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STATIONARY DISTRIBUTION AND PERMANENCE OF A STOCHASTIC DELAY PREDATOR-PREY LOTKA-VOLTERRA MODEL WITH LÉVY JUMPS

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Abstract In this paper, we propose and investigate an impulsive stochastic predator-prey Lotka-Volterra model with infinite delay and Lévy jumps. Sufficient criteria for permanence in time average and the threshold between stability in time average and extinction are provided. For the corresponding case without impulse, the easily substantiated sufficient criteria for stability in distribution are derived. Our results demonstrate that, first of all, the coefficients related to infinite delay have some effects on permanence in time average and stability in distribution; then impulsive perturbations play a prominent part in keeping the permanence in time average despite the unfavourable factor Lévy jumps causes.

Keywords Predator-prey Lotka-Volterra model, permanence in time average, stability in distribution, Lévy jumps, infinite delay, impulsive perturbations.

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1. Introduction

In the past few decades, delay population model driven by functional differential equations has attracted a great deal of attention [7, 10]. In 1931, Brelot proposed the classical predator-prey system with infinite delay(see page 200 in [7]):

$$\begin{cases} \dfrac{dY_1(t)}{dt} = Y_1(t)\Big(r_1 - c_{11}Y_1(t) - c_{12}\displaystyle\int_{-\infty}^{t} F_2(t-s)Y_2(s)ds\Big), \\ \dfrac{dY_2(t)}{dt} = Y_2(t)\Big(r_2 + c_{21}\displaystyle\int_{-\infty}^{t} F_1(t-s)Y_1(s)ds - c_{22}Y_2(t)\Big), \end{cases} \tag{1.1}$$

where r_1 is the growth rate of prey Y_1, c_{11} stands for the strength of competition among individuals of Y_1, c_{12} denotes the capture rate, r_2 is the growth rate of predator Y_2, c_{21} represents the conversion rate, c_{22} is the strength of competition among individuals of Y_2. All the parameters in model (1.1) are positive constants. There exists vast research and achievements of model (1.1) and its various of forms

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[7,10,11,35,48]. As the idea of weak kernel proposed by MacDonald [5], we consider $F_i(t) = \vartheta_i e^{-\vartheta_i t}, \vartheta_i > 0, i = 1, 2$. Then, by applying integral formula, model (1.1) could be described as follows

$$\begin{cases} \dfrac{dY_1(t)}{dt} =Y_1(t)\Big(r_1 - c_{11}Y_1(t) - c_{12}\displaystyle\int_{-\infty}^{0} Y_2(t+\varsigma)d\eta_2(\varsigma)\Big), \\ \dfrac{dY_2(t)}{dt} =Y_2(t)\Big(r_2 + c_{21}\displaystyle\int_{-\infty}^{0} Y_1(t+\varsigma)d\eta_1(\varsigma) - c_{22}Y_2(t)\Big), \end{cases} \tag{1.2}$$

where $\eta_i(\varsigma) = e^{\vartheta_i \varsigma}$ is a probability measure on $(-\infty, 0]$. By using the method of Theorem 2.1 in [48], we easily derive the conclusion that model (1.2) exists a positive equilibrium $x^* = (x_1^*, x_2^*) = (\Theta_1/\Theta, \Theta_2/\Theta)$ which is globally asymptotically stable if $\Theta_3 > 0$, where $\Theta = c_{11}c_{22} + c_{12}c_{21}, \Theta_1 = r_1c_{22} + r_2c_{12}, \Theta_2 = r_1c_{21} + r_2c_{11}, \Theta_3 = 2c_{11}c_{22} - c_{12}c_{21}$.

In reality, population models inescapably undergo white noise(see e.g., [12,13,36,39,41,45,49]. Particularly, Wu et al. [41] researched the effects of white noise on population models with infinite delay for the first time. In addition, [2,21–23,50] pointed out Lévy jumps can reasonably describe random discontinuous phenomenon many population models confront. Following the research approach, Mao et al. [36] setted up the sufficient criteria for the existence and uniqueness of solutions to neutral stochastic functional differential equations with infinite delay and Lévy jumps. Liu et al. [13] discussed a general stochastic non-autonomous logistic model with infinite delay and Lévy jumps. Consequently, introducing the two random perturbations mentioned above, model (1.2) become the following form:

$$\begin{cases} dY_1(t) =Y_1(t)\Big(r_1 - c_{11}Y_1(t) - c_{12}\displaystyle\int_{-\infty}^{0} Y_2(t+\varsigma)d\eta_2(\varsigma)\Big)dt \\ \qquad\qquad + \nu_1 Y_1(t)dB_1(t) + Y_1(t^-)\displaystyle\int_{\mathbb{U}} \Xi_1(u)\Gamma(dt,du), \\ dY_2(t) =Y_2(t)\Big(r_2 + c_{21}\displaystyle\int_{-\infty}^{0} Y_1(t+\varsigma)d\eta_1(\varsigma) - c_{22}Y_2(t)\Big)dt \\ \qquad\qquad + \nu_2 Y_2(t)dB_2(t) + Y_2(t^-)\displaystyle\int_{\mathbb{U}} \Xi_2(u)\Gamma(dt,du). \end{cases} \tag{1.3}$$

Here, for $j = 1, 2$, $B_j(t)$ denotes a white noise and ν_j^2 is its intensity, $Y_j(t^-) = \lim\limits_{s\uparrow t} Y_j(s)$, $\mathbb{U} \subseteq (0, +\infty)$, $\tilde{\Gamma}(dt, du) = \Gamma(dt, du) - \lambda(du)dt$, $\Gamma(dt, du)$ represents a Poisson counting measure, λ is the characteristic measure of $\Gamma(dt, du)$ with $\lambda(\mathbb{U}) < \infty$. Obviously, model (1.3) is a special case of model (2.1) in Mao et al. [36].

Recently, many scholars have shown much interest and enthusiasm in explaining discontinuous phenomena by impulsive perturbations and have many achievements(see e.g., [5,30–32,52]). Taking account of the impulsive perturbations, model

(1.3) is converted into the form

$$\begin{cases} dY_1(t) = Y_1(t)\Big(r_1 - c_{11}Y_1(t) - c_{12}\displaystyle\int_{-\infty}^{0} Y_2(t+\varsigma)d\eta_2(\varsigma)\Big)dt + \nu_1 Y_1(t)dB_1(t) \\ \qquad\qquad + Y_1(t^-)\displaystyle\int_{\mathbb{U}} \Xi_1(u)\Gamma(dt,du), \quad t \neq t_k, \quad k \in N, \\ Y_1(t_k^+) - Y_1(t_k) = J_k Y_1(t_k), \quad k \in N, \\ dY_2(t) = Y_2(t)\Big(r_2 + c_{21}\displaystyle\int_{-\infty}^{0} Y_1(t+\varsigma)d\eta_1(\varsigma) - c_{22}Y_2(t)\Big)dt \\ \qquad\qquad + \nu_2 Y_2(t)dB_2(t) + Y_2(t^-)\displaystyle\int_{\mathbb{U}} \Xi_2(u)\Gamma(dt,du), \quad t \neq t_k, \quad k \in N, \\ Y_2(t_k^+) - Y_2(t_k) = L_k Y_2(t_k), \quad k \in N, \end{cases} \tag{1.4}$$

where $t_k (k \in N)$ is monotone increasing and $t_k \to +\infty$.

Several papers have been devoted to the study of stability in distribution of stochastic functional differential equations, see [3, 4, 6, 29, 44, 46, 53]. So far as anyone can tell, permanence and stability in distribution are always hot topics in the area of mathematical ecology. Recently, based on theory of Has'minskii [8] and Markov semigroup method [33], Jiang et al. [16, 17, 48] first investigate stationary distribution of stochastic population model with infinite delay. Nevertheless, appropriate measure can not easily be found for lots of stochastic delay population models because their forms are very complex and varied [39]. Moreover, the theory of Has'minskii can not be applied to stochastic population model with Lévy jumps [8, 48, 51]. To make up for the deficiency each other, Liu et al. [18, 19] proposed an asymptotic approach to study the distribution of the stochastic population system with finite delay. However, to our best knowledge, no scholars extend the method to one with infinite delay except for our work [20, 23]. In this paper, our aim is concerned with applying the asymptotic approach to establish the sufficient conditions for stability in distribution of model (1.3). In addition, we derive the sufficient conditions for permanence in time average, stability in time average, extinction, and the threshold between stability in time average and extinction of model (1.4). In model (1.4), we set the initial positive value $\eta = (\eta_1, \eta_2)$ which pertains to the phase space C_g (see [10, 11]), where $C_g = \{\psi \in C((-\infty, 0]; \mathbb{R}^2) : \| \psi \|_{c_g} = \sup_{-\infty < s \leq 0} e^{\mathbf{q}s}|\psi(s)| < +\infty\}$, $g(s) = e^{-\mathbf{q}s}$, $\mathbf{q} > 0$, $|\psi(s)| = \sqrt{\psi_1^2(s) + \psi_2^2(s)}$, $(\psi_1(s), \psi_2(s)) \in \mathbb{R}^2$. And definite the following notations: $\mathbb{R}^2_+ = \{g = (g_1, g_2) \in \mathbb{R}^2 |\ g_j > 0, j = 1, 2\}$, $\langle g(t)\rangle = t^{-1}\int_0^t g(s)ds$.

