Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1353–1370
Website:http://www.jaac-online.com
DOI:10.11948/20210081

EXPLICIT SOLUTIONS TO A HIERARCHY OF DISCRETE COUPLING KORTEWEG-DE VRIES EQUATIONS*

Qiulan Zhao[1,†], Yadong Zhong[1] and Xinyue Li[1]

Abstract To get a hierarchy of discrete coupling Korteweg-de Vries equations, we consider from a discrete four-by-four matrix spectral problem. Then we can get the Lax pair of the KdV equations. Finally we present the explicit solutions of the KdV equations by constructing theirs Darboux transformations with the help of the corresponding Lax pairs.

Keywords KdV equation, Darboux transformation, explicit solutions.

MSC(2010) 35Q53, 37K10.

1. Introduction

Since the Korteweg and de Vries described a model for shallow water waves in 1895 [7], the Korteweg-de Vries (KdV) equation [3, 21, 25, 27] has attracted many researchers attentions in recent years because of its nice mathematical and physical features. The KdV equation is a universal nonlinear system which arises whenever there is a balance of weak dispersion and quadratic nonlinearity, the discrete KdV equation was studied in Ref. [5], the coupled KdV equation was studied in Ref. [24], the super KdV equation was studied in Ref. [9]. Studying KdV equation is a helpful work to understand the complex behaviours and many researchers continue to explore this way [2, 28]. Then in our paper, we will study the following equations by constructing the Lax pair [23, 31] based on Ref. [29]

$$\begin{cases} u_{n,t_1}+h_{n,t_1}=(u_n+h_n)(1-E)(u_n+h_n)(u_{n-1}+h_{n-1}),\\ v_{n,t_1}=v_n(1-E)(u_n+h_n)(u_{n-1}+h_{n-1})-(u_n+h_n)(1-E)\\ \qquad\quad\times[(u_{n-1}+h_{n-1})v_n+(u_n+h_n)v_{n-1}-(u_{n-1}+h_{n-1})(u_n+h_n)]. \end{cases} \tag{1.1}$$

We can choose different suitable values of h_n in Eq. (1.1), then we can get different equations from Eq. (1.1). For example, if we choose $h_n=0$, and taking $v_n=-\frac{1}{2}u_n$, then we can get the famous discrete KdV equation [5]

$$u_{n,t}=u_n^2(u_{n-1}-u_{n+1}).$$

In Ref. [29], Xu studied a four-by-four matrix spectral problem, then he got a hierarchy of integrable lattice equations. Finally, he found the lattice equations

[†]The corresponding author. Email: qlzhao@sdust.edu.cn(Q. L. Zhao)
[1]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China
*The authors were supported by the National Nature Science Foundation of China (No. 11701334) and the "Jingying" Project of Shandong University of Science and Technology.

were all integrable in Liouville sense. But he didn't find the explicit solutions of the lattice equations. So the main idea of our paper is to find the explicit solutions of the KdV equations based on a similar matrix spectral problem.

There are many methods have been developed to obtain explicit solutions, where include the inverse scattering transform method [1], the Bäcklund transformation [4], the Darboux transformation [6, 10–12, 16, 20, 26], the Hirota bilinear method [14, 15, 17–19], the Jacobi elliptic function expansion method [8], the Lie symmetry method [22]. In this paper, we will choose the Darboux transformation to discuss the KdV equations. The Darboux transformation is a powerful tool to get the explicit solutions from seed solutions. Then the solutions are analyzed in Figs. 1, 2 and 3.

In this paper, starting from the using of Lax pair, we have obtained the KdV equations, then we choose different values in the KdV equations and we can get some different equations. Finally, the Darboux transformations of the KdV equations are given and explicit solutions are obtained.

This paper is organized as follows. In section 2, a discrete matrix spectral is introduced and its Lax pair will be derived. In section 3, the Darboux transformations of the KdV equations are constructed with the help of Lax pair and explicit solutions are shown in section 4. The last section contains some discussions.

2. The discrete coupling Korteweg-de Vries equations

In this section, starting from the following discrete 4×4 spectral problem

$$E\varphi_n = U_n\varphi_n, U_n = \begin{pmatrix} 0 & u_n + h_n & 0 & v_n \\ u_n + h_n & \lambda & v_n & \lambda \\ 0 & 0 & 0 & u_n + h_n \\ 0 & 0 & u_n + h_n & \lambda \end{pmatrix}, \varphi_n = \begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \\ \varphi_{3,n} \\ \varphi_{4,n} \end{pmatrix}, \quad (2.1)$$

where $\lambda_t = 0$, E is the shift operator defined by $Ef(n,t) = f(n+1,t)$ and $E^{-1}f(n,t) = f(n-1,t)$.

We solve the stationary zero-curvature equation

$$(E\Gamma_n)U_n - U_n\Gamma_n = 0, \quad (2.2)$$

taking $\Gamma_n = \begin{pmatrix} a_n & b_n & e_n & f_n \\ c_n & -a_n & g_n & -e_n \\ 0 & 0 & a_n & b_n \\ 0 & 0 & c_n & -a_n \end{pmatrix}$, then Eq. (2.2) becomes

$$\begin{cases} b_{n+1} - c_n = 0, \\ (u_n + h_n)(a_{n+1} + a_n) + \lambda b_{n+1} = 0, \\ -(u_n + h_n)(a_{n+1} + a_n) - \lambda c_n = 0, \\ (u_n + h_n)c_{n+1} - (u_n + h_n)b_n + \lambda(a_n - a_{n+1}) = 0, \\ v_n b_{n+1} - v_n c_n + (u_n + h_n)f_{n+1} + (u_n + h_n)g_n = 0, \\ v_n(a_n + a_{n+1}) + (u_n + h_n)(e_n + e_{n+1}) + \lambda b_{n+1} + \lambda f_{n+1} = 0, \\ -v_n(a_n + a_{n+1}) - (u_n + h_n)(e_n + e_{n+1}) - \lambda c_n - \lambda g_n = 0, \\ -v_n b_n + v_n c_{n+1} - \lambda(a_{n+1} - a_n) - \lambda(e_{n+1} - e_n) + (u_n + h_n)(g_{n+1} - f_n) = 0. \end{cases} \tag{2.3}$$

Substituting $a_n = \Sigma_{j=1}^{\infty} a_n^{(j)} \lambda^{-2m}$, $b_n = \Sigma_{j=1}^{\infty} b_n^{(j)} \lambda^{-2m+1}$, $c_n = \Sigma_{j=1}^{\infty} c_n^{(j)} \lambda^{-2m+1}$, $e_n = \Sigma_{j=1}^{\infty} e_n^{(j)} \lambda^{-2m}$, $f_n = \Sigma_{j=1}^{\infty} f_n^{(j)} \lambda^{-2m+1}$ and $g_n = \Sigma_{j=1}^{\infty} g_n^{(j)} \lambda^{-2m+1}$ into Eq. (2.3), we can obtain the following recursion relations

$$\begin{cases} b_{n+1}^{(m)} - c_n^{(m)} = 0, \\ (u_n + h_n)(a_{n+1}^{(m)} + a_n^{(m)}) + b_{n+1}^{(m+1)} = 0, \\ -(u_n + h_n)(a_{n+1}^{(m)} + a_n^{(m)}) - c_n^{(m+1)} = 0, \\ (u_n + h_n)c_{n+1}^{(m)} - (u_n + h_n)b_n^{(m)} + (a_n^{(m)} - a_{n+1}^{(m)}) = 0, \\ v_n b_{n+1}^{(m)} - v_n c_n^{(m)} + (u_n + h_n)f_{n+1}^{(m)} + (u_n + h_n)g_n^{(m)} = 0, \\ v_n(a_n^{(m)} + a_{n+1}^{(m)}) + (u_n + h_n)(e_n^{(m)} + e_{n+1}^{(m)}) + b_{n+1}^{(m+1)} + f_{n+1}^{(m+1)} = 0, \\ -v_n(a_n^{(m)} + a_{n+1}^{(m)}) - (u_n + h_n)(e_n^{(m)} + e_{n+1}^{(m)}) - c_n^{(m+1)} - g_n^{(m+1)} = 0, \\ -v_n b_n^{(m)} + v_n c_{n+1}^{(m)} - (a_{n+1}^{(m)} - a_n^{(m)}) - (e_{n+1}^{(m)} - e_n^{(m)}) + (u_n + h_n)(g_{n+1}^{(m)} - f_n^{(m)}) = 0. \end{cases} \tag{2.4}$$

