily get that $\omega_{20} = 1.2696$, $\tau_{20} = 0.2752$ and $\mathrm{Re}[\frac{ds}{d\tau_2}]^{-1}|_{\omega=\omega_{20},\tau_2=\tau_{20}} = 2.3826 \neq 0$. When $\tau_2 = 0.2 < 0.2752$, the positive equilibrium J^* of system (4.1) is locally asymptotically stable (see Fig.4). When $\tau_2 = 0.35 > 0.2752$, system (4.1) occurs periodic solution (see Fig.5).

When $\tau_1 > 0$, $\tau_2 > 0$ and $\tau_1 \neq \tau_2$, we can get the graph of $F(\omega)$ (see Fig.6). It has two different positive real roots 1.1965 and 1.8217. Then the feasible region is $\Omega = (1.1965, 1.8217)$. When $\omega \in \Omega$, we can calculate the stability switching curves and the directions of change in stability of system (4.1) (see Fig.7). Fig.8 is an enlargement of the lower left corner of Fig.7. The range of stability region of system



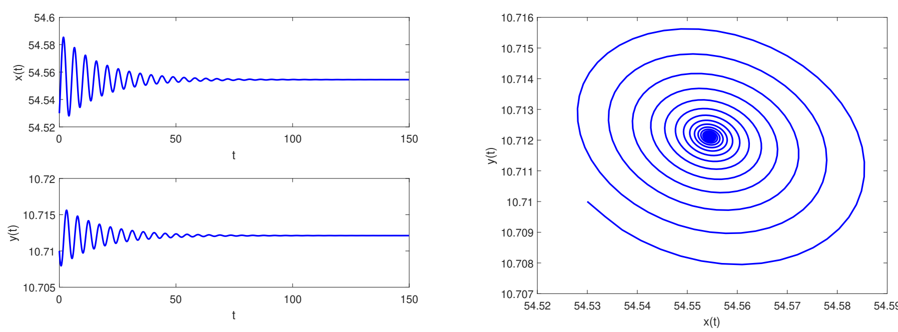
(a) (b)

Figure 2. The positive equilibrium J^* of system (4.1) is locally asymptotically stable when $\tau_1 = 0.8 < \tau_{10}$ and $\tau_2 = 0$ with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.



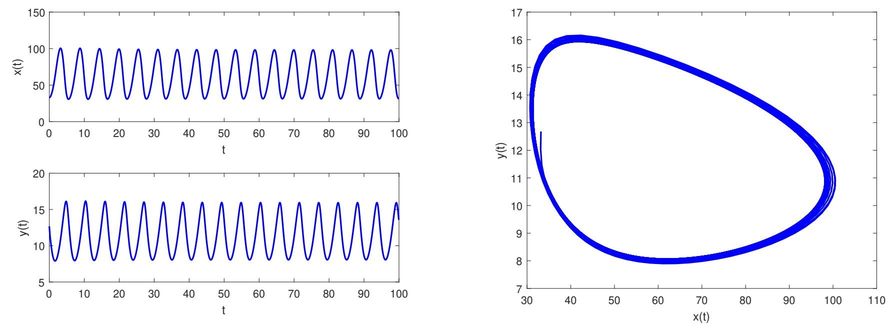
(a) (b)

Figure 3. The system (4.1) appears Hopf bifurcation when $\tau_1 = 1 > \tau_{10}$ and $\tau_2 = 0$ with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.



(a) (b)

Figure 4. The positive equilibrium J^* of system (4.1) is locally asymptotically stable when $\tau_1 = 0$ with $\tau_2 = 0.2 < \tau_{20}$ with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.



(a) (b)

Figure 5. The system (4.1) appears Hopf bifurcation when $\tau_1 = 0$ and $\tau_2 = 0.35 > \tau_{20}$ with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.