For model (1.4), we make the following hypotheses:

(B1):$1 + J_k > 0, 1 + L_k > 0$ and there exists a positive constant χ which satisfies $\prod_{0<t_k<t}(1 + J_k) < 2\chi c_{11}$ and $c_{22}\prod_{0<t_k<t}(1 + L_k) > \frac{\chi}{2}c_{21}^2$.

(B2):For $i = 1, 2$, $\upsilon_{\mathbf{q}} = \int_{-\infty}^{0} e^{-2\mathbf{q}\varsigma}d\eta_i(\varsigma) < +\infty, \mathbf{q} > 0, \vartheta_i > 2\mathbf{q}$.

(B3):$-1 < \Xi_j(u) < \kappa, u \in \mathbb{U}, j = 1, 2$, where $\kappa > 0$.

The noteworthy contributions of this paper can be stated as follows:

$\star$ Distinguishing from existing approach [48], the used method is a combined the asymptotic method to investigate the distribution [18, 19] and the phase space C_g [10, 11].

$\star$ Different from previous literature [20], impulsive perturbations is introduced into model (1.1) and make it more different and complicated to copy with compared with model (1.3) in [20]. And, the factor improves its availability and explains biological significance perfectly well.

The remainder of this paper is organized as follows. Section 2 derives the existence and uniqueness of the positive solution of our model. Then in section 3, our main results are provided. Finally, some numerical simulations are introduced to validate the theoretical results in section 4.

2. Permanence and Extinction

Lemma 2.1. *Under Hypotheses (B1)-(B3), for any given initial value $\eta \in C_g$, model (1.4) admits a unique global solution $(Y_1(t), Y_2(t))$ on $\mathbb{R}^2_+$ for all $t \geq 0$ almost surely(a.s.).*

Proof. Enlightened by Ref. [24,42,43], the content of proof are as follow. Consider the following stochastic differential equation with infinite delay:

$$\begin{cases} dZ_1(t) = Z_1(t)\Big[r_1 - \prod\limits_{0<t_k<t}(1+J_k)c_{11}Z_1(t) - c_{12}\int_{-\infty}^{0}\prod\limits_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)\Big]dt \\ \qquad + \nu_1 Z_1(t)dB_1(t) + Z_1(t^-)\int_{\mathbb{U}}\Xi_1(u)\Gamma(dt,du), \\ dZ_2(t) = Z_2(t)\Big[r_2 + c_{21}\int_{-\infty}^{0}\prod\limits_{0<t_k<t+\varsigma}(1+J_k)Z_1(t+\varsigma)d\eta_1(\varsigma) \\ \qquad - \prod\limits_{0<t_k<t}(1+L_k)c_{22}Z_2(t)\Big]dt + \nu_2 Z_2(t)dB_2(t) + Z_2(t^-)\int_{\mathbb{U}}\Xi_2(u)\Gamma(dt,du) \end{cases} \tag{2.1}$$

with the same initial value as model (1.4). To proceed, we should certify that model (2.1) exists a unique positive solution $Z(t) = (Z_1(t), Z_2(t))$ for all $t \geq 0$ with probability 1. Define a C^2-function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ as follows: $V(Z) = Z_1 - 1 - \ln Z_1 + Z_2 - 1 - \ln Z_2$. When $Z(t) \in \mathbb{R}^2_+$, one finds that

$$\begin{aligned} dV(Z) =& (Z_1-1)\Big(r_1 - \prod_{0<t_k<t}(1+J_k)c_{11}Z_1 - c_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)\Big)dt \\ &+ \nu_1(Z_1-1)dB_1(t) + 0.5\nu_1^2 dt + Z_1\int_{\mathbb{U}}\Xi_1(u)\lambda(du)dt \\ &- \int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)dt + Z_1\int_{\mathbb{U}}\Xi_1(u)\tilde{\Gamma}(dt,du) - \int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(dt,du) \\ &+ (Z_2-1)\Big(r_2 + c_{21}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+J_k)Z_1(t+\varsigma)d\eta_1(\varsigma) - \prod_{0<t_k<t}(1+L_k)c_{22}Z_2\Big)dt \\ &+ \nu_2(Z_2-1)dB_2(t) + 0.5\nu_2^2 dt + Z_2\int_{\mathbb{U}}\Xi_2(u)\lambda(du)dt - \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)dt \\ &+ Z_2\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(dt,du) - \int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(dt,du) \\ =& \Big[r_1 Z_1 - \prod_{0<t_k<t}(1+J_k)c_{11}Z_1^2 + \prod_{0<t_k<t}(1+J_k)c_{11}Z_1 - c_{12}Z_1 \end{aligned}$$

$$\times\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)+c_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)$$
$$+r_2Z_2+c_{21}Z_2\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+J_k)Z_1(t+\varsigma)d\eta_1(\varsigma)-\prod_{0<t_k<t}(1+L_k)c_{22}Z_2^2-r_2$$
$$-c_{21}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+J_k)Z_1(t+\varsigma)d\eta_1(\varsigma)+\prod_{0<t_k<t}(1+L_k)c_{22}Z_2-r_1$$
$$+0.5\nu_1^2+0.5\nu_2^2+Z_1\int_{\mathbb{U}}\Xi_1(u)\lambda(du)-\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)$$
$$+Z_2\int_{\mathbb{U}}\Xi_2(u)\lambda(du)-\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)\Big]dt+\nu_1(Z_1-1)dB_1(t)$$
$$+\nu_2(Z_2-1)dB_2(t)+Z_1\int_{\mathbb{U}}\Xi_1(u)\tilde{\Gamma}(dt,du)-\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(dt,du)$$
$$+Z_2\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(dt,du)-\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(dt,du)$$
$$\le\Big[r_1Z_1-\prod_{0<t_k<t}(1+J_k)c_{11}Z_1^2+\prod_{0<t_k<t}(1+J_k)c_{11}Z_1$$
$$+c_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)+r_2Z_2+\frac{\chi}{2}c_{21}^2Z_2^2$$
$$+\frac{1}{2\chi}\int_{-\infty}^{0}\Big(\prod_{0<t_k<t+\varsigma}(1+J_k)\Big)^2Z_1^2(t+\varsigma)d\eta_1(\varsigma)-\prod_{0<t_k<t}(1+L_k)c_{22}Z_2^2$$
$$-r_2-c_{21}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+J_k)Z_1(t+\varsigma)d\eta_1(\varsigma)+\prod_{0<t_k<t}(1+L_k)c_{22}Z_2$$
$$-r_1+0.5\nu_1^2+0.5\nu_2^2+Z_1\int_{\mathbb{U}}\Xi_1(u)\lambda(du)-\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)$$
$$+Z_2\int_{\mathbb{U}}\Xi_2(u)\lambda(du)-\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)\Big]dt+\nu_1(Z_1-1)dB_1(t)$$
$$+\nu_2(Z_2-1)dB_2(t)+Z_1\int_{\mathbb{U}}\Xi_1(u)\tilde{\Gamma}(dt,du)-\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(dt,du)$$
$$+Z_2\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(dt,du)-\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(dt,du)$$
$$=\Big[\Big(r_1+c_{11}\prod_{0<t_k<t}(1+J_k)\Big)Z_1-\prod_{0<t_k<t}(1+J_k)c_{11}Z_1^2$$
$$+c_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)+\Big(\frac{\chi}{2}c_{21}^2-\prod_{0<t_k<t}(1+L_k)c_{22}\Big)Z_2^2$$
$$+\frac{1}{2\chi}\int_{-\infty}^{0}\Big(\prod_{0<t_k<t+\varsigma}(1+J_k)\Big)^2Z_1^2(t+\varsigma)d\eta_1(\varsigma)+\Big(\prod_{0<t_k<t}(1+L_k)c_{22}+r_2\Big)Z_2$$
$$-r_2-r_1+0.5\nu_1^2+0.5\nu_2^2+Z_1\int_{\mathbb{U}}\Xi_1(u)\lambda(du)-\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)$$
$$+Z_2\int_{\mathbb{U}}\Xi_2(u)\lambda(du)-\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)\Big]dt+\nu_1(Z_1-1)dB_1(t)$$

$$
\begin{aligned}
&+\nu_2(Z_2-1)dB_2(t)+Z_1\int_{\mathbb{U}}\Xi_1(u)\tilde{\Gamma}(dt,du)-\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(dt,du)\\
&+Z_2\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(dt,du)-\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(dt,du)\\
=&G(Z)dt+\frac{1}{2\chi}\int_{-\infty}^{0}\Big(\prod_{0<t_k<t+\varsigma}(1+J_k)\Big)^2 Z_1^2(t+\varsigma)d\eta_1(\varsigma)dt\\
&+c_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)dt-\frac{1}{2\chi}\Big(\prod_{0<t_k<t}(1+J_k)\Big)^2 Z_1^2dt\\
&-c_{12}\Big(\prod_{0<t_k<t}(1+L_k)\Big)Z_2dt+\nu_1(Z_1-1)dB_1(t)+\nu_2(Z_2-1)dB_2(t)\\
&+Z_1\int_{\mathbb{U}}\Xi_1(u)\tilde{\Gamma}(dt,du)-\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(dt,du)+Z_2\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(dt,du)\\
&-\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(dt,du),
\end{aligned}
$$

where

$$
\begin{aligned}
G(Z)=&-\Big[c_{11}\prod_{0<t_k<t}(1+J_k)-\frac{1}{2\chi}\Big(\prod_{0<t_k<t}(1+J_k)\Big)^2\Big]Z_1^2+\Big(r_1+c_{11}\prod_{0<t_k<t}(1+J_k)\\
&+\int_{\mathbb{U}}\Xi_1(u)\lambda(du)\Big)Z_1-\Big(c_{22}\prod_{0<t_k<t}(1+L_k)-\frac{\chi}{2}c_{21}^2\Big)Z_2^2-\Big(\int_{\mathbb{U}}\Xi_2(u)\lambda(du)-r_2\\
&-c_{12}\prod_{0<t_k<t}(1+L_k)-c_{22}\prod_{0<t_k<t}(1+L_k)\Big)Z_2-r_2-r_1+0.5\nu_2^2+0.5\nu_2^2\\
&-\int_{\mathbb{U}}(\ln(1+\Xi_1(u))\lambda(du)-\int_{\mathbb{U}}(\ln(1+\Xi_2(u))\lambda(du).
\end{aligned}
$$