Especially, if we choose the suitable initial values $a_n^{(0)} = -\frac{1}{2}$, $b_n^{(0)} = 0$, $c_n^{(0)} = 0$, $e_n^{(0)} = -\frac{1}{2}$, $f_n^{(0)} = 0$ and $g_n^{(0)} = 0$, the above recursion relations can determine other functions such as $a_n^{(m)}$, $b_n^{(m)}$, $c_n^{(m)}$, $e_n^{(m)}$, $f_n^{(m)}$ and $g_n^{(m)}$, $m \geq 0$ and the first few quantities are given by

$$\begin{aligned} &a_n^{(1)} = (u_n + h_n)(u_{n-1} + h_{n-1}), b_n^{(1)} = u_{n-1} + h_{n-1}, c_n^{(1)} = u_n + h_n, \\ &e_n^{(1)} = (u_{n-1} + h_{n-1})(v_n - u_n - h_n) + (u_n + h_n)v_{n-1}, f_n^{(1)} = v_{n-1}, g_n^{(1)} = v_n. \end{aligned}$$

Then we define

$$(\lambda^m \Gamma_n)_+ = \Sigma_{j=0}^{m} \begin{pmatrix} a_n^{(j)}\lambda^{m-j} & b_n^{(j)}\lambda^{m-j} & e_n^{(j)}\lambda^{m-j} & f_n^{(j)}\lambda^{m-j} \\ c_n^{(j)}\lambda^{m-j} & -a_n^{(j)}\lambda^{m-j} & g_n^{(j)}\lambda^{m-j} & -e_n^{(j)}\lambda^{m-j} \\ 0 & 0 & a_n^{(j)}\lambda^{m-j} & b_n^{(j)}\lambda^{m-j} \\ 0 & 0 & c_n^{(j)}\lambda^{m-j} & -a_n^{(j)}\lambda^{m-j} \end{pmatrix}, \tag{2.5}$$

and $V_n^{(m)} = (\lambda^m \Gamma_n)_+ + \triangle_n^{(m)}$, where

$$\triangle_n^{(m)} = \begin{pmatrix} -2a_n^{(m)} & 0 & -2e_n^{(m)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2a_n^{(m)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

Now we consider the following auxiliary spectral problem

$$\varphi_{n,t_m} = V_n^{(m)} \varphi_n, m \geq 0. \tag{2.7}$$

The compatibility conditions of Eq. (2.1) and Eq. (2.7) are

$$U_{n,t_m} = (EV_n^{(m)})U_n^{(m)} - U_n^{(m)} V_n^{(m)}, m \geq 0, \tag{2.8}$$

which lead to the following integrable lattice hierarchy

$$\begin{cases} u_{n,t_m} + h_{n,t_m} = (u_n + h_n)(a_n^{(m)} - a_{n+1}^{(m)}), \\ v_{n,t_m} = v_n(a_n^{(m)} - a_{n+1}^{(m)}) + (u_n + h_n)(e_n^{(m)} - e_{n+1}^{(m)}), \end{cases} \tag{2.9}$$

then when $m = 1$, we have

$$\begin{cases} u_{n,t_1} + h_{n,t_1} = (u_n + h_n)(1 - E)(u_n + h_n)(u_{n-1} + h_{n-1}), \\ v_{n,t_1} = v_n(1 - E)(u_n + h_n)(u_{n-1} + h_{n-1}) - (u_n + h_n)(1 - E) \\ \qquad \times [(u_{n-1} + h_{n-1})v_n + (u_n + h_n)v_{n-1} - (u_{n-1} + h_{n-1})(u_n + h_n)]. \end{cases} \tag{2.10}$$

Which is the Eq. (1.1), and the time part of the Lax pair of Eq. (1.1) is

$$\varphi_{n,t_1} = V_n \varphi_n = \begin{pmatrix} -\frac{1}{2}\lambda - r_n & u_{n-1} + h_{n-1} & -\frac{1}{2}\lambda - s_n & v_n \\ u_n + h_n & \frac{1}{2}\lambda - r_n & v_n & \frac{1}{2}\lambda - s_n \\ 0 & 0 & -\frac{1}{2}\lambda - r_n & u_{n-1} + h_{n-1} \\ 0 & 0 & u_n + h_n & \frac{1}{2}\lambda - r_n \end{pmatrix} \varphi_n, \tag{2.11}$$

where $r_n = (u_n + h_n)(u_{n-1} + h_{n-1})$, $s_n = (u_{n-1} + h_{n-1})v_n + (u_n + h_n)v_{n-1} - (u_{n-1} + h_{n-1})(u_n + h_n)$.

Now we consider some different values of h_n in Eq. (1.1).

First, if we choose $h_n = 0$, the Eq. (2.9) becomes

$$\begin{cases} u_{n,t_m} = u_n(a_n^{(m)} - a_{n+1}^{(m)}), \\ v_{n,t_m} = v_n(a_n^{(m)} - a_{n+1}^{(m)}) + u_n(e_n^{(m)} - e_{n+1}^{(m)}), \end{cases} \tag{2.12}$$

when $m = 1$, we have

$$\begin{cases} u_{n,t_1} = u_n^2(u_{n-1} - u_{n+1}), \\ v_{n,t_1} = u_n^2(u_{n+1} - u_{n-1} + v_{n-1} - v_{n+1}) + 2u_n v_n(u_{n-1} - u_{n+1}). \end{cases} \tag{2.13}$$

Then if we take $v_n = -\frac{1}{2}u_n$, we can get the famous discrete KdV equation.

Second, in the condition $h_n = \varepsilon v_n$, the Eq. (2.9) can be

$$\begin{cases} u_{n,t_m} = u_n(a_n^{(m)} - a_{n+1}^{(m)}) - \varepsilon(u_n + \varepsilon v_n)(e_n^{(m)} - e_{n+1}^{(m)}), \\ v_{n,t_m} = v_n(a_n^{(m)} - a_{n+1}^{(m)}) + (u_n + \varepsilon v_n)(e_n^{(m)} - e_{n+1}^{(m)}), \end{cases} \tag{2.14}$$

then when $m = 1$, we have

$$\begin{cases} u_{n,t_1} = u_n(1-E)(u_n + \varepsilon v_n)(u_{n-1} + \varepsilon v_{n-1}) - \varepsilon(u_n + \varepsilon v_n)(1-E) \\ \qquad \times [(u_{n-1} + \varepsilon v_{n-1})v_n + (u_n + \varepsilon v_n)v_{n-1} - (u_{n-1} + \varepsilon v_{n-1})(u_n + \varepsilon v_n)], \\ v_{n,t_1} = v_n(1-E)(u_n + \varepsilon v_n)(u_{n-1} + \varepsilon v_{n-1}) - (u_n + \varepsilon v_n)(1-E) \\ \qquad \times [(u_{n-1} + \varepsilon v_{n-1})v_n + (u_n + \varepsilon v_n)v_{n-1} - (u_{n-1} + \varepsilon v_{n-1})(u_n + \varepsilon v_n)]. \end{cases} \tag{2.15}$$

To see Eq. (2.15) better, we can choose $\varepsilon = 1$, so we get

$$\begin{cases} u_{n,t_1} = (u_n^2 - v_n^2)(v_{n+1} - v_{n-1}) + 2(u_n v_n + u_n^2)(u_{n+1} - u_{n-1}), \\ v_{n,t_1} = (v_n^2 - u_n^2)(u_{n+1} - u_{n-1}) + 2(u_n v_n + v_n^2)(v_{n+1} - v_{n-1})). \end{cases} \tag{2.16}$$

Third, let $h_n = \varepsilon u_n v_n$, the Eq. (2.9) can become

$$\begin{cases} u_{n,t_m} = \dfrac{u_n}{1+\varepsilon v_n}(a_n^{(m)} - a_{n+1}^{(m)}) - \varepsilon u_n^2(e_n^{(m)} - e_{n+1}^{(m)}), \\ v_{n,t_m} = v_n(a_n^{(m)} - a_{n+1}^{(m)}) + (u_n + \varepsilon u_n v_n)(e_n^{(m)} - e_{n+1}^{(m)}), \end{cases} \tag{2.17}$$

we can see the following equations when $m = 1$

$$\begin{cases} u_{n,t_1} = \dfrac{u_n}{1+\varepsilon v_n}(1-E)(u_n + \varepsilon u_n v_n)(u_{n-1} + \varepsilon u_{n-1}v_{n-1}) - \varepsilon u_n^2(1-E) \\ \qquad \times [(u_{n-1} + \varepsilon u_{n-1}v_{n-1})v_n + (u_n + \varepsilon u_n v_n)v_{n-1} \\ \qquad - (u_{n-1} + \varepsilon u_{n-1}v_{n-1})(u_n + \varepsilon u_n v_n)], \\ v_{n,t_1} = v_n(1-E)(u_n + \varepsilon u_n v_n)(u_{n-1} + \varepsilon u_{n-1}v_{n-1}) - (u_n + \varepsilon u_n v_n)(1-E) \\ \qquad \times [(u_{n-1} + \varepsilon u_{n-1}v_{n-1})v_n + (u_n + \varepsilon u_n v_n)v_{n-1} \\ \qquad - (u_{n-1} + \varepsilon u_{n-1}v_{n-1})(u_n + \varepsilon u_n v_n)]. \end{cases} \tag{2.18}$$

