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THE ASYMPTOTIC BEHAVIOR OF STRONG SOLUTIONS TO THE CHEMOTAXIS MODEL IN THE CRITICAL FRAMEWORK*

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Abstract The Keller-Segel model is an effective mathematical model (derived by Keller and Segel), which is used to describe the phenomenon of chemotaxis in biological sciences. The purpose of this paper is to investigate the asymptotic behavior of solutions in the L^p framework by developing the pure energy approach (independent of spectral analysis). Precisely, a new low-frequency regularity of initial data is posted, which enables us to establish the Lyapunov-type inequality in time for energy norms. As a result, the large-time behavior of strong solutions near constant equilibrium can be obtained. The proof crucially depends on non standard product estimates and interpolations. It's worth noting that the smallness requirement of low frequencies is no longer needed.

Keywords Chemotaxis model, critical Besov spaces, decay estimates, pure energy approach.

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1. Introduction and main results

The chemotaxis model with logarithmic sensitivity reads as

$$\begin{cases} \partial_t u = \operatorname{div}\left(D\nabla u - \chi u \nabla \ln v\right), \\ \partial_t v = -\mu u v - \sigma v \end{cases} \tag{1.1}$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d (d \geq 2)$, which derived from the classical Keller-Segel model (see [10–12]) and is one of the models describing the chemotaxis phenomenon in biology. Here $u(t,x)$ and $v(t,x)$ are the density of a cellular population and the concentration of a chemical signal, respectively. The constant $D > 0$ stands for the diffusion coefficient of cellular population. The constant $\chi \neq 0$ is the coefficient of chemotactic sensitivity, where $|\chi|$ measures the strength of chemical signals. The constant $\mu \neq 0$ is the density-dependent production/degradation rate of chemical signal, and $\sigma \geq 0$ is the natural degradation rate of chemical signal.

This paper is devoted to the hyperbolic-parabolic case $\chi\mu > 0$ (see [25, 26] for details). We note that the case includes two scenarios: $\chi > 0$ and $\mu > 0$, or $\chi < 0$

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and $\mu<0$. The former case indicates that cells are attracted and to consume the chemical and the later case describes the movement of a chemotactic population that deposits a chemical signal to modify the local environment for succeeding passages (see [19]). In order to eliminate the singularity caused by $\ln v$, a couple of new variables in terms of the Hopf-Cole type transformation was introduced (see [9, 14, 16, 27]): $\varrho=u,\ w=-\nabla\ln v$. Under the rescaled and dimensionless variables: $\widetilde{t}=\frac{\chi\mu}{D}t,\ \widetilde{x}=\frac{\sqrt{\chi\mu}}{D}x$ and $\widetilde{w}=\mathrm{sign}(\chi)\sqrt{\frac{\chi}{\mu}}w$, system (1.1) becomes

$$\begin{cases}\partial_t\varrho-\Delta\varrho=\mathrm{div}(\varrho w),\\ \partial_t w-\nabla\varrho=0,\end{cases}\tag{1.2}$$

after dropping the tilde accent.

To go directly to the theme of this paper, let us now review some previous results closely related which motivated us to start this study. For other related results, the readers may refer to [8,13,15–17,21,28]and references therein. In fact, observe that (1.2) is obviously invariant for all $\lambda>0$ by the following transformation

$$\varrho(t,x)\rightsquigarrow\lambda^2\varrho(\lambda^2t,\lambda x)\quad\text{and}\quad w(t,x)\rightsquigarrow\lambda w(\lambda^2t,\lambda x).\tag{1.3}$$

Hence, one takes advantage of functional spaces to investigate (1.2), which are endowed with norms enjoying the scaling invariance (1.3). This trick is now classic and has been used by different authors. For example, Hao [9] proved the global existence and uniqueness of strong solutions to (1.2) in the L^2 critical regularity framework by using the compactness arguments. Xu, Li & Wang [24] developed the approach of [6] to obtain the time-decay results of the constructed solutions in [9] in the L^2 critical framework. A natural question is what is to extend the results of [9, 24] to more general L^p critical Besov spaces. To this end, we introduce an appropriate transformation, which was initiated by Deng & Li [7] and then developed by [23]. That is, let $q=\Lambda^{-1}\mathrm{div}\,w$ with $\Lambda^s f\triangleq\mathcal{F}^{-1}(|\xi|^s\mathcal{F}f)$ for $s\in\mathbb{R}$, we get from (1.3) that

$$\begin{cases}\partial_t\varrho-\Delta\varrho=-\mathrm{div}(\varrho\nabla\Lambda^{-1}q),\\ \partial_t q+\Lambda\varrho=0.\end{cases}\tag{1.4}$$