Using the conditions $\prod_{0<t_k<t}(1+J_k)<2\chi c_{11}$ and $c_{22}\prod_{0<t_k<t}(1+L_k)>\frac{\chi}{2}c_{21}^2$, one can obtain $G(Z)$ is capped. Obviously, $(Y_1(t),Y_2(t))=(\prod_{0<t_k<t}(1+J_k)Z_1(t),\prod_{0<t_k<t}(1+L_k)Z_2(t))$ is the solution of model (1.4)(see [24,42]). The rest of proof is analogous to Lemma 2.1 and Theorem 2.1 in Ref. [24], we leave out it here. □

Theorem 2.1. *Suppose that Hypotheses (B1)-(B3) hold, then model (1.4) has the following property.*

(I)If $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)<0$ *and* $\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)<0$, *then both* Y_1 *and* Y_2 *tend to zero a.s., i.e.,* $\lim_{t\to+\infty}Y_i(t)=0$ *a.s.,* $i=1,2$.

(II) If $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)>0$ *and* $\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)<0$, *then* Y_2 *tends to zero a.s. and* Y_1 *is stability in time average a.s., i.e.,*

$$
\lim_{t\to+\infty}\langle Y_1(t)\rangle=\frac{r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)}{c_{11}},\quad a.s..
$$

(III) If $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)<0$ *and*

$\liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\right]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)>0$, *then* Y_1 *tends to zero a.s. and* Y_2 *is permanence in time average a.s., i.e.,*

$$\begin{aligned}
&\liminf_{t\to+\infty}\langle Y_2(t)\rangle\\
&\geq\frac{\liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\right]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)}{c_{22}},\quad a.s.,\\
&\limsup_{t\to+\infty}\langle Y_2(t)\rangle\\
&\leq\frac{\limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\right]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)}{c_{22}},\quad a.s..
\end{aligned}$$

Proof. Employing Itô's formula [25,26,40] to the first equation of model (2.1), we derive

$$\begin{aligned}
\ln Z_1(t)-\ln Z_1(0)=&\Big(r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)\Big)t\\
&-c_{11}\int_0^t\sum_{0<t_k<s}\ln(1+J_k)Z_1(s)ds\\
&-c_{12}\int_0^t\int_{-\infty}^0\prod_{0<t_k<s+\varsigma}(1+L_k)Z_2(s+\varsigma)d\eta_2(\varsigma)ds\\
&+\nu_1B_1(t)+\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du)\\
=&\Big(r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)\Big)t\\
&-c_{11}\int_0^tY_1(s)ds-c_{12}\int_0^t\int_{-\infty}^0Y_2(s+\varsigma)d\eta_2(\varsigma)ds\\
&+\nu_1B_1(t)+\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{0<t_k<t}\ln(1+J_k)+\ln Z_1(t)-\ln Z_1(0)\\
=&\sum_{0<t_k<t}\ln(1+J_k)+\Big(r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)\Big)t\\
&-c_{11}\int_0^tY_1(s)ds-c_{12}\int_0^t\int_{-\infty}^0Y_2(s+\varsigma)d\eta_2(\varsigma)ds+\nu_1B_1(t)\\
&+\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du).
\end{aligned}$$

In other words, we get

$$\ln Y_1(t)-\ln Y_1(0)$$

$$
\begin{aligned}
&= \sum_{0<t_k<t} \ln(1+J_k) + \Big(r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du)\Big)t - c_{11}\int_0^t Y_1(s)ds \\
&\quad - c_{12}\int_0^t \int_{-\infty}^0 Y_2(s+\varsigma)d\eta_2(\varsigma)ds + \nu_1 B_1(t) + \int_0^t \int_{\mathbb{U}} \ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du).
\end{aligned}
\tag{2.2}
$$

For $j=1,2$, direct calculation obtains

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^0 Y_j(s+\varsigma)d\eta_j(\varsigma)ds \\
&= \int_0^t \Big[\int_{-\infty}^{-s} Y_j(s+\varsigma)d\eta_j(\varsigma)ds + \int_{-s}^0 Y_j(s+\varsigma)d\eta_j(\varsigma)\Big]ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)} Y_j(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_j(\varsigma) + \int_{-t}^0 d\eta_j(\varsigma)\int_{-\varsigma}^t Y_j(s+\varsigma)ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)} Y_j(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_j(\varsigma) + \int_{-t}^0 d\eta_j(\varsigma)\int_0^{t+\varsigma} Y_j(s)ds.
\end{aligned}
\tag{2.3}
$$

Using the Hypotheses (B2), for $j=1,2$, we obtain

$$
\begin{aligned}
&\int_0^t ds \int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)} Y_j(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_j(\varsigma) \le ||\eta||_{c_g}\int_0^t e^{-\mathbf{q}s}ds\int_{-\infty}^0 e^{-\mathbf{q}\varsigma}d\eta_j(\varsigma) \\
&\le ||\eta||_{c_g}\int_0^t e^{-\mathbf{q}s}ds\Big(\int_{-\infty}^0 e^{-2\mathbf{q}\varsigma}d\eta_j(\varsigma)\Big)^{\frac{1}{2}} \le \frac{1}{\mathbf{q}}||\eta||_{c_g}(\upsilon_{\mathbf{q}})^{\frac{1}{2}}(1-e^{-\mathbf{q}t}).
\end{aligned}
\tag{2.4}
$$

Putting (2.3), (2.4) into (2.2) gives that

$$
\begin{aligned}
\ln Y_1(t) - \ln Y_1(0) \ge &\sum_{0<t_k<t} \ln(1+J_k) + \Big(r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du)\Big)t \\
&- c_{11}\int_0^t Y_1(s)ds - c_{12}\int_{-t}^0 d\eta_2(\varsigma)\int_0^{t+\varsigma} Y_2(s)ds \\
&- c_{12}\frac{1}{\mathbf{q}}||\eta||_{c_g}(\upsilon_{\mathbf{q}})^{\frac{1}{2}}(1-e^{-\mathbf{q}t}) \\
&+ \nu_1 B_1(t) + \int_0^t \int_{\mathbb{U}} \ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du).
\end{aligned}
\tag{2.5}
$$

Similar methods could be adopted in the second equation of model (2.1). Then we have

$$
\begin{aligned}
\ln Y_2(t) - \ln Y_2(0) = &\sum_{0<t_k<t} \ln(1+L_k) + \Big(r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}} \ln(1+\Xi_2(u))\lambda(du)\Big)t \\
&- c_{22}\int_0^t Y_2(s)ds + c_{21}\int_0^t \int_{-\infty}^0 Y_1(s+\varsigma)d\eta_1(\varsigma)ds \\
&+ \nu_2 B_2(t) + \int_0^t \int_{\mathbb{U}} \ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du) \\
= &\sum_{0<t_k<t} \ln(1+L_k) + \Big(r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}} \ln(1+\Xi_2(u))\lambda(du)\Big)t
\end{aligned}
$$

$$- c_{22}\int_0^t Y_2(s)ds + c_{21}\int_{-t}^0 d\eta_1(\varsigma)\int_0^{t+\varsigma} Y_1(s)ds$$
$$+ c_{21}\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_1(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_1(\varsigma) + \nu_2 B_2(t)$$
$$+ \int_0^t\int_{\mathbb{U}} \ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du). \tag{2.6}$$

(I): Suppose that $r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du) < 0$ and

$$\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du) < 0.$$

By (2.2), we get

$$t^{-1}\ln\frac{Y_1(t)}{Y_1(0)} \le \frac{1}{t}\sum_{0<t_k<t}\ln(1+J_k) + r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du) + t^{-1}\nu_1 B_1(t)$$
$$+ t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du).$$

Whence, if $r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du) < 0$, then

$$\limsup_{t\to+\infty} t^{-1}\ln Y_1(t) \le r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du) < 0.$$

Consequently, $\lim_{t\to+\infty} Y_1(t) = 0,\ a.s.$. As in the previous analysis, by (2.6), we can show that if $\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du) < 0$, then $\lim_{t\to+\infty} Y_2(t) = 0,\ a.s.$.