3. The N-fold Darboux transformation

In this section, we first introduce a gauge transformation

$$\widetilde{\varphi}_n = T_n \varphi_n, \tag{3.1}$$

where $\widetilde{\varphi}_n$ satisfies Eqs. (2.1) and (2.11) with U_n and V_n replaced by $\widetilde{U}_n$ and $\widetilde{V}_n$, as

$$\begin{cases} \widetilde{\varphi}_{n+1} = \widetilde{U}_n \widetilde{\varphi}_n, \widetilde{U}_n = T_{n+1} U_n T_n^{-1}, \\ \widetilde{\varphi}_{n,t} = \widetilde{V}_n \widetilde{\varphi}_n, \widetilde{V}_n = (T_{n,t} + T_n V_n) T_n^{-1}. \end{cases} \tag{3.2}$$

Then let T_n be of form

$$T_n = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ 0 & 0 & T_{11} & T_{12} \\ 0 & 0 & T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} \lambda^N + \Sigma_{i=0}^{N-1} a_n^{(i)} \lambda^i & \Sigma_{i=0}^{N-1} b_n^{(i)} \lambda^i & \lambda^N + \Sigma_{i=0}^{N-1} e_n^{(i)} \lambda^i & \Sigma_{i=0}^{N-1} f_n^{(i)} \lambda^i \\ -\Sigma_{i=0}^{N-1} b_{n+1}^{(i)} \lambda^i & \lambda^N + 1 & -\Sigma_{i=0}^{N-1} f_{n+1}^{(i)} \lambda^i & \lambda^N + 1 \\ 0 & 0 & \lambda^N + \Sigma_{i=0}^{N-1} a_n^{(i)} \lambda^i & \Sigma_{i=0}^{N-1} b_n^{(i)} \lambda^i \\ 0 & 0 & -\Sigma_{i=0}^{N-1} b_{n+1}^{(i)} \lambda^i & \lambda^N + 1 \end{pmatrix}, \tag{3.3}$$

where N is a natural number, $T_{i,j}$ are the functions with respect to n, t and all are independent of λ, which can be determined later. These blocks $b_n^{(N-1)}$, $b_{n+1}^{(N-1)}$, $f_n^{(N-1)}$, $f_{n+1}^{(N-1)}$ satisfy the following constraint relationship,

$$s_n + r_n = \frac{1}{4} b_n^{(N-1)} f_{n+1}^{(N-1)} + \frac{1}{4} b_{n+1}^{(N-1)} f_n^{(N-1)}. \tag{3.4}$$

$\lambda_i (\lambda_i \neq \lambda_k, i \neq j, i = 1, 2, ..., 2N)$ are roots of the (4N)th order polynomial $detT_n$, i.e., $detT_n = \prod_{i=1}^{4N} (\lambda - \lambda_i)$ and $detT_n(\lambda_i) = 0$. Then assumed that $\phi_n = (\phi_{1,n}, \phi_{2,n}, \phi_{3,n}, \phi_{4,n})^T$ and $\psi_n = (\psi_{1,n}, \psi_{2,n}, \psi_{3,n}, \psi_{4,n})^T$ are two basic solutions of the Eqs. (2.1) and (2.11), where they are linearly independent. We define that

$$\varphi_n = \begin{pmatrix} \phi_{1,n} & \psi_{1,n} \\ \phi_{2,n} & \psi_{2,n} \\ \phi_{3,n} & \psi_{3,n} \\ \phi_{4,n} & \psi_{4,n} \end{pmatrix}, \tag{3.5}$$

where φ_n is a solution of the Eqs. (2.1) and (2.11). Then we have

$$\widetilde{\varphi}_n = T_n \varphi_n = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ 0 & 0 & T_{11} & T_{12} \\ 0 & 0 & T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \phi_{1,n} & \psi_{1,n} \\ \phi_{2,n} & \psi_{2,n} \\ \phi_{3,n} & \psi_{3,n} \\ \phi_{4,n} & \psi_{4,n} \end{pmatrix} = \begin{pmatrix} T_{11}\phi_{1,n} + T_{12}\phi_{2,n} + T_{13}\phi_{3,n} + T_{14}\phi_{4,n} & T_{11}\psi_{1,n} + T_{12}\psi_{2,n} + T_{13}\psi_{3,n} + T_{14}\psi_{4,n} \\ T_{21}\phi_{1,n} + T_{22}\phi_{2,n} + T_{23}\phi_{3,n} + T_{24}\phi_{4,n} & T_{21}\psi_{1,n} + T_{22}\psi_{2,n} + T_{23}\psi_{3,n} + T_{24}\psi_{4,n} \\ T_{11}\phi_{3,n} + T_{12}\phi_{4,n} & T_{11}\psi_{3,n} + T_{12}\psi_{4,n} \\ T_{21}\phi_{3,n} + T_{22}\phi_{4,n} & T_{21}\psi_{3,n} + T_{22}\psi_{4,n} \end{pmatrix}, \tag{3.6}$$

where the colum vectors of φ_n are linearly independent. Thus we exists constants κ_j so that

$$\kappa_1\widetilde{\phi}_n + \kappa_2\widetilde{\psi}_n = 0, \tag{3.7}$$

i.e.,

$$\begin{cases} \kappa_1(T_{11}\phi_{1,n} + T_{12}\phi_{2,n} + T_{13}\phi_{3,n} + T_{14}\phi_{4,n}) \\ \ + \kappa_2(T_{11}\psi_{1,n} + T_{12}\psi_{2,n} + T_{13}\psi_{3,n} + T_{14}\psi_{4,n}) = 0, \\ \kappa_1(T_{21}\phi_{1,n} + T_{22}\phi_{2,n} + T_{23}\phi_{3,n} + T_{24}\phi_{4,n}) \\ \ + \kappa_2(T_{21}\psi_{1,n} + T_{22}\psi_{2,n} + T_{23}\psi_{3,n} + T_{24}\psi_{4,n}) = 0, \\ \kappa_1(T_{11}\phi_{3,n} + T_{12}\phi_{4,n}) + \kappa_2(T_{11}\psi_{3,n} + T_{12}\psi_{4,n}) = 0, \\ \kappa_1(T_{21}\phi_{3,n} + T_{22}\phi_{4,n}) + \kappa_2(T_{21}\psi_{3,n} + T_{22}\psi_{4,n}) = 0. \end{cases} \tag{3.8}$$

Let

$$\begin{cases} \alpha_j[n] = \dfrac{\phi_{2,n}(t,\lambda_j) - \kappa_j\psi_{2,n}(t,\lambda_j)}{\phi_{1,n}(t,\lambda_j) - \kappa_j\psi_{1,n}(t,\lambda_j)}, \\ \beta_j[n] = \dfrac{\phi_{3,n}(t,\lambda_j) - \kappa_j\psi_{3,n}(t,\lambda_j)}{\phi_{1,n}(t,\lambda_j) - \kappa_j\psi_{1,n}(t,\lambda_j)}, \\ \gamma_j[n] = \dfrac{\phi_{4,n}(t,\lambda_j) - \kappa_j\psi_{4,n}(t,\lambda_j)}{\phi_{1,n}(t,\lambda_j) - \kappa_j\psi_{1,n}(t,\lambda_j)}, \end{cases} \tag{3.9}$$

we can obtain

$$\begin{cases} T_{11} + T_{12}\alpha_j[n] + T_{13}\beta_j[n] + T_{14}\gamma_j[n] = 0, \\ T_{21} + T_{22}\alpha_j[n] + T_{23}\beta_j[n] + T_{24}\gamma_j[n] = 0, \\ T_{11}\beta_j[n] + T_{12}\gamma_j[n] = 0, \\ T_{21}\beta_j[n] + T_{22}\gamma_j[n] = 0. \end{cases} \tag{3.10}$$