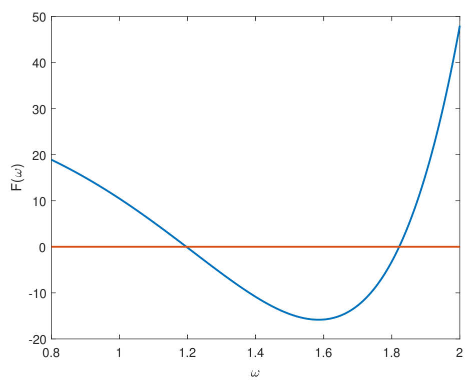


Figure 6. Graph of F(ω) when $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.

(4.1) is the green part of Fig.8. We finding any one point $O = (0.315, 0.2433)$ on the stability switching curve in Fig.8. When $\tau_1 = 0.315$, and $\tau_2 = 0.24 < 0.2433$, the positive equilibrium J^* of system (4.1) is locally asymptotically stable (see Fig.9). When $\tau_1 = 0.315$, and $\tau_2 = 0.25 > 0.2433$, system (4.1) occurs Hopf bifurcation (see Fig.10).

In order to better compare the range of stability region of system (4.1) in two different cases: (i) $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$ and (ii) $\gamma_1 = \gamma_2 = 1$, then when $\gamma_1 = \gamma_2 = 1$, the stability switching curves of system (4.1) (see Fig.11) and an enlargement of the lower left corner of Fig.11 (see Fig.12) are shown. The green part of Fig.12 is the range of stability region of system (4.1) with $\gamma_1 = \gamma_2 = 1$. Comparing Fig.8 and Fig.12, we can find that the system (4.1) with fractional order has widely stable region than the system (4.1) with integer order.

5. Conclusion

In this paper,we have investigated a fractional predator-prey system with two delays and incommensurate orders. Taking time delay as bifurcation parameter, the

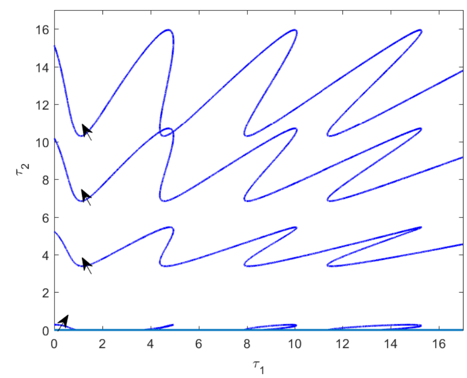


Figure 7. Plot of the stability switching curves when $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.

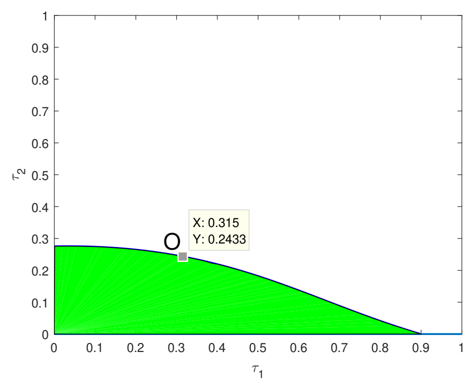


Figure 8. The stable region of system (4.1) with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.

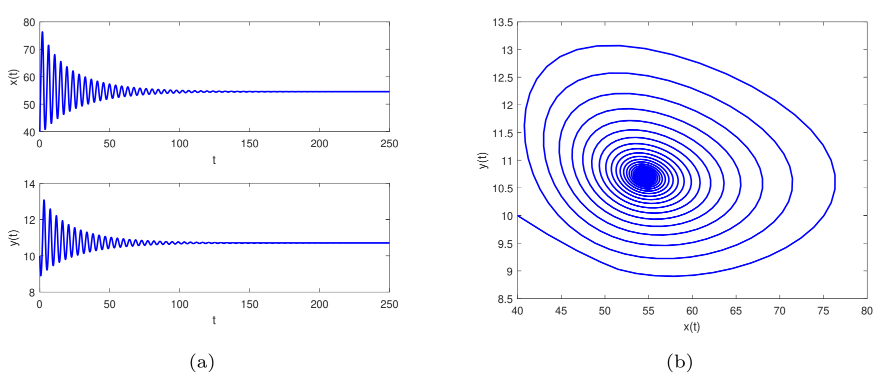
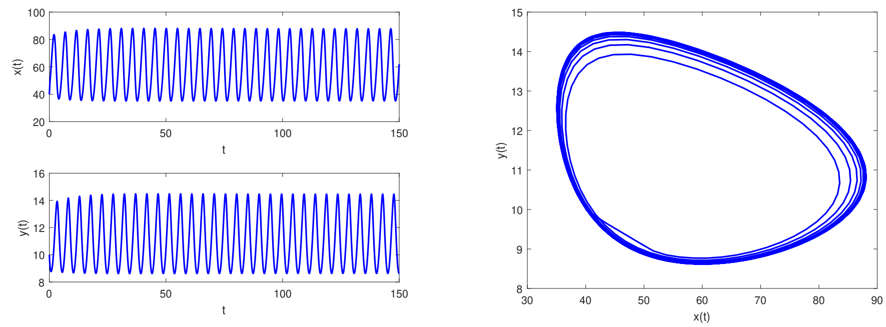