(II): Assume that $r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}}(\ln(1+\Xi_1(u)))\lambda(du) > 0$ and

$$\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du) + r_2 - 0.5\nu_2^2 < 0.$$

Since $\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du) + r_2 - 0.5\nu_2^2 < 0$, then by (I), we have $\lim_{t\to+\infty} Y_2(t) = 0,\ a.s.$. Therefore, for arbitrary $\varepsilon > 0$, there is $T > 0$ such that for $t \ge T$,

$$t^{-1}c_{12}\int_{-t}^0 d\mu_2(\varsigma)\int_0^{t+\varsigma} Y_2(s)ds \le t^{-1}c_{12}\int_0^t Y_2(s)ds \le \frac{\varepsilon}{4},$$
$$t^{-1}c_{12}\frac{1}{\mathbf{q}}||\xi||_{c_g}(\upsilon_{\mathbf{q}})^{\frac{1}{2}}(1-e^{-\mathbf{q}t}) \le \frac{\varepsilon}{4}$$

and

$$t^{-1}\ln Y_1(0) \le \varepsilon/2.$$

Plugging the three inequalities above into (2.5), we obtain for $t \ge T$,

$$\ln Y_1(t) \ge \Big[\sum_{0<t_k<t}\ln(1+J_k) + r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du) - \varepsilon\Big]t$$
$$- c_{11}\int_0^t Y_1(s)ds + \nu_1 B_1(t) + \int_0^t\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du). \tag{2.7}$$

And

$$\begin{aligned}\ln Y_1(t) \leq &\Big[\sum_{0<t_k<t} \ln(1+J_k) + r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du)\Big]t \\ &- c_{11}\int_0^t Y_1(s)ds + \nu_1 B_1(t) + \int_0^t \int_{\mathbb{U}} \ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du).\end{aligned} \tag{2.8}$$

By means of the condition $r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du) > 0$, for a sufficiently small $\varepsilon > 0$, we then have $r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} (\ln(1+\Xi_1(u)))\lambda(du) - \varepsilon > 0$. Using (I) and (II) in Lemma 4.1 in Appendix to (2.7), (2.8) and the arbitrariness of ε respectively, we conclude that

$$\lim_{t\to+\infty} \langle Y_1(t)\rangle = \frac{r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du)}{c_{11}}, \quad a.s..$$

Analogous to the above analysis, the conclusion of (III) can be obtained and its proof are left out. □

Theorem 2.2. *For model (1.4), we let the Hypotheses (B1)-(B3) hold. Suppose that the conditions* $r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du) - c_{12}H_2 > 0$ *and* $\liminf_{t\to+\infty} \Big[t^{-1} \sum_{0<t_k<t} \ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}} \ln(1+\Xi_2(u))\lambda(du) - c_{21}H_1 > 0$ *hold, then for any initial data* $\xi \in C_g$*, the solution* $(Y_1(t), Y_2(t))$ *of Eq.(1.4) has the properties that*

$$\begin{aligned}&\liminf_{t\to+\infty}\langle Y_1(t)\rangle \geq h_1, a.s., \quad \liminf_{t\to+\infty}\langle Y_2(t)\rangle \geq h_2, a.s.,\\ &\limsup_{t\to+\infty}\langle Y_1(t)\rangle \leq H_1, a.s., \quad \limsup_{t\to+\infty}\langle Y_2(t)\rangle \leq H_2, a.s.,\end{aligned}$$

where

$$\begin{aligned}H_1 &= \frac{r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du)}{c_{11}},\\ H_2 &= \frac{\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du) + c_{21}H_1}{c_{22}},\\ h_1 &= \frac{r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du) - c_{12}H_2}{c_{11}},\\ h_2 &= \frac{\liminf_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du) - c_{21}H_1}{c_{22}}.\end{aligned}$$

The means model (1.4) will be permanence in time average a.s..

Proof. For the solution $(Y_1(t), Y_2(t))$ of model (1.4), $j = 1, 2$, one can yield

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^0 Y_j(s+\varsigma)d\eta_j(\varsigma)ds \\
=&\int_0^t \Big[\int_{-\infty}^{-s} Y_j(s+\varsigma)d\eta_j(\varsigma)ds + \int_{-s}^0 Y_j(s+\varsigma)d\eta_j(\varsigma)\Big]ds \\
=&\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_j(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_j(\varsigma)+\int_{-t}^0 d\eta_j(\varsigma)\int_0^{t+\varsigma} Y_j(s)ds \\
=&\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_j(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_j(\varsigma)+\int_{-t}^0 d\eta_j(\varsigma)\int_0^{t} Y_j(s)ds \\
&+\int_{-t}^0 d\eta_j(\varsigma)\int_t^{t+\varsigma} Y_j(s)ds \\
=&\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_j(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_j(\varsigma)+\int_0^t Y_j(s)ds-\int_{-\infty}^{-t} d\eta_j(\varsigma)\int_0^{t} Y_j(s)ds \\
&+\int_{-t}^0 d\eta_j(\varsigma)\int_t^{t+\varsigma} Y_j(s)ds \\
=&\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_j(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_j(\varsigma)+\int_0^t Y_j(s)ds-\int_{-\infty}^{-t} d\eta_j(\varsigma)\int_0^{t} Y_j(s)ds \\
&-\int_{-t}^0 d\eta_j(\varsigma)\int_{t+\varsigma}^{t} Y_j(s)ds. \qquad (2.9)
\end{aligned}
$$

Plugging (2.9) into (2.2), we have

$$
\begin{aligned}
&\ln Y_1(t) - \ln Y_1(0) \\
=&\sum_{0<t_k<t} \ln(1+J_k) + \Big(r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du)\Big)t \\
&- c_{11}\int_0^t Y_1(s)ds - c_{12}\int_0^t Y_2(s)ds - c_{12}\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_2(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_2(\varsigma) \\
&+ c_{12}\int_{-t}^0 d\eta_2(\varsigma)\int_{t+\varsigma}^t Y_2(s)ds + c_{12}\int_{-\infty}^{-t} d\eta_2(\varsigma)\int_0^t Y_2(s)ds + \nu_1 B_1(t) \\
&+ \int_0^t \int_{\mathbb{U}} \ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du). \qquad (2.10)
\end{aligned}
$$

Similarly, together with (2.6) and (2.9), we obtain

$$
\begin{aligned}
\ln Y_2(t) - \ln Y_2(0) =& \sum_{0<t_k<t} \ln(1+L_k) + \Big(r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}} \ln(1+\Xi_2(u))\lambda(du)\Big)t \\
&- c_{22}\int_0^t Y_2(s)ds + c_{21}\int_0^t Y_1(s)ds \\
&+ c_{21}\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_1(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_1(\varsigma) \\
&- c_{21}\int_{-t}^0 d\eta_1(\varsigma)\int_{t+\varsigma}^t Y_1(s)ds - c_{21}\int_{-\infty}^{-t} d\eta_1(\varsigma)\int_0^t Y_1(s)ds
\end{aligned}
$$

$$+\nu_2 B_2(t)+\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du). \tag{2.11}$$

Making use of the conditions $r_1-0.5\nu_1^2+\int_{\mathbb{U}}(\ln(1+\Xi_1(u)))\lambda(du)>0$ and (2.8), we obtain

$$\limsup_{t\to+\infty}\langle Y_1(t)\rangle \le \frac{r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)}{c_{11}}=H_1, \quad a.s.. \tag{2.12}$$

When (2.12) and (2.4) are used in (2.11), we have

$$\begin{aligned}
&t^{-1}\ln Y_2(t)-t^{-1}\ln Y_2(0)\\
=&\Big(t^{-1}\sum_{0<t_k<t}\ln(1+L_k)+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)\Big)-c_{22}\langle Y_2(t)\rangle+c_{21}\langle Y_1(t)\rangle\\
&+t^{-1}c_{21}\int_0^t ds\int_{-\infty}^{-s}e^{\mathbf{q}(s+\varsigma)}Y_1(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_1(\varsigma)\\
&-t^{-1}c_{21}\int_{-t}^{0}d\eta_1(\varsigma)\int_{t+\varsigma}^{t}Y_1(s)ds-t^{-1}c_{21}\int_{-\infty}^{-t}d\eta_1(\varsigma)\int_0^t Y_1(s)ds\\
&+t^{-1}\nu_2B_2(t)+t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_1(u))\tilde{\Gamma}(ds,du).\\
\le&\Big(t^{-1}\sum_{0<t_k<t}\ln(1+L_k)+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)\Big)-c_{22}\langle Y_2(t)\rangle\\
&+c_{21}\limsup_{t\to+\infty}\langle Y_1(t)\rangle+c_{21}\varepsilon+t^{-1}\nu_2B_2(t)+t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du)\\
\le&\Big(\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)+c_{21}H_1+c_{21}\varepsilon\Big)\\
&-c_{22}\langle Y_2(t)\rangle+t^{-1}\nu_2B_2(t)+t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du)
\end{aligned} \tag{2.13}$$

for sufficiently large t. Applying (II) in Lemma 4.1 in Appendix to (2.13) and the arbitrariness of ε, we derive