Then with the help of Eq. (3.6), we have

$$\begin{cases} \Sigma_{i=0}^{N-1}(a_n^{(i)}\lambda_j^i + \alpha_j[n]b_n^{(i)}\lambda_j^i + \beta_j[n]e_n^{(i)}\lambda_j^i + \gamma_j[n]f_n^{(i)}\lambda_j^i) = -(1+\beta_j[n])\lambda^N, \\ \Sigma_{i=0}^{N-1}(-b_{n+1}^{(i)}\lambda_j^i - \beta_j[n]f_{n+1}^{(i)}\lambda_j^i) = -(\alpha_j[n](\lambda^N+1) + \gamma_j[n](\lambda^N+1)), \\ \Sigma_{i=0}^{N-1}(\beta_j[n]a_n^{(i)}\lambda^j + \gamma_j[n]b_n^{(i)}\lambda^j) = -\beta_j[n]\lambda^N, \\ \Sigma_{i=0}^{N-1}(-\beta_j[n]b_{n+1}^{(i)}\lambda_j^i) = -\gamma_j[n](\lambda^N+1). \end{cases} \tag{3.11}$$

Theorem 3.1. *The matrix* $\widetilde{U}_n = T_{n+1}U_nT_n^{-1}$ *has the same form as matrix* U_n*, the* $\widetilde{U}_n$ *can be written as*

$$\widetilde{U}_n = \begin{pmatrix} 0 & \widetilde{u}_n + \widetilde{h}_n & 0 & \widetilde{v}_n \\ \widetilde{u}_n + \widetilde{h}_n & \lambda & \widetilde{v}_n & \lambda \\ 0 & 0 & 0 & \widetilde{u}_n + \widetilde{h}_n \\ 0 & 0 & \widetilde{u}_n + \widetilde{h}_n & \lambda \end{pmatrix}, \tag{3.12}$$

which the transformation formulae between old and new potentials are given by

$$\begin{cases}\widetilde{u}_n+\widetilde{h}_n=u_n+h_n+b_{n+1}^{(N-1)},\\ \widetilde{v}_n=v_n+f_{n+1}^{(N-1)}.\end{cases}\tag{3.13}$$

Proof. Let $T_n^{-1}=\frac{T_n^*}{detT_n}$ and

$$F(\lambda)=T_{n+1}U_nT_n^*=\begin{pmatrix} f_{11}(\lambda,n) & f_{12}(\lambda,n) & f_{13}(\lambda,n) & f_{14}(\lambda,n)\\ f_{21}(\lambda,n) & f_{22}(\lambda,n) & f_{23}(\lambda,n) & f_{24}(\lambda,n)\\ 0 & 0 & f_{11}(\lambda,n) & f_{12}(\lambda,n)\\ 0 & 0 & f_{21}(\lambda,n) & f_{22}(\lambda,n)\end{pmatrix}.\tag{3.14}$$

It is easy to obtain that $f_{22}(\lambda,n)$ and $f_{24}(\lambda,n)$ are the (4N+1)th order polynomials in λ, $f_{11}(\lambda,n)$, $f_{12}(\lambda,n)$, $f_{13}(\lambda,n)$, $f_{14}(\lambda,n)$, $f_{21}(\lambda,n)$ and $f_{23}(\lambda,n)$ are the (4N)th order polynomials in λ. In addition, by Eqs. (2.1) and (3.6), we get

$$\begin{cases}\alpha_j[n+1]=\dfrac{u_n+h_n+\lambda_j\alpha_j[n]+\beta_j[n]v_n+\lambda_j\gamma_j[n]}{\alpha_j[n](u_n+h_n)+\gamma_j[n]v_n},\\ \beta_j[n+1]=\dfrac{\gamma_j[n](u_n+h_n)}{\alpha_j[n](u_n+h_n)+\gamma_j[n]v_n}, \qquad\qquad j=1,2,...2N,\\ \gamma_j[n+1]=\dfrac{\beta_j[n](u_n+h_n)+\lambda_j\gamma_j[n]}{\alpha_j[n](u_n+h_n)+\gamma_j[n]v_n},\end{cases}\tag{3.15}$$

that $\lambda_j(j=1,2,...,2N)$ are also roots of $f_{st}(s=1,2,t=1,2,3,4)$, and Eq. (3.14) can be written as

$$T_{n+1}U_nT_n^*=detT_n\cdot P_n,\tag{3.16}$$

with

$$P_n=\begin{pmatrix} P_{11}^{(0)}(n) & P_{12}^{(0)}(n) & P_{13}^{(0)}(n) & P_{14}^{(0)}(n)\\ P_{21}^{(0)}(n) & P_{22}^{(1)}(n)\lambda+P_{22}^{(0)}(n) & P_{23}^{(0)}(n) & P_{24}^{(1)}(n)\lambda+P_{24}^{(0)}(n)\\ 0 & 0 & P_{11}^{(0)}(n) & P_{12}^{(0)}(n)\\ 0 & 0 & P_{21}^{(0)}(n) & P_{22}^{(1)}(n)\lambda+P_{22}^{(0)}(n)\end{pmatrix},\tag{3.17}$$

where $P_{lm}^{j}(n)(l=1,2,m=1,2,3,4,j=0,1)$ are functions which are indepent of λ, here we get

$$T_{n+1}U_n=P_nT_n.\tag{3.18}$$

By comparing the coefficients of the same power of λ in both sides of Eq. (3.18), we have

$$\begin{aligned}&P_{11}^{(0)}=0, P_{12}^{(0)}=u_n+h_n+b_{n+1}^{(N-1)}=\widetilde{u}_n+\widetilde{h}_n, P_{13}^{(0)}=0,\\ &P_{14}^{(0)}=v_n+f_{n+1}^{(N-1)}=\widetilde{v}_n, P_{21}^{(0)}=u_n+h_n+b_{n+1}^{(N-1)}=\widetilde{u}_n+\widetilde{h}_n,\\ &P_{22}^{(1)}=1, P_{22}^{(0)}(n)=0, P_{23}^{(0)}=v_n+f_{n+1}^{(N-1)}=\widetilde{v}_n, P_{24}^{(1)}=1, P_{24}^{(0)}(n)=0.\end{aligned}\tag{3.19}$$

□

Theorem 3.2. *The matrix $\widetilde{V}_n=(T_{n,t}+T_nV_n)T_n^{-1}$ has the same form as matrix V_n, the $\widetilde{V}_n$ can be written as*

$$\widetilde{V}_n=\begin{pmatrix} -\frac{1}{2}\lambda-\widetilde{r}_n & \widetilde{u}_{n-1}+\widetilde{h}_{n-1} & -\frac{1}{2}\lambda-\widetilde{s}_n & \widetilde{v}_n \\ \widetilde{u}_n+\widetilde{h}_n & \frac{1}{2}\lambda-\widetilde{r}_n & \widetilde{v}_n & \frac{1}{2}\lambda-\widetilde{s}_n \\ 0 & 0 & -\frac{1}{2}\lambda-\widetilde{r}_n & \widetilde{u}_{n-1}+\widetilde{h}_{n-1} \\ 0 & 0 & \widetilde{u}_n+\widetilde{h}_n & \frac{1}{2}\lambda-\widetilde{r}_n \end{pmatrix}. \tag{3.20}$$

Proof. Let

$$G(\lambda)=(T_{n,t}+T_nV_n)T_n^*=\begin{pmatrix} g_{11}(\lambda,n) & g_{12}(\lambda,n) & g_{13}(\lambda,n) & g_{14}(\lambda,n) \\ g_{21}(\lambda,n) & g_{22}(\lambda,n) & g_{23}(\lambda,n) & g_{24}(\lambda,n) \\ 0 & 0 & g_{11}(\lambda,n) & g_{12}(\lambda,n) \\ 0 & 0 & g_{13}(\lambda,n) & g_{14}(\lambda,n) \end{pmatrix}, \tag{3.21}$$

where $g_{11}(\lambda,n)$, $g_{13}(\lambda,n)$, $g_{22}(\lambda,n)$ and $g_{24}(\lambda,n)$ are the (4N+1)th order polynomials in λ, $g_{12}(\lambda,n)$, $g_{14}(\lambda,n)$, $g_{21}(\lambda,n)$ and $g_{23}(\lambda,n)$ are the (4N)th order polynomials in λ, and where $\lambda_j(j=1,2,...,2N)$ are also roots of $g_{st}(s=1,2,t=1,2,3,4)$, so the Eq. (3.21) can be written as

$$(T_{n,t}+T_nV_n)T_n^*=detT_n\cdot Q_n, \tag{3.22}$$

with

$$Q_n=\begin{pmatrix} Q_{11}^{(1)}(n)\lambda+Q_{11}^{(0)}(n) & Q_{12}^{(0)}(n) & Q_{13}^{(1)}(n)\lambda+Q_{13}^{(0)}(n) & Q_{14}^{(0)}(n) \\ Q_{21}^{(0)}(n) & Q_{22}^{(1)}(n)\lambda+Q_{22}^{(0)}(n) & Q_{23}^{(0)}(n) & Q_{24}^{(1)}(n)\lambda+Q_{24}^{(0)}(n) \\ 0 & 0 & Q_{11}^{(1)}(n)\lambda+Q_{11}^{(0)}(n) & Q_{12}^{(0)}(n) \\ 0 & 0 & Q_{21}^{(0)}(n) & Q_{22}^{(1)}(n)\lambda+Q_{22}^{(0)}(n) \end{pmatrix}, \tag{3.23}$$