Indeed, it is mention that (1.4) is equivalent to (1.3) because of $\mathrm{curl}\,w=0$ and $w=-\nabla\Lambda^{-1}q$. System (1.4) is supplemented with the initial data

$$(\varrho(0,x),q(0,x))=(\varrho_0(x),q_0(x)),x\in\mathbb{R}^d,\tag{1.5}$$

which are assumed to be close to some constant state $(\bar\varrho,0)$ with $\bar\varrho>0$, at infinity. Recently, Xu & Li [23] considered $\bar\varrho=1$ for simplicity, and established global strong solutions to (1.4)-(1.5) in more general L^p framework. For convenience, we state it as follows.

Theorem 1.1. *Let $d\geq2$ and p fulfill*

$$2\leq p\leq\min(4,2d/(d-2))\ \textit{and, additionally, } p\neq4\ \textit{if}\ d=2.$$

There exists a small positive constant $c=c(p,d)$ and a universal integer $k_0\in\mathbb{Z}$ such that if $a_0^h\triangleq(\varrho_0-1)^h\in\dot B^{d/p-2}_{p,1}$ and $q_0^h\in\dot B^{d/p-1}_{p,1}$ with $(a_0,q_0)^\ell$ in $\dot B^{d/2-2}_{2,1}$

satisfy

$$\mathcal{E}_{p,0} \triangleq \|(a_0, q_0)\|^{\ell}_{\dot{B}^{d/2-2}_{2,1}} + \|a_0\|^{h}_{\dot{B}^{d/p-2}_{p,1}} + \|q_0\|^{h}_{\dot{B}^{d/p-1}_{p,1}} \le c,$$

then Cauchy problem (1.4)-(1.5) *admits a unique global-in-time solutions* (ϱ, q) *with* $\varrho = 1 + a$ *and* (a, q) *in the space* E_p *defined by:*

$$(a,q)^{\ell} \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{d/2-2}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{d/2}_{2,1}), \quad a^h \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{d/p-2}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{d/p}_{p,1}),$$
$$q^h \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{d/p-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{d/p-1}_{p,1}).$$

Furthermore, we have for some constant $C = C\,(p, d)$,

$$\mathcal{E}_p(t) \le C\mathcal{E}_{p,0}$$

for any $t > 0$, *where*

$$\begin{aligned}\mathcal{E}_p(t) \triangleq\ & \|(a,q)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{d/2-2}_{2,1})} + \|(a,q)\|^{\ell}_{L^1_t(\dot{B}^{d/2}_{2,1})} + \|(a,\Lambda q)\|^{h}_{\widetilde{L}^{\infty}_t(\dot{B}^{d/p-2}_{p,1})} \\ & + \|(\Lambda a, q)\|^{h}_{L^1_t(\dot{B}^{d/p-1}_{p,1})}.\end{aligned} \tag{1.6}$$

We would like to mention that above norm notations for tempered distributions will be given in Appendix.

In the present paper, our aim is to develop the method of [22] so as to establish the asymptotic behavior of the constructed solutions in Theorem 1.1. For that purpose, let us rewrite (1.4) as the nonlinear perturbation form of $(1, 0)$, looking at the nonlinearities as source terms. Consequently, in terms of (a, q) with $\varrho = a + 1$, system (1.4) becomes

$$\begin{cases} \partial_t a - \Delta a - \Lambda q = f, \\ \partial_t q + \Lambda a = 0 \end{cases} \tag{1.7}$$

with $f = -\mathrm{div}\,(a\nabla\Lambda^{-1}q)$.

The main results in this paper are stated as follows.