$$\begin{aligned}
&\limsup_{t\to+\infty}\langle Y_2(t)\rangle\\
\le&\frac{\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)+c_{21}H_1}{c_{22}}\\
=&H_2, \quad a.s..
\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.10), and (2.4), we have

$$\begin{aligned}
&t^{-1}\ln Y_1(t)-t^{-1}\ln Y_1(0)\\
=&\Big(\frac{1}{t}\sum_{0<t_k<t}\ln(1+J_k)+r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)+c_{21}H_1\Big)
\end{aligned}$$

$$
\begin{aligned}
&- c_{11}\langle Y_1(t)\rangle - c_{12}\langle Y_2(t)\rangle - c_{12}t^{-1}\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_2(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_2(\varsigma)\\
&+ c_{12}t^{-1}\int_{-t}^0 d\eta_2(\varsigma)\int_{t+\varsigma}^t Y_2(s)ds + c_{12}\int_{-\infty}^{-t} d\eta_2(\varsigma)\int_0^t Y_2(s)ds + t^{-1}\nu_1 B_1(t)\\
&+ t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du)\\
\geq&\Big(r_1 - 0.5\nu_1^2 - c_{12}H_2 + \int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du) - c_{12}\varepsilon\Big) - c_{11}\langle Y_1(t)\rangle\\
&+ t^{-1}\nu_1 B_1(t) + t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du)\\
=&\Big(r_1 - 0.5\nu_1^2 - c_{12}H_2 - c_{12}\varepsilon\Big) - c_{11}\langle Y_1(t)\rangle + t^{-1}\nu_1 B_1(t)\\
&+ t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du).
\end{aligned}
\tag{2.15}
$$

From (II) in Lemma 4.1 in Appendix, (2.15) and the arbitrariness of ε, we derive

$$
\liminf_{t\to+\infty}\langle Y_1(t)\rangle \geq \frac{r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du) - c_{12}H_2}{c_{11}} = h_1 \quad a.s.. \tag{2.16}
$$

Similar to the previous case, by (2.4) and (2.11), we find

$$
\begin{aligned}
&t^{-1}\ln Y_2(t) - t^{-1}\ln Y_2(0)\\
=&(t^{-1}\sum_{0<t_k<t}\ln(1+L_k) + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du))\\
&- c_{22}\langle Y_2(t)\rangle + c_{21}\langle Y_1(t)\rangle + c_{21}\int_0^t ds\int_{-\infty}^{-s} e^{\mathbf{q}(s+\varsigma)}Y_1(s+\varsigma)e^{-\mathbf{q}(s+\varsigma)}d\eta_1(\varsigma)\\
&- c_{21}t^{-1}\int_{-t}^0 d\eta_1(\varsigma)\int_{t+\varsigma}^t Y_1(s)ds - c_{21}t^{-1}\int_{-\infty}^{-t} d\eta_1(\varsigma)\int_0^t Y_1(s)ds.\\
&+ t^{-1}\nu_2 B_2(t) + t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du)\\
\geq&\Big(\liminf_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)\\
&- c_{21}H_1 - c_{21}\varepsilon H_1\Big) - c_{22}\langle Y_2(t)\rangle + t^{-1}\nu_2 B_2(t)\\
&+ t^{-1}\int_0^t\int_{\mathbb{U}}\ln(1+\Xi_2(u))\tilde{\Gamma}(ds,du).
\end{aligned}
\tag{2.17}
$$

Applying (II) in Lemma 4.1 in Appendix to (2.17), we obtain

$$
\begin{aligned}
&\liminf_{t\to+\infty}\langle Y_2(t)\rangle\\
\geq&\frac{\liminf_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)}{c_{22}}\\
&- \frac{c_{21}H_1}{c_{22}} = h_2, \quad a.s..
\end{aligned}
\tag{2.18}
$$

Thus we have finished the proof of this Theorem 2.3. □

Corollary 2.1. *When Hypotheses (B1)-(B3) hold, then model (1.3) has the following property.*
(I)If $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)<0$ *and* $r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)<0$, *then both* Y_1 *and* Y_2 *tend to zero a.s., i.e.,* $\lim_{t\to+\infty} Y_i(t)=0$ *a.s.*, $i=1,2$.

(II) If $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)>0$ *and* $\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)<0$, *then* Y_2 *tends to zero a.s. and* Y_1 *is stability in time average a.s., i.e.,*

$$\lim_{t\to+\infty}\langle Y_1(t)\rangle=\frac{r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)}{c_{11}},\quad a.s..$$

(III) If $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)<0$ *and* $-r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)>0$, *then* Y_1 *tends to zero a.s. and* Y_2 *is stability in time average a.s., i.e.,*

$$\lim_{t\to+\infty}\langle Y_2(t)\rangle=\frac{r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)}{c_{22}},\qquad a.s..$$

Corollary 2.2. *For model (1.3), we let the Hypotheses (B1)-(B3) hold. If the conditions* $c_{11}>0.5$, $c_{22}>0.5c_{21}^2$, $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)-a_{12}M_2>0$ *and* $r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)-a_{21}M_1>0$ *hold, then for any initial data* $\xi\in C_g$,

$$\liminf_{t\to+\infty}\langle Y_1(t)\rangle\ge m_1, a.s.,\quad \liminf_{t\to+\infty}\langle Y_2(t)\rangle\ge m_2, a.s.,$$

$$\limsup_{t\to+\infty}\langle Y_1(t)\rangle\le M_1, a.s.,\quad \limsup_{t\to+\infty}\langle Y_2(t)\rangle\le M_2, a.s.,$$

where

$$\begin{aligned}
M_1=&\frac{r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)}{c_{11}},\\
M_2=&\frac{r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)+c_{21}M_1}{c_{22}},\\
m_1=&\frac{r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)-c_{12}M_2}{c_{11}},\\
m_2=&\frac{r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)-c_{21}M_1}{c_{22}}.
\end{aligned}$$

That is, model (1.3) will be permanence in time average a.s..

Remark 2.1. The coefficient c_{12} in model (1.4) denotes the intensity of one infinite delay. According to Theorem 2.3, we easily derive that c_{12} plays an negative effect on permanence in time average of population $Y_1(t)$ in model (1.4) under the Hypothese (B2).

Remark 2.2. In view of Theorem 2.2, we find that $r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)$ is the threshold between extinction and stability in time average of population $Y_1(t)$ in model (1.4) when Hypotheses (B1)-(B3) and $\limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+$

$L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}} \ln(1+\Xi_2(u))\lambda(du) < 0$ hold. In addition, we can also conclude that $\liminf_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)$ is the threshold between extinction and stability in time average of population $Y_2(t)$ in model (1.4) under Hypotheses (B1)-(B3), $\liminf_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big] = \limsup_{t\to+\infty}\Big[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\Big]$ and $r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du) < 0$.

Remark 2.3. In the light of (I)-(III) in Theorem 2.2 and Theorem 2.3, for model (1.4), it can be seen that the impulse can achieve favorable effect for permanence in time average when the impulsive perturbations are within a controllable range.

3. Stability in distribution

In this section, we shall study sufficient conditions for the stability in distribution of Eq.(1.3).

Hypothesis (B4). For model (1.4), there exist $l_i > 0$ and $L_i > 0$ s.t. $k_1 \le \prod_{0<t_k<t}(1+J_k) \le K_1$ and $k_2 \le \prod_{0<t_k<t}(1+L_k) \le K_2$ for all $t > 0$, respectively.

Definition 3.1. If there is a unique probability measure μ with nowhere zero density in $\mathbb{R}_+^2$ such that for arbitrary $Y(\theta) = (Y_1(\theta), Y_2(\theta)) = \xi \in C_g$, the transition probability $p(t,\xi,\cdot)$ of $Y(t) = (Y_1(t), Y_2(t))$ converges weakly to μ with $t \to +\infty$, then model (1.3) is said to be stable in distribution.

Lemma 3.1. *When the Hypotheses (B1)-(B4) hold, then model (1.4) is global attractivity under the conditions $k_1c_{11} > K_1c_{21}$ and $k_2c_{22} > K_2c_{12}$.*

Proof. Let $(Y_1(t), Y_2(t))$ and $(Y_1^*(t), Y_2^*(t))$ be two arbitrary solutions of Eq.(1.4) with initial values $\eta \in C_g$, $\eta^* \in C_g$, respectively. Suppose that the solution of Eq.

$$\begin{cases} dZ_1(t) = Z_1(t)\Big[r_1 - \prod\limits_{0<t_k<t}(1+J_k)c_{11}Z_1(t) - c_{12}\int_{-\infty}^0 \prod\limits_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)\Big]dt \\ \qquad\qquad + \nu_1 Z_1(t)dB_1(t) + Z_1(t^-)\int_{\mathbb{U}}\Xi_1(u)\Gamma(dt,du), \\ dZ_2(t) = Z_2(t)\Big[r_2 + c_{21}\int_{-\infty}^0 \prod\limits_{0<t_k<t+\varsigma}(1+J_k)Z_1(t+\varsigma)d\eta_1(\varsigma) - \prod\limits_{0<t_k<t}(1+L_k)c_{22}Z_2(t)\Big]dt \\ \qquad\qquad + \nu_2 Z_2(t)dB_2(t) + Z_2(t^-)\int_{\mathbb{U}}\Xi_1(u)\Gamma(dt,du) \end{cases} \tag{3.1}$$

is $(Z_1(t), Z_2(t))$ and the same initial values $\eta \in C_g$ as Eq.(1.4). What is more, the solution of Eq.