where $Q_{lm}^{j}(n)(l=1,2,m=1,2,3,4,j=0,1)$ are functions which are independent of λ, thus we see

$$T_{n,t}+T_nV_n=Q_n\cdot T_n. \tag{3.24}$$

By comparing the coefficients of the same power of λ in both sides of Eq. (3.24), we have

$$\begin{aligned} Q_{11}^{(0)}&=Q_{22}^{(0)}=-(u_n+h_n+b_{n+1}^{(N-1)})(u_{n-1}+h_{n-1}+b_n^{(N-1)})=-\widetilde{r}_n, \\ Q_{13}^{(0)}&=Q_{24}^{(0)}=-(u_{n-1}+h_{n-1}+b_n^{(N-1)})(v_n+f_{n+1}^{(N-1)}) \\ &\qquad -(u_n+h_n+b_{n+1}^{(N-1)})(v_{n-1}+f_n^{(N-1)}) \\ &\qquad +(u_{n-1}+h_{n-1}+b_n^{(N-1)})(u_n+h_n+b_{n+1}^{(N-1)})=-\widetilde{s}_n, \\ Q_{14}^{(0)}&=v_{n-1}+f_n^{(N-1)}=\widetilde{v}_{n-1},\quad Q_{21}^{(0)}=u_n+h_n+b_{n+1}^{(N-1)}=\widetilde{u}_n+\widetilde{h}_n, \\ Q_{23}^{(0)}&=v_n+f_{n+1}^{(N-1)}=\widetilde{v}_n,\quad Q_{12}^{(0)}=u_{n-1}+h_{n-1}+b_n^{(N-1)}=\widetilde{u}_{n-1}+\widetilde{h}_{n-1}, \end{aligned} \tag{3.25}$$

$$Q_{24}^{(1)} = Q_{22}^{(1)} = \frac{1}{2}, Q_{11}^{(1)} = Q_{13}^{(1)} = -\frac{1}{2}.$$

Thus we complete the proof. □

In conclusion, the transformations Eqs. (3.1) and (3.13) are the N-fold Darboux transformation we are looking for.

Now we considered some different values of h_n in the N-fold Darboux transformation to see how it changed.

If we choose $h_n = 0$, the transformation formulae (3.13) become

$$\begin{cases} \widetilde{u}_n = u_n + b_{n+1}^{(N-1)}, \\ \widetilde{v}_n = v_n + f_{n+1}^{(N-1)}. \end{cases} \tag{3.26}$$

In the case $h_n = \varepsilon v_n$, we have the following transformation formulae between old and new potentials

$$\begin{cases} \widetilde{u}_n = u_n + b_{n+1}^{(N-1)} - \varepsilon f_{n+1}^{(N-1)}, \\ \widetilde{v}_n = v_n + f_{n+1}^{(N-1)}. \end{cases} \tag{3.27}$$

In the condition $h_n = \varepsilon u_n v_n$, the transformation formulae between old and new potentials become

$$\begin{cases} \widetilde{u}_n = \dfrac{u_n(1+\varepsilon v_n) + b_{n+1}^{(N-1)}}{1+\varepsilon(v_n + f_{n+1}^{(N-1)})}, \\ \widetilde{v}_n = v_n + f_{n+1}^{(N-1)}. \end{cases} \tag{3.28}$$

Remark 3.1. In this paper, the Darboux transformation we have constructed is a Darboux transformation with some constraint condition. The relationship (3.4) among $b_n^{(N-1)}, b_{n+1}^{(N-1)}, f_n^{(N-1)}$ and $f_{n+1}^{(N-1)}$ is obtained by comparing the coefficients twice in the process above.

4. Explicit solutions

In this section, we apply the explicit solutions of the KdV equations by the N-fold Darboux transformation in Eqs. (3.1) and (3.13). We select three proper values of h_n in Eq. (1.1) as $h_n = 0$, $h_n = V_n$ and $h_n = U_n V n$.

In the condition of the $h_n = 0$, the Lax pair of the Eq. (2.13) are as follows

$$U_n = \begin{pmatrix} 0 & u_n & 0 & v_n \\ u_n & \lambda & v_n & \lambda \\ 0 & 0 & 0 & u_n \\ 0 & 0 & u_n & \lambda \end{pmatrix}, \tag{4.1}$$

$$V_n = \begin{pmatrix} -\frac{1}{2}\lambda - u_n u_{n-1} & u_{n-1} & -\frac{1}{2}\lambda - u_{n-1}v_n - u_n v_{n-1} + u_n u_{n-1} & v_n \\ u_n & \frac{1}{2}\lambda - u_n u_{n-1} & v_n & \frac{1}{2}\lambda - u_{n-1}v_n - u_n v_{n-1} + u_n u_{n-1} \\ 0 & 0 & -\frac{1}{2}\lambda - u_n u_{n-1} & u_{n-1} \\ 0 & 0 & u_n & \frac{1}{2}\lambda - u_n u_{n-1} \end{pmatrix}.$$

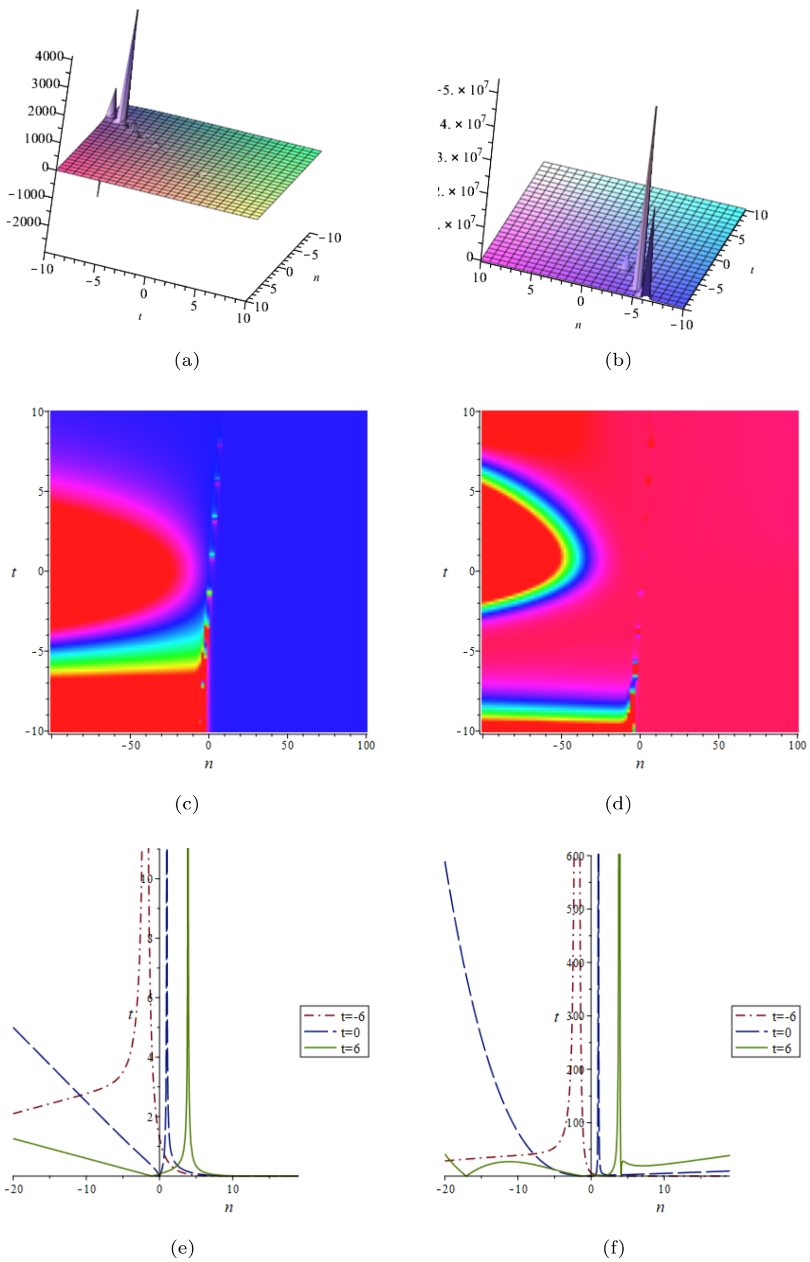

Figure 1. Solutions (51) with parameters $\lambda_1 = 0.5$, $\lambda_2 = 1$, $\kappa_1 = -1$, $\kappa_2 = 1$. Figs. (a)-(c) and (e) the component $\widetilde{u}_n$, Figs. (b), (d) and (f) the component $\widetilde{v}_n$.