Figure 9. The positive equilibrium J^* of system (4.1) is locally asymptotically stable when $\tau_1 = 0.315$ and $\tau_2 = 0.24 < 0.2433$ with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.

stability and the existence conditions of Hopf bifurcation of system (1.2) have been discussed in four cases:



(a) (b)

Figure 10. The system (4.1) appears Hopf bifurcation when $\tau_1 = 0.315$ and $\tau_2 = 0.25 > 0.2433$ with $\gamma_1 = 0.9$ and $\gamma_2 = 0.95$.

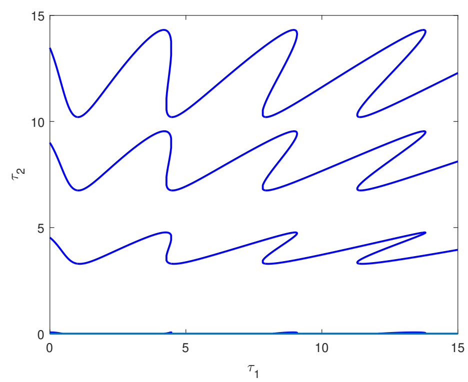


Figure 11. Plot of the stability switching curves when $\gamma_1 = \gamma_2 = 1$.

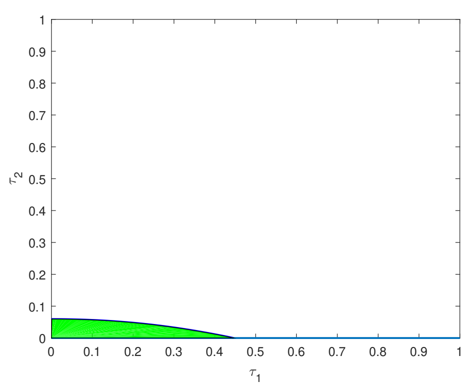


Figure 12. The stable region of system (4.1) with $\gamma_1 = \gamma_2 = 1$.

(i) When $\tau_1 = \tau_2 = 0$, the local stability of the positive equilibrium of the system (1.2) is analyzed by Routh-Hurwitz criterion.

(ii) When $\tau_1 > 0$, $\tau_2 = 0$ or $\tau_1 = 0$, $\tau_2 > 0$, the critical value of Hopf bifurcation at the positive equilibrium of the system (1.2) is calculated by taking τ_1 and τ_2 as bifurcation parameter, respectively.

(iii) When $\tau_1 > 0$, $\tau_2 > 0$, and $\tau_1 \neq \tau_2$, applying the method of [10, 16], we can calculate the stability switching curves and the directions of crossing, and obtain the change in the stability of the positive equilibrium of the system (1.2) and the existence of Hopf bifurcation as two delays change simultaneously.

From the discussion of the above cases, we can get the following results:

(i) Delay has an important effect on the stability of system. When the delay crosses a critical value, Hopf bifurcation occurs in the system (1.2).

(ii) Contrasted the system (1.2) with integer order and fractional order, the latter has widely stability region. The order has an effect on the stability of the system.

Acknowledgements

We would like to thank the reviewers for their valuable comments and suggestions, which can significantly improve the quality of our paper indeed.

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