$$\begin{cases} dZ_1(t) = Z_1(t)\Big[r_1 - \prod_{0<t_k<t}(1+J_k)c_{11}Z_1(t) \\ \qquad - c_{12}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)Z_2(t+\varsigma)d\eta_2(\varsigma)\Big]dt \\ \qquad + \nu_1 Z_1(t)dB_1(t) + Z_1(t^-)\int_{\mathbb{U}}\Xi_1(u)\Gamma(dt,du), \\ dZ_2(t) = Z_2(t)\Big[r_2 + c_{21}\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+J_k)Z_1(t+\varsigma)d\eta_1(\varsigma) \\ \qquad - \prod_{0<t_k<t}(1+L_k)c_{22}Z_2(t)\Big]dt + \nu_2 Z_2(t)dB_2(t) \\ \qquad + Z_2(t^-)\int_{\mathbb{U}}\Xi_2(u)\Gamma(dt,du) \end{cases} \tag{3.2}$$

is $(Z_1^*(t), Z_2^*(t))$ and the same initial values $\eta^* \in C_g$ as Eq.(1.4). Then one can get

$$Y_1(t) = \prod_{0<t_k<t}(1+J_k)Z_1(t), \quad Y_2(t) = \prod_{0<t_k<t}(1+L_k)Z_2(t),$$
$$Y_1^*(t) = \prod_{0<t_k<t}(1+J_k)Z_1^*(t), \quad Y_2^*(t) = \prod_{0<t_k<t}(1+L_k)Z_2^*(t).$$

Define

$$V(t) = \sum_{j=1}^{2}|\ln(Z_j(t)) - \ln(Z_j^*(t))| + c_{21}K_1\int_{-\infty}^{0}\int_{t+\varsigma}^{t}|Z_1(s) - Z_1^*(s)|dsd\eta_1(\varsigma)$$
$$+ c_{12}K_2\int_{-\infty}^{0}\int_{t+\varsigma}^{t}|Z_2(s) - Z_2^*(s)|dsd\eta_2(\varsigma).$$

Computing $D^+V(t)$, and by the Itô's formula, we can derive

$$\begin{aligned} &D^+V(t) \\ =&\sum_{j=1}^{2}\mathrm{sgn}(Z_j(t) - Z_j^*(t))d(\ln(Z_j(t)) - \ln(Z_j^*(t))) \\ &+ c_{21}K_1\int_{-\infty}^{0}|Z_1(t) - Z_1^*(t)|d\eta_1(\varsigma)dt - c_{21}K_1\int_{-\infty}^{0}|Z_1(t+\varsigma) - Z_1^*(t+\varsigma)|d\eta_1(\varsigma)dt \\ &+ c_{12}K_2\int_{-\infty}^{0}|Z_2(t) - Z_2^*(t)|d\eta_2(\varsigma)dt - c_{12}K_2\int_{-\infty}^{0}|Z_2(t+\varsigma) - Z_2^*(t+\varsigma)|d\eta_2(\varsigma)dt \\ =&\mathrm{sgn}(Z_1(t) - Z_1^*(t))\Big(- c_{11}\prod_{0<t_k<t}(1+J_k)(Z_1(t) - Z_1^*(t)) \\ &- c_{12}\Big(\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+L_k)(Z_2(t+\varsigma) - Z_2^*(t+\varsigma))d\eta_2(\varsigma)\Big)\Big)dt \\ &+ \mathrm{sgn}(Z_2(t) - Z_2^*(t))\Big(- c_{22}\prod_{0<t_k<t}(1+L_k)(Z_2(t) - Z_2(t)) \end{aligned}$$

$$
\begin{aligned}
&+c_{21}\Big(\int_{-\infty}^{0}\prod_{0<t_k<t+\varsigma}(1+J_k)(Z_1(t+\varsigma)-Z_1^*(t+\varsigma))d\eta_1(\varsigma)\Big)\Big)dt\\
&+c_{21}K_1\int_{-\infty}^{0}|Z_1(t)-Z_1^*(t)|d\eta_1(\varsigma)dt-c_{21}K_1\int_{-\infty}^{0}|Z_1(t+v\varsigma)-Z_1^*(t+\varsigma)|d\eta_1(\varsigma)dt\\
&+c_{12}K_2\int_{-\infty}^{0}|Z_2(t)-Z_2^*(t)|d\eta_2(\varsigma)dt-c_{12}K_2\int_{-\infty}^{0}|Z_2(t+\varsigma)-Z_2^*(t+\varsigma)|d\eta_2(\varsigma)dt\\
\leq&-c_{11}k_1|Z_1(t)-Z_1^*(t)|dt+c_{12}K_2\int_{-\infty}^{0}|Z_2(t+\varsigma)-Z_2^*(t+\varsigma)|d\eta_2(\varsigma)dt\\
&-c_{22}k_2|Z_2(t)-Z_2^*(t)|dt+c_{21}K_1\int_{-\infty}^{0}|Z_1(t+\varsigma)-Z_1^*(s+\varsigma)|d\eta_2(\varsigma)dt\\
&+c_{21}K_1\int_{-\infty}^{0}|Z_1(t)-Z_1^*(t)|d\eta_1(\varsigma)dt-c_{21}K_1\int_{-\infty}^{0}|Z_1(t+\varsigma)-Z_1^*(t+\varsigma)|d\eta_1(\varsigma)dt\\
&+c_{12}K_2\int_{-\infty}^{0}|Z_2(t)-Z_2^*(t)|d\eta_2(\varsigma)dt-c_{12}K_2\int_{-\infty}^{0}|Z_2(t+\varsigma)v-Z_2^*(t+\varsigma)|d\eta_2(\varsigma)dt\\
=&-(c_{11}k_1-c_{21}K_1)|Z_1(t)-Z_1^*(t)|dt-(c_{22}k_2-c_{12}K_2)|Z_2(t)-Z_2^*(t)|dt.
\end{aligned}
$$

Then it leads to

$$
\begin{aligned}
\mathbb{E}V(t)\leq&V(0)-\int_0^t(c_{11}k_1-c_{21}K_1)\mathbb{E}|Z_1(s)-Z_1^*(s)|ds\\
&-\int_0^t(c_{22}k_2-c_{12}K_2)\mathbb{E}|Z_2(s)-Z_2^*(s)|ds.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
&\mathbb{E}V(t)+\int_0^t(c_{11}k_1-c_{21}K_1)\mathbb{E}|Z_1(s)-Z_1^*(s)|ds\\
&+\int_0^t(c_{22}k_2-c_{12}K_2)\mathbb{E}|Z_2(s)-Z_2^*(s)|ds\leq V(0)<\infty.
\end{aligned}
$$

This together with $k_1c_{11}>K_1c_{21}$ and $k_2c_{22}>K_2c_{12}$ implies

$$\mathbb{E}|Z_j(t)-Z_j^*(t)|\in L^1[0,+\infty),\qquad j=1,2.$$

The rest of proof is standard, we leave out it. □

Corollary 3.1. *Let the Hypotheses (B1)-(B3) hold. If* $c_{11}>c_{21}$ *and* $c_{22}>c_{12}$, *then model (1.3) is global attractivity.*

Theorem 3.1. *Suppose that the Hypotheses (B1)-(B3) hold. If* $c_{11}>c_{21},c_{22}>c_{12}$, *then model (1.3) is stability in distribution.*

Proof. Now we claim that there exist three positive constants $K_1(p)$, p and K_2, such that

$$\limsup_{t\to+\infty}\mathbb{E}(Y_1^p(t))\leq K_1(p), \tag{3.3}$$

$$\limsup_{t\to+\infty}\mathbb{E}(Y_2(t))\leq K_2. \tag{3.4}$$

The proof of inequality (3.3) is homologous to that of Lemma 3.1 in Li et al. [27], hence could be left out. Now we confirm inequality (3.4). By Eq.(1.3), from Itô's formula, one can see that

$$\begin{aligned} de^tY_2(t) =& e^tY_2(t)\Big[1+\Big(r_2-c_{22}Y_2(t)+c_{21}\int_{-\infty}^{0}Y_1(t+\varsigma)d\eta_1(\varsigma)\Big)\Big] \\ &+\nu_2e^tY_2(t^-)dB_2(t)+e^tY_2(t^-)\int_{\mathbb{U}}\Xi_2(u)\Gamma(dt,du). \end{aligned} \tag{3.5}$$