Then taking trivial solution $u_n = 0$ and $v_n = 1$ of the Eq. (2.13) and solving the Lax pair, we get

$$\begin{aligned}
\phi_n &= \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \\ \phi_{3,n} \\ \phi_{4,n} \end{pmatrix} = \begin{pmatrix} \lambda^{n-1} e^{\frac{1}{2}\lambda t} \\ (\frac{1}{2}\lambda^{n+1} t + (n-1)\lambda^{n+1}) e^{\frac{1}{2}\lambda t} \\ 0 \\ \lambda^n e^{\frac{1}{2}\lambda t} \end{pmatrix}, \\
\psi_n &= \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \\ \psi_{3,n} \\ \psi_{4,n} \end{pmatrix} = \begin{pmatrix} (-\frac{1}{2}\lambda^{n+1} t + (n-1)\lambda^{n+1}) e^{-\frac{1}{2}\lambda t} \\ \lambda^{n-1} e^{-\frac{1}{2}\lambda t} \\ \lambda^n e^{-\frac{1}{2}\lambda t} \\ 0 \end{pmatrix}.
\end{aligned} \tag{4.2}$$

By Eqs. (3.9) and (3.15), we can get $\alpha_j[n]$, $\beta_j[n]$, $\gamma_j[n]$, $\alpha_j[n+1]$, $\beta_j[n+1]$ and $\gamma_j[n+1]$, and from Eq. (3.11), when $N = 1$, we have

$$\begin{aligned}
b_n^{(0)} &= \frac{(\lambda_2 - \lambda_1)\beta_1[n]\beta_2[n]}{\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n])}, \\
f_n^{(0)} &= \frac{(\lambda_2\beta_2[n]\gamma_1[n] - \lambda_2\beta_1[n]^2\gamma_2[n] + \lambda_2\beta_1[n]\beta_2[n]\gamma_1[n] - \lambda_1\beta_2[n]\gamma_1[n] - \alpha_1[n]\beta_1[n]\beta_2[n](\lambda_2 - \lambda_1))\lambda_2\beta_2[n]}{(\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n]))^2} \\
&\quad - \frac{(\lambda_1\beta_2[n]^2\gamma_1[n] - \lambda_1\beta_2[n]\gamma_2[n] - \lambda_1\beta_1[n]\beta_2[n]\gamma_2[n] + \lambda_2\beta_1[n]\gamma_2[n] - \alpha_2[n]\beta_1[n]\beta_2[n](\lambda_2 - \lambda_1))\lambda_1\beta_1[n]}{(\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n]))^2}.
\end{aligned} \tag{4.3}$$

Then the solutions of the Eq. (2.13) are

$$\begin{cases} \widetilde{u}_n = b_{n+1}^{(0)} = \dfrac{(\lambda_2 - \lambda_1)\beta_1[n+1]\beta_2[n+1]}{\lambda_1\lambda_2(\beta_1[n+1]\gamma_2[n+1] - \beta_2[n+1]\gamma_1[n+1])}, \\ \widetilde{v}_n = 1 + f_{n+1}^{(0)}. \end{cases} \tag{4.4}$$

The solutions (4.4) with parameters $\lambda_1 = 0.5$, $\lambda_2 = 1$, $\kappa_1 = -1$, $\kappa_2 = 1$ are showing in Fig. 1.

When $h_n = v_n$, we have the Lax pair of the Eq. (2.16)

$$\begin{aligned}
U_n &= \begin{pmatrix} 0 & u_n + v_n & 0 & v_n \\ u_n + v_n & \lambda & v_n & \lambda \\ 0 & 0 & 0 & u_n + v_n \\ 0 & 0 & u_n + v_n & \lambda \end{pmatrix}, \\
V_n &= \begin{pmatrix} -\frac{1}{2}\lambda - r_n & u_{n-1} + v_{n-1} & -\frac{1}{2}\lambda - s_n & v_n \\ u_n + v_n & \frac{1}{2}\lambda - r_n & v_n & \frac{1}{2}\lambda - s_n \\ 0 & 0 & -\frac{1}{2}\lambda - r_n & u_{n-1} + v_{n-1} \\ 0 & 0 & u_n + v_n & \frac{1}{2}\lambda - r_n \end{pmatrix},
\end{aligned} \tag{4.5}$$

here $r_n = (u_n + v_n)(u_{n-1} + v_{n-1})$, $s_n = (u_n + v_n)v_{n-1} - (u_{n-1} + v_{n-1})u_n$, then taking trivial solution $u_n = 0$ and $v_n = 0$ and solving the Lax pair, we get

$$\begin{aligned} \phi_n &= \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \\ \phi_{3,n} \\ \phi_{4,n} \end{pmatrix} = \begin{pmatrix} 0 \\ (\frac{1}{2}\lambda^{n+1}t + (n-1)\lambda^{n+1})e^{\frac{1}{2}\lambda t} \\ 0 \\ \lambda^n e^{\frac{1}{2}\lambda t} \end{pmatrix}, \\ \psi_n &= \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \\ \psi_{3,n} \\ \psi_{4,n} \end{pmatrix} = \begin{pmatrix} (-\frac{1}{2}\lambda^{n+1}t + (n-1)\lambda^{n+1})e^{-\frac{1}{2}\lambda t} \\ 0 \\ \lambda^n e^{-\frac{1}{2}\lambda t} \\ 0 \end{pmatrix}. \end{aligned} \tag{4.6}$$

By Eqs. (3.9) and (3.15), we can get $\alpha_j[n]$, $\beta_j[n]$, $\gamma_j[n]$, $\alpha_j[n+1]$, $\beta_j[n+1]$ and $\gamma_j[n+1]$, and from Eq. (3.11), when $N = 1$, we have

$$\begin{aligned} b_n^{(0)} &= \frac{(\lambda_2 - \lambda_1)\beta_1[n]\beta_2[n]}{\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n])}, \\ f_n^{(0)} &= \frac{(\lambda_2\beta_2[n]\gamma_1[n] - \lambda_2\beta_1[n]^2\gamma_2[n] + \lambda_2\beta_1[n]\beta_2[n]\gamma_1[n] - \lambda_1\beta_2[n]\gamma_1[n] - \alpha_1[n]\beta_1[n]\beta_2[n](\lambda_2 - \lambda_1))\lambda_2\beta_2[n]}{(\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n]))^2} \\ &\quad - \frac{(\lambda_1\beta_2[n]^2\gamma_1[n] - \lambda_1\beta_2[n]\gamma_2[n] - \lambda_1\beta_1[n]\beta_2[n]\gamma_2[n] + \lambda_2\beta_1[n]\gamma_2[n] - \alpha_2[n]\beta_1[n]\beta_2[n](\lambda_2 - \lambda_1))\lambda_1\beta_1[n]}{(\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n]))^2}. \end{aligned} \tag{4.7}$$

Then the solutions of the Eq. (2.16) are

$$\begin{cases} \widetilde{u}_n = b_{n+1}^{(0)} - f_{n+1}^{(0)}, \\ \widetilde{v}_n = f_{n+1}^{(0)}. \end{cases} \tag{4.8}$$

The solutions (4.8) with parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = -1$, $\kappa_2 = 1$ are showing in Fig. 2.

Finally, if we take $h_n = u_n v_n$, the Lax pair becomes

$$\begin{aligned} U_n &= \begin{pmatrix} 0 & u_n + u_n v_n & 0 & v_n \\ u_n + u_n v_n & \lambda & v_n & \lambda \\ 0 & 0 & 0 & u_n + u_n v_n \\ 0 & 0 & u_n + u_n v_n & \lambda \end{pmatrix}, \\ V_n &= \begin{pmatrix} -\frac{1}{2}\lambda - r_n & u_{n-1} + u_{n-1}v_{n-1} & -\frac{1}{2}\lambda - s_n & v_n \\ u_n + u_n v_n & \frac{1}{2}\lambda - r_n & v_n & \frac{1}{2}\lambda - s_n \\ 0 & 0 & -\frac{1}{2}\lambda - r_n & u_{n-1} + u_{n-1}v_{n-1} \\ 0 & 0 & u_n + u_n v_n & \frac{1}{2}\lambda - r_n \end{pmatrix}, \end{aligned} \tag{4.9}$$

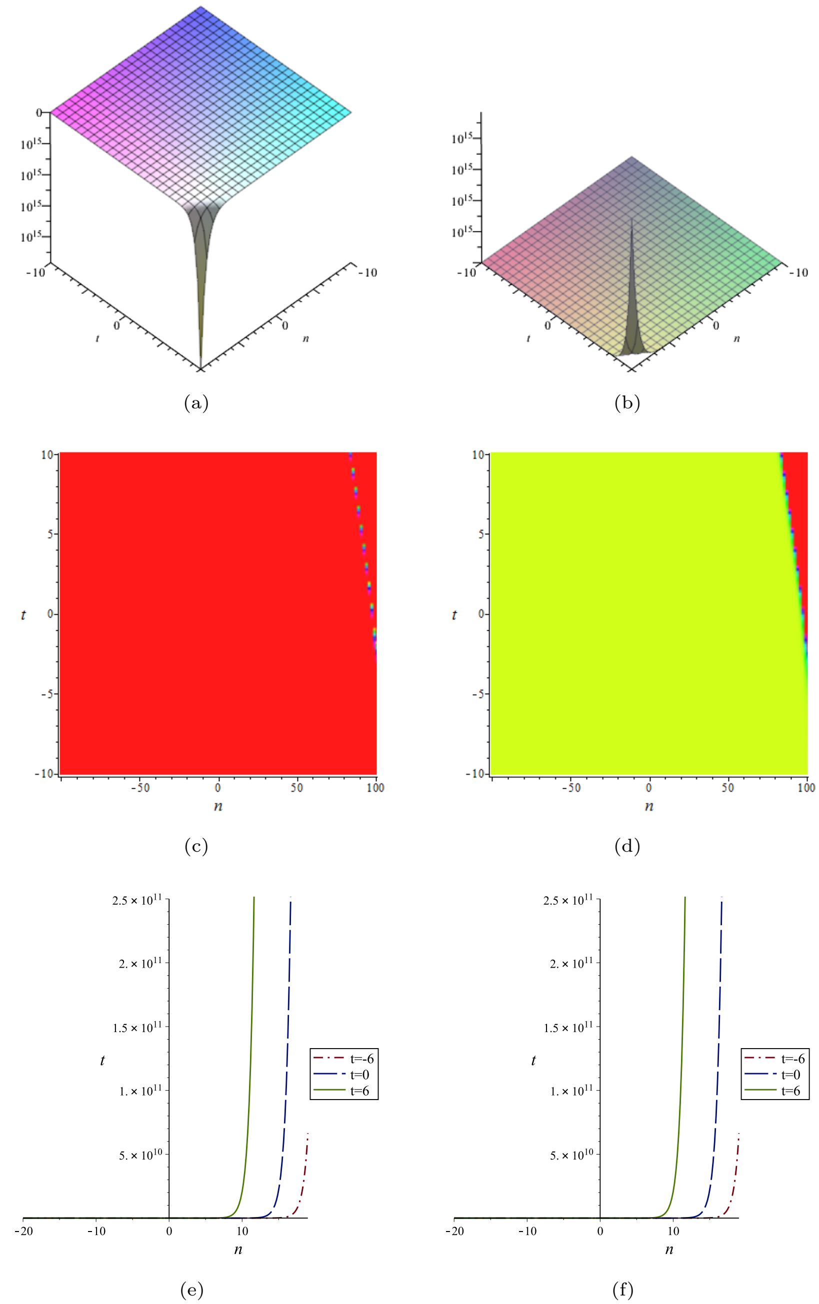

Figure 2. Solutions (55) with parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = -1$, $\kappa_2 = 1$. Figs. (a), (c) and (e) the component $\widetilde{u}_n$, Figs. (b), (d) and (f) the component $\widetilde{v}_n$.

here $r_n = (u_n + u_n v_n)(u_{n-1} + u_{n-1}v_{n-1})$, $s_n = (u_{n-1} + u_{n-1}v_{n-1})v_n + (u_n + u_n v_n)v_{n-1} - (u_{n-1} + u_{n-1}v_{n-1})(u_n + u_n v_n)$.

Then taking trivial solution $u_n = 0$ and $v_n = 0$ and solving the Lax pair, we get

$$
\begin{aligned}
\phi_n &= \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \\ \phi_{3,n} \\ \phi_{4,n} \end{pmatrix} = \begin{pmatrix} 0 \\ (\frac{1}{2}\lambda^{n+1}t + (n-1)\lambda^{n+1})e^{\frac{1}{2}\lambda t} \\ 0 \\ \lambda^n e^{\frac{1}{2}\lambda t} \end{pmatrix}, \\
\psi_n &= \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \\ \psi_{3,n} \\ \psi_{4,n} \end{pmatrix} = \begin{pmatrix} (-\frac{1}{2}\lambda^{n+1}t + (n-1)\lambda^{n+1})e^{-\frac{1}{2}\lambda t} \\ 0 \\ \lambda^n e^{-\frac{1}{2}\lambda t} \\ 0 \end{pmatrix}.
\end{aligned} \tag{4.10}
$$

By Eqs. (3.9) and (3.15), we can get $\alpha_j[n]$, $\beta_j[n]$, $\gamma_j[n]$, $\alpha_j[n+1]$, $\beta_j[n+1]$ and $\gamma_j[n+1]$, and from Eq. (3.11), when $N = 1$, we have

$$
\begin{aligned}
b_n^{(0)} =& \frac{(\lambda_2 - \lambda_1)\beta_1[n]\beta_2[n]}{\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n])}, \\
f_n^{(0)} =& \frac{(\lambda_2\beta_2[n]\gamma_1[n] - \lambda_2\beta_1[n]^2\gamma_2[n] + \lambda_2\beta_1[n]\beta_2[n]\gamma_1[n] - \lambda_1\beta_2[n]\gamma_1[n] - \alpha_1[n]\beta_1[n]\beta_2[n](\lambda_2-\lambda_1))\lambda_2\beta_2[n]}{(\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n]))^2} \\
& - \frac{(\lambda_1\beta_2[n]^2\gamma_1[n] - \lambda_1\beta_2[n]\gamma_2[n] - \lambda_1\beta_1[n]\beta_2[n]\gamma_2[n] + \lambda_2\beta_1[n]\gamma_2[n] - \alpha_2[n]\beta_1[n]\beta_2[n](\lambda_2-\lambda_1))\lambda_1\beta_1[n]}{(\lambda_1\lambda_2(\beta_1[n]\gamma_2[n] - \beta_2[n]\gamma_1[n]))^2}.
\end{aligned} \tag{4.11}
$$

Then the solutions of the Eq. (2.18) are

$$
\begin{cases}
\widetilde{u}_n = \dfrac{b_{n+1}^{(0)}}{1 + f_{n+1}^{(0)}}, \\
\widetilde{v}_n = f_{n+1}^{(0)}.
\end{cases} \tag{4.12}
$$

The solutions (4.12) with parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = -1$, $\kappa_2 = 1$ are showing in Fig. 3.

In order to understand explicit solutions well, we analyze the solutions (4.4) in Fig. 1, the solutions (4.8) in Fig. 2 and illustrate solutions (4.12) in Fig. 3.

When the parameters are suitably chosen, these solutions can be graphically illustrated. So we present the three-dimension graphs and density profiles of the solutions $\widetilde{u}_n$ and $\widetilde{v}_n$. Fig. 1 shows the solutions (4.4) with the parameters $\lambda_1 = 0.5$, $\lambda_2 = 1$, $\gamma_1 = -1$, $\gamma_2 = 1$. Fig. 2 shows the solutions (4.8) with the parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\gamma_1 = -1$, $\gamma_2 = 1$. Fig. 3 shows the solutions (4.12) with the parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\gamma_1 = -1$, $\gamma_2 = 1$. In Figs. 1, 2 and 3 where the first line displays the space-time distributions, the second line displays the density profiles and the third line displays the wave propagations at different time for components $\widetilde{u}_n$ and $\widetilde{v}_n$. From Fig. 1, it can be observed that the solitary waves move from right to left. In Fig. 2, the solitary waves of $\widetilde{u}_n$ and $\widetilde{v}_n$ are very similar. Fig. 2 shows that the solitary wave of $\widetilde{v}_n$ is similar to the solitary wave of $\widetilde{v}_n$ in Fig. 2, but the solitary wave of $\widetilde{u}_n$ is different from the solitary wave of $\widetilde{u}_n$ in Fig. 2.