Integrating Eq.(3.5) from 0 to t on both sides, we have

$$\begin{aligned} & e^tY_2(t)-Y_2(0) \\ =& \int_0^t e^sY_2(s)\Big[1+\Big(r_2-c_{22}Y_2(s)+c_{21}\int_{-\infty}^{0}Y_1(s+\varsigma)d\eta_1(\varsigma)\Big)\Big]ds \\ &+\int_0^t \nu_2e^sY_2(s^-)dB_2(s)+\int_0^t e^sY_2(s^-)\int_{\mathbb{U}}\Xi_2(u)\Gamma(ds,du) \\ \leq& \int_0^t e^s\Big(Y_2(s)\Big[1+\Big(r_2+\int_{\mathbb{U}}\Xi_2(u)\lambda(du)-c_{22}Y_2(s)\Big)\Big]+0.5c_{21}Y_2^2(s)\Big)ds \\ &+0.5c_{21}\int_0^t e^s\int_{-\infty}^{0}Y_1^2(s+\varsigma)d\eta_1(\varsigma)ds+\int_0^t \nu_2e^sY_2(s^-)dB_2(s) \\ &+\int_0^t e^sY_2(s^-)\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(ds,du) \\ =& \int_0^t e^s\Big(\Big(1+r_2+\int_{\mathbb{U}}\Xi_2(u)\lambda(du)\Big)Y_2(s)-\Big(c_{22}-\frac{\gamma}{2}\rho_{21}\Big)Y_2^2(s)\Big)ds \\ &+\frac{1}{2\gamma}c_{21}\int_0^t e^s\int_{-\infty}^{0}Y_1^2(s+\varsigma)d\eta_1(\varsigma)ds+\int_0^t \nu_2(\vartheta(s))e^sY_2(s^-)dB_2(s) \\ &+\int_0^t e^sY_2(s^-)\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(ds,du) \\ \leq& (e^t-1)K+0.5\rho_{21}\int_0^t e^s\int_{-\infty}^{0}Y_1^2(s+\varsigma)d\eta_1(\varsigma)ds+\int_0^t \nu_2(\vartheta(s))e^sY_2(s^-)dB_2(s) \\ &+\int_0^t e^sY_2(s^-)\int_{\mathbb{U}}\Xi_2(u)\tilde{\Gamma}(ds,du), \end{aligned}$$

where $\gamma>0$ is sufficiently small number satisfying $k_2c_{22}-\dfrac{\gamma}{2}c_{21}>0$. Then

$$\begin{aligned} e^t\mathbb{E}Y_2(t)-Y_2(0) \leq& (e^t-1)K++\frac{1}{2\gamma}c_{21}\int_0^t e^s\int_{-\infty}^{0}\mathbb{E}Y_1^2(s+\varsigma)d\eta_1(\varsigma)ds \\ =& (e^t-1)K++\frac{1}{2\gamma}c_{21}(e^t-1)K_1^*(2). \end{aligned}$$

Hence

$$\limsup_{t\to\infty}\mathbb{E}Y_2(t)\leq K_2.$$

The proof of inequality (3.4) is therefore complete. Therefore, for $i=1,2$, $\mathbb{E}(Y_i(t))$ is uniformly continuous(see Lemma 2 in [37], Lemma 3.1 in [27], Lemma 3.2 in [28]).

With the help of the Corollary 3.2 and Barbălat's work [1], the rest of proof is similar to Lemma 3.2 in [38] and hence we skip the details to save space. □

Remark 3.1. The parameters c_{12} and c_{21} in model (1.3) stand for the intensities of two infinite delays, respectively. By Theorem 3.4, we find the coefficients c_{12} and c_{21} related to infinite delay are unfavorable to stability in distribution of model (1.3).

4. Numerical Examples

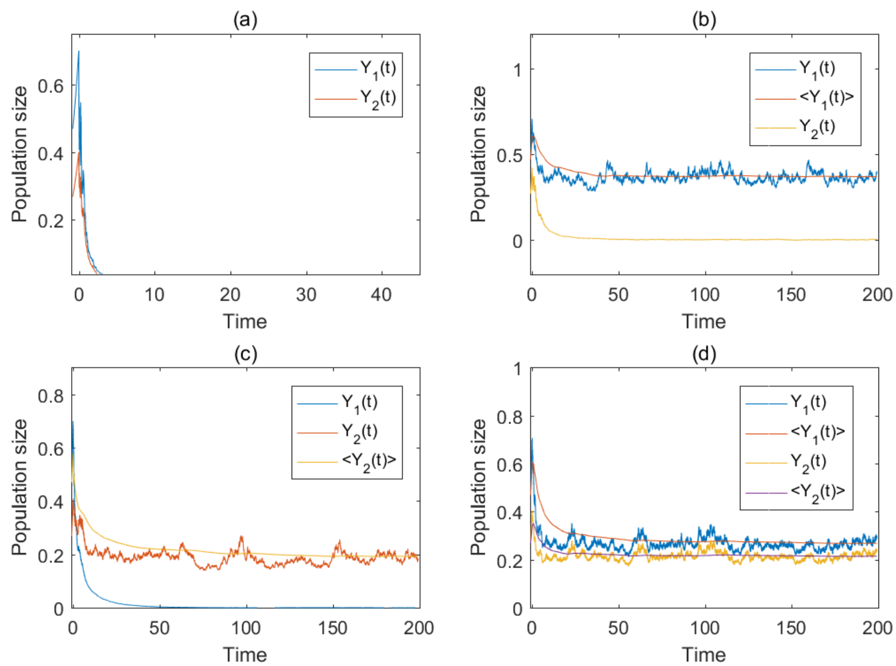

Figure 1. Step size $\Delta t = 0.001$. The horizontal axis in this and following figures represent the time t. (a) is with $L_k = 0, \Xi_1(u) = -0.5$; (b) is with $L_k = 0, \Xi_1(u) = -0.17$; (c) is with $L_k = e^7 - 1, \Xi_1(u) = -0.5$; (d) is with $L_k = e^4 - 1$, $\Xi_1(u) = -0.3$, $\Xi_2(u) = -0.3$.

To demonstrate our theoretic results, numerical simulations are given by the Euler scheme [9] to discretize model (1.4).

In Fig.1(a)-Fig.1(c), we set the initial data $(\eta_1, \eta_2) = (0.7e^s, 0.4e^s)$. Selecting parameters $r_1 = 0.6, r_2 = 0.1, c_{11} = 0.5, c_{22} = 0.22, c_{12} = c_{21} = 0.1, \nu_1^2 = \nu_2^2 = 0.12, \Xi_2(u) = -0.5, J_k = 0, t_k = 10k$. Then the positive equilibrium $(\frac{\Theta_1}{\Theta}, \frac{\Theta_2}{\Theta}) = (1.1083, 0.91667)$ of model (1.2) is globally asymptotically stable. The nothing but distinction in Fig.1(a)-Fig.1(c) is that L_k and $\Xi_1(u)$ are not identical. In Fig.1(a), we choose $L_k = 0$ and $\Xi_1(u) = -0.5$. From (I) of Theorem 2.2, the population $Y_1(t)$ and $Y_2(t)$ in model (1.4) tend to zero a.s.. In Fig.1(b), we choose $L_k = 0, \Xi_1(u) = -0.17$. From (II) of Theorem 2.2, for model (1.4), the population Y_2 tends to zero a.s., and

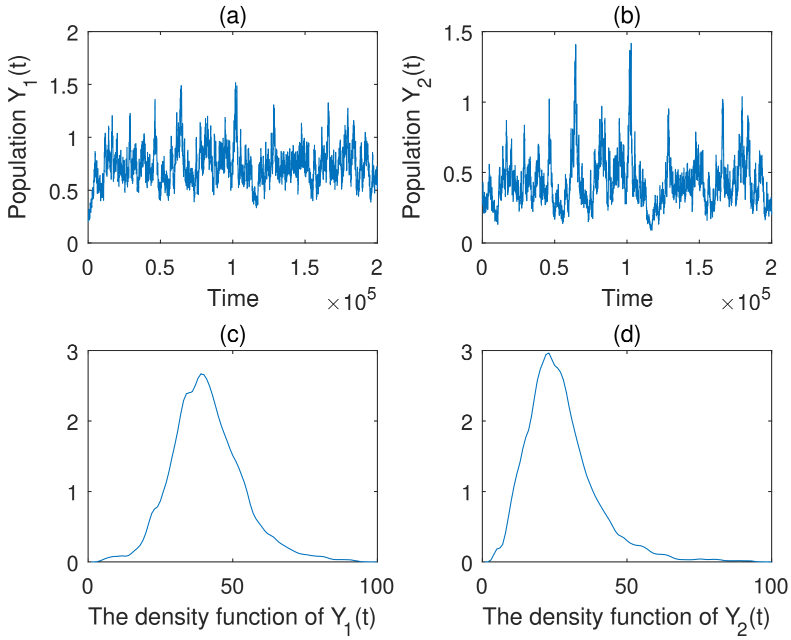

Figure 2. Step size $\Delta t = 0.001$. (a) and (b) is a sample path of model (1.3); (c) is the probability density function of $Y_1(t)$ at time $t = 1000$; (d) is the probability density function of $Y_2(t)$ at time $t = 1000$.

Y_1 is stability in time average a.s., i.e.,

$$\lim_{t\to+\infty}\langle Y_1(t)\rangle = \frac{r_1 - 0.5\nu_1^2 + \int_{\mathbb{U}} \ln(1+\Xi_1(u))\lambda(du)}{c_{11}} = 0.36, \quad a.s..$$

In Fig.1(c), we choose $L_k = e^7 - 1$ and $\Xi_1(u) = -0.5$. In view of (III) of Theorem 2.2, for model (1.4), population $Y_1(t)$ tends to zero a.s., $Y_2(t)$ is stability in time average a.s., i.e.,

$$\begin{aligned}&\lim_{t\to+\infty}\langle Y_2(t)\rangle\\ =&\frac{\lim\limits_{t\to+\infty}\left[t^{-1}\sum\limits_{0<t_k<t}\ln(1+L_k) + r_2 - 0.5\nu_2^2 + \int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)\right]}{c_{22}}\\ =&0.18, \quad a.s..\end{aligned}$$

In Fig.1(d), we consider $L_k = e^4 - 1$, $\Xi_1(u) = -0.3$ and $\Xi_2(u) = -0.3$. By computing, we can get

$$\limsup_{t\to+\infty}\langle Y_1(t)\rangle \le H_1 = \frac{r_1 - 0.5\nu_1^2 + \displaystyle\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)}{c_{11}} = 0.36, \quad a.s.,$$

$$\limsup_{t\to+\infty}\langle Y_2(t)\rangle \le H_2$$

$$= \frac{\limsup_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\right]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)}{c_{22}}$$

$$+\frac{c_{21}H_1}{c_{22}}=0.59,\quad a.s.,$$

$$\liminf_{t\to+\infty}\langle Y_1(t)\rangle$$

$$\ge h_1=\frac{r_1-0.5\nu_1^2+\int_{\mathbb{U}}\ln(1+\Xi_1(u))\lambda(du)-c_{12}H_2}{c_{11}}=0.25,\quad a.s.,$$

$$\liminf_{t\to+\infty}\langle Y_2(t)\rangle$$

$$\ge h_2=\frac{\liminf_{t\to+\infty}\left[t^{-1}\sum_{0<t_k<t}\ln(1+L_k)\right]+r_2-0.5\nu_2^2+\int_{\mathbb{U}}\ln(1+\Xi_2(u))\lambda(du)-c_{21}H_1}{c_{22}}$$

$$=0.218,\quad a.s..$$

In view of Theorem 2.3, population $(Y_1(t), Y_2(t))$ in model (1.4) is permanence in time average a.s..

In Fig.2, we consider the initial value $(\eta_1,\eta_2)=(0.3e^s,0.4e^s), r_1=0.62, r_2=0.54, c_{11}=0.8, c_{22}=0.7, c_{12}=0.3, c_{21}=0.2, \nu_1^2=0.5, \nu_2^2=0.2, \Xi_1(u)=\Xi_2(u)=-0.3$. From the Theorem 3.4, model (1.3) is stability in distribution.

Many population models always experience sudden changes in their structure and coefficients, for example, Zhu and Yin [47] pointed out that the growth rates of some species in the dry season will be much different from those in the rainy season, and one may make use of a continuous-time Markov chain $\zeta(t)$ with a finite state space $1,\cdots,m$ to explain these abrupt changes. For the sake of future research, we establish the following definition.

Definition 1. For the following impulsive stochastic functional differential equation with Markovian switching(ISFFDM):

$$\begin{cases} dY(t) = G_1\Big(t,\varsigma(t),Y(t),\int_{-\tau_1}^{0}Y(t+\varrho)d\nu_1(\varrho),\int_{-\infty}^{0}Y(t+\theta)d\mu_1(\theta)\Big)dt \\ \qquad\qquad + G_2\Big(t,\varsigma(t),Y(t),\int_{-\tau_2}^{0}Y(t+\varrho)d\nu_2(\varrho),\int_{-\infty}^{0}Y(t+\theta)d\mu_2(\theta)\Big)dB(t), \\ \qquad\qquad t\ne t_{\mathbf{k}},\quad \mathbf{k}\in N, \\ Y(t_{\mathbf{k}}^+)-Y(t_{\mathbf{k}})=I_kY(t_{\mathbf{k}}),\quad \mathbf{k}\in N, \end{cases} \tag{4.1}$$

where $Y(t+\theta), -\infty<\theta\le 0$, is C_g-value stochastic process, $C_g=\{\psi\in C((-\infty,0];\mathbb{R}^d):\|\psi\|_{c_g}=\sup_{-\infty<s\le 0}e^{\mathbf{q}s}|\psi(s)|<+\infty\}$, $g(s)=e^{-\mathbf{q}s}, \mathbf{q}>0$, $|\psi(s)|=\sqrt{\psi_1^2(s)+\cdots+\psi_d^2(s)}$, $(\psi_1(s),\psi_2(s),\cdots,\psi_d(s))\in\mathbb{R}^d$. For $i=1,2$, $\nu_i(\varrho)$ is a measure on $(-\tau_i,0]$, where $\tau_i (i=1,2)$ are constant. $I_{\mathbf{k}}>-1$, $\varsigma(t)$ stands for the regime switching [14, 34]. For $i=1,2$, $\mu_i(\theta)$ is a measure on $(-\infty,0]$, $0<t_{\mathbf{1}}<t_{\mathbf{2}}<\cdots$, $\lim_{\mathbf{k}\to+\infty}t_{\mathbf{k}}=+\infty$. The initial condition $Y_0\in C_g$ and $\varsigma(0)=0$, where $Y_0=\varpi=\{\varpi(\theta):-\infty<\theta\le 0\}$ is an $\mathcal{F}_0$-measurable C_g-valued random variable such that $\vartheta\in\mathcal{M}^2((-\infty,0];\mathbb{R}^d)$ which is the family of all $\mathcal{F}_0$-measurable, $\mathbb{R}^d$-valued processes $\psi(t), t \in (-\infty, 0]$ such that $\mathbb{E}\int_{-\infty}^{0} |\psi(t)|^2 dt < +\infty$. An $\mathbb{R}^d$-value stochastic process $Y(t)$ defined on $\mathbb{R}$ is called a solution of Eq. (4.1) with initial value above if $Y(t)$ satisfies the following criterion:

(i)$Y(t)$ is $\mathcal{F}_t$-adapted and continuous on $(0, t_{\mathbf{1}})$ and $(t_{\mathbf{k}}, t_{\mathbf{k+1}}), \mathbf{k} \in N; G_1(t, \varsigma(t), Y(t), \int_{-\tau_1}^{0} Y(t+\varrho)d\nu_1(\varrho), \int_{-\infty}^{0} Y(t+\theta)d\mu_1(\theta)) \in \mathcal{L}^1(\overline{\mathbb{R}}_+; \mathbb{R}^d)$ and $G_2(t, \varsigma(t), Y(t), \int_{-\tau_2}^{0} Y(t+\varrho)d\nu_2(\varrho), \int_{-\infty}^{0} Y(t+\theta)d\mu_2(\theta)) \in \mathcal{L}^2(\overline{\mathbb{R}}_+; \mathbb{R}^{d\times m})$. Here, for the explanations of $\mathcal{L}^1(\overline{\mathbb{R}}_+; \mathbb{R}^d)$ and $\mathcal{L}^2(\overline{\mathbb{R}}_+; \mathbb{R}^{d\times m})$, see [24]. $B(t)$ depicts a m-dimension standard Brownian motion.

(ii)for each $t_{\mathbf{k}}, \mathbf{k} \in N, Y(t_{\mathbf{k}}^+) = \lim\limits_{t\to t_{\mathbf{k}}^+} Y(t)$ and $Y(t_{\mathbf{k}}) = Y(t_{\mathbf{k}}^-) = \lim\limits_{t\to t_{\mathbf{k}}^-} Y(t)$ a.s..

(iii)$Y(t)$ obeys Eq. (4.1) for almost every $t \in [0, \infty)\backslash t_{\mathbf{k}}$ and satisfies the impulsive criterion at each $t = t_{\mathbf{k}}, \mathbf{k} \in N$ a.s..

Remark 4.1. Liu and Wang [15] proposed a new definition of a solution of an impulsive stochastic differential equation (ISDE). We give the definition 1, which extends the definition of a solution of ISDE to ISFFDM. In addition, it also generalizes the definition 5 in Ref. [23].

Appendix

Assumption 1. There is a positive constant c such that $\int_{\mathbb{Y}}[\ln(1+\gamma(u))]^2\lambda(du) < c$.

Lemma 4.1 (Liu et al. [21]). *Suppose that $z(t) \in C(\Omega \times [0, +\infty),\ \mathbb{R}_+)$ and let Assumption 1 hold.*
(i) If there exist two positive constants T and ρ_0 such that
$\ln z(t) \le \rho t - \rho_0 \int_0^t z(s)ds + \alpha B(t) + \sum\limits_{i=1}^{2} \delta_i \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\tilde{N}(ds, du)$ *a.s. for all $t \ge T$, where $\rho, \alpha, \delta_i, i = 1, 2$, are constants, then*

$$\begin{cases} \limsup\limits_{t\to+\infty}\langle z(t)\rangle \le \rho/\rho_0 & a.s., \quad if \quad \rho \ge 0; \\ \lim\limits_{t\to+\infty} z(t) = 0 & a.s., \quad if \quad \rho < 0. \end{cases} \tag{4.2}$$

(ii) If there exist three positive constants T, ρ and ρ_0 such that $\ln z(t) \ge \rho t - \rho_0 \int_0^t z(s)ds + \alpha B(t) + \sum\limits_{i=1}^{2} \delta_i \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\tilde{N}(ds, du)$ for all $t \ge T$, then $\liminf\limits_{t\to+\infty}\langle z(t)\rangle \ge \rho/\rho_0$ a.s..

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