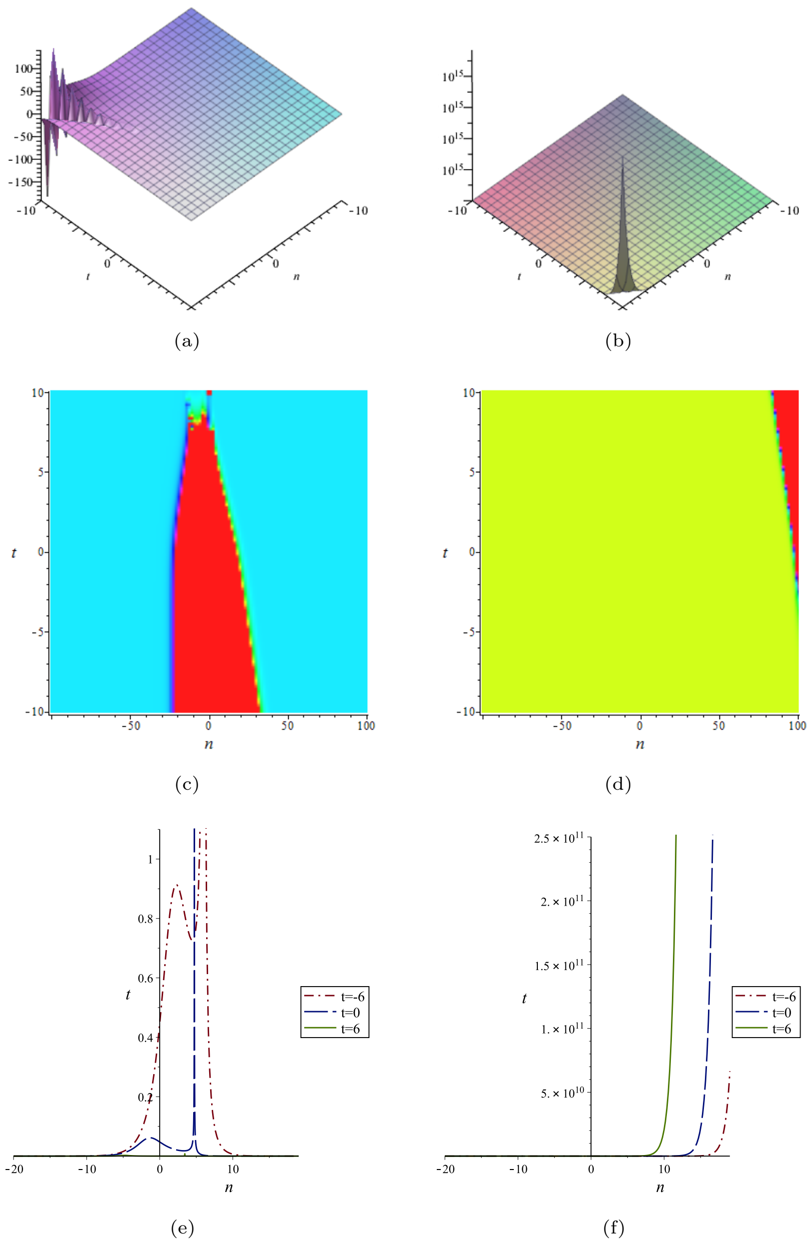

Figure 3. Solutions (59) with parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = -1$, $\kappa_2 = 1$. Figs. (a), (c) and (e) the component $\widetilde{u}_n$, Figs. (b), (d) and (f) the component $\widetilde{v}_n$.

5. Conclusions

In this paper, starting from a 4×4 discrete matrix spectral problem (2.1) with three potential functions, we have successfully constructed a new integrable lattice hierarchy (2.9) and a special N-fold Darboux transformation for the typical Eqs. (2.14), (2.16), (2.18). Explicit solutions have been represented in Figs. 1–3 with proper parameters. According to the integrable hierarchy, a set of integrable discrete equations can be found. In Ref. [30], a general scheme of conservation law based on Lax pair for discrete integrable equations is proposed. By this method, the symmetries and conserved quantities of some equations can be derived.

References

[1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.

[2] E. Cerpa, E. Crepeau and C. Moreno, *On the boundary controllability of the Korteweg-de Vries equation on a star-shaped network*, IMA J. Math. Control I., 2019, 1–15.

[3] E. Fan and H. Zhang, *New exact solutions to a system of coupled KdV equations*, Phys. Lett. A, 1998, 245(5), 389–392.

[4] C. Gu, *Soliton Theory and Its Application*, Zhejiang Science and Technology Press, Zhejiang, 1990.

[5] R. Hirota, *Nonlinear Partial Difference Equations. I. A Difference Analogue of the Korteweg-de Vries Equation*, J. Phys. Soc. Jap., 1977, 43(4), 1424–1433.

[6] J. Ji and Z. Zhu, *On a nonlocal modied Korteweg-de Vries equation: Integrability, Darboux transformation and soliton solutions*, Commun. Nonlinear Sci., 2017, 42, 699–708.

[7] D. J. Korteweg and de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Philos. Mag., 1895, 5(39), 422–443.

[8] V. S. Kumar, H. Rezazadeh, M. Eslami, F. Izadi and M. S. Osman, *Jacobi Elliptic Function Expansion Method for Solving KdV Equation with Conformable Derivative and Dual-Power Law Nonlinearity*, Int. J. Appl. Comput. Math., 2019, 5, 127.

[9] B. A. Kupershmidt, *A super Korteweg-de Vries equation: An integrable system*, Phys. Lett. A, 1984, 102(5–6), 213–215.

[10] X. Li, Q. Zhao and Q. Yang, *Integrable asymmetric AKNS model with multi-component*, Commun. Nonlinear Sci. Numer. Simulat., 2020, 91, 105434.

[11] L. Liu, X. Wen and D. Wang, *A new lattice hierarchy: Hamiltonian structures, symplectic map and N-fold Darboux transformation*, Appl. Math. Model., 2019, 67, 201–218.

[12] L. Liu, X. Wen and D. Wang, *A new lattice hierarchy: Hamiltonian structures, symplectic map and N-fold Darboux transformation*, Appl. Math. Model., 2019, 67, 201–218.

[13] W. Ma, *A generating scheme for conservation laws of discrete zero curvature equations and its application*, Comput. Math. Appl., 2019, 78, 3422–3428.

[14] W. Ma, *N-soliton solutions and the Hirota conditions in (1+1)-dimensions*, Int. J. Nonlin. Sci. Num., 2021.

[15] W. Ma, *N-soliton solutions and the Hirota conditions in (2+1)-dimensions*, Opt. Quant. Electron., 2020, 52(12).

[16] W. Ma, *A Darboux transformation for the Volterra lattice equation*, Anal. Math. Phys., 2019, 9(4), 1711–1718.

[17] W. Ma, *N-soliton solution and the Hirota condition of a (2+1)-dimensional combined equation*, Math. Comput. Simulat., 2021, 190, 270–279.

[18] W. Ma, *N-soliton solution of a combined pKP-BKP equation*, J. Geom. Phys., 2021, 165, 104191.

[19] W. Ma, X. Yong and X. Lv, *Soliton solutions to the B-type Kadomtsev–Petviashvili equation under general dispersion relations*, Wave Motion, 2021, 103, 102719.

[20] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, 1991.

[21] R. M. Miura, *Korteweg-de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation*, J. Math. Phys., 1968, 9(8), 1202–1204.

[22] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1993.

[23] S. Shen, C. Li, Y. Jin and W. Ma, *Completion of the Ablowitz-Kaup-Newell-Segur integrable coupling*, J. Math. Phys., 2018, 59, 103503.

[24] T. Tsuchida and M. Wadati, *The Coupled Modified Korteweg-de Vries Equations*, J. Phys. Soc. Jap., 1998, 67(4), 1175–1187.

[25] H. D. Wahlquist and F. B. Estabrook, *Bäcklund transformation for solutions of the Korteweg-de Vries equation*, Phys. Rev. Lett., 1973, 31(23), 1386–1390.

[26] X. Wen and C. Yuan, *Controllable rogue wave and mixed interaction solutions for the coupled Ablowitz-Ladik equations with branched dispersion*, Appl. Math. Lett., 2022, 123, 107591.

[27] M. Wang, *Exact solutions for a compound KdV-Burgers equation*, Phys. Lett. A, 1996, 213, 279–287.

[28] J. Wu, *A new combined soliton solution of the modified Korteweg-de Vries equation*, Pramana J. Phys., 2020, 94, 123.

[29] X. Xu, *A hierarchy of Liouville integrable discrete Hamiltonian equations*, Phys. Lett. A, 2008, 372, 3683–3693.

[30] C. Yuan and X. Wen, *Integrability, discrete kink multi-soliton solutions on an inclined plane background and dynamics in the modifed exponential Toda lattice equation*, Nonlinear Dyn., 2021, 105, 643–669.

[31] Y. Zhang, W. Ma and O. Unsal, *A novel kind of AKNS integrable couplings and their Hamiltonian structures*, Turk. J. Math., 2017, 41, 1467–1476.