Theorem 1.2. *Let the real number* σ_1 *fulfill*

$$2 - d/2 < \sigma_1 \le \sigma_0 \ \textit{with}\ \sigma_0 \triangleq 2d/p - d/2 + 1. \tag{1.8}$$

If in addition $(a_0, q_0)^{\ell} \in \dot{B}^{-\sigma_1}_{2,\infty}$ *such that* $\|(a_0, q_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}}$ *is bounded, then it holds that*

$$\|\Lambda^{\sigma}(a,q)\|_{L^p} \lesssim (1+t)^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{\sigma_1+\sigma}{2}} \ \textit{if}\ -\widetilde{\sigma}_1 < \sigma \le d/p - 2,$$

for all $t \ge 0$, *where* $\widetilde{\sigma}_1 = \sigma_1 + d(1/2 - 1/p)$.

Furthermore, one has the following optimal time-decay estimates of $\dot{B}^{-\sigma_1}_{2,\infty}$-$L^r$ type.

Corollary 1.1. *Under the additional assumption* (1.8), *the global solution constructed in Theorem 1.1 fulfills*

$$\|\Lambda^m(a,q)\|_{L^r} \lesssim (1+t)^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{\sigma_1+m}{2}}$$

for $p \le r \le \infty$, $m \in \mathbb{R}$ *and* $-\widetilde{\sigma}_1 < m + d(1/p - 1/r) \le d/p - 2$.

Remark 1.1. Compared to [23,24], the innovative ingredient is that the smallness of low frequencies of initial data is no longer needed in Theorem 1.2 and Corollary 1.1. In addition, the proof of Corollary 1.1 follows from the embedding, (2.34) and (2.36) directly, and is omitted for brevity.

The rest of the paper unfolds as follows. Section 2 is devoted to the proof of Theorem 1.2. In the last section (Appendix), we briefly recall Littlewood-Paley decomposition, Besov spaces and related analysis tools for the reader's convenience.

Notation: Throughout the paper, $C > 0$ stands for a generic "constant". For brevity, we write $f \lesssim g$ instead of $f \leq Cg$. The notation $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$. For any Banach space X and $f, g \in X$, we agree that $\|(f,g)\|_X \triangleq \|f\|_X + \|g\|_X$. For all $T > 0$ and $\theta \in [1, +\infty]$, we denote by $L^\theta_T(X) \triangleq L^\theta([0,T]; X)$ the set of measurable functions $f : [0,T] \to X$ such that $t \mapsto \|f(t)\|_X$ are in $L^\theta(0,T)$.

2. The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2 taking for granted the global existence result in Theorem 1.1. Throughout the proof, our task is to establish a Lyapunov-type inequality in time for energy norms by using a pure energy argument. Based on which we further obtain the time-decay estimates of strong solutions. For clarity, we divide into several steps.

2.1. Low-frequency and high-frequency analysis

In this section, we first give the low-frequency and high-frequency estimates, which plays the key role in deriving the Lyapunov-type inequality for energy norms.

2.1.1. Low-frequency estimates

Proposition 2.1. *let k_0 be some integer. Then it holds that for all $t \geq 0$,*

$$\frac{d}{dt}\|(a,q)^\ell\|_{\dot{B}^{d/2-2}_{2,1}} + \|(a,q)^\ell\|_{\dot{B}^{d/2}_{2,1}} \lesssim \|f^\ell\|_{\dot{B}^{d/2-2}_{2,1}}. \tag{2.1}$$

Proof. Applying the operator $\dot{\Delta}_k \dot{S}_{k_0}$ to (1.7) yields

$$\begin{cases} \partial_t a^\ell_k - \Delta a^\ell_k - \Lambda q^\ell_k = f^\ell_k, \\ \partial_t q^\ell_k + \Lambda a^\ell_k = 0, \end{cases}$$

where $z^\ell_k \triangleq \dot{\Delta}_k \dot{S}_{k_0} z$. Taking advantage of the standard energy approach, we have

$$\frac{1}{2}\frac{d}{dt}\big(\|a^\ell_k\|^2_{L^2} + \|q^\ell_k\|^2_{L^2}\big) + \|\Lambda a^\ell_k\|^2_{L^2} = (f^\ell_k | a^\ell_k), \tag{2.2}$$

$$-\frac{d}{dt}(a^\ell_k | \Lambda q^\ell_k) + \|\Lambda q^\ell_k\|^2_{L^2} = -(\Delta a^\ell_k | \Lambda q^\ell_k) + \|\Lambda a^\ell_k\|^2_{L^2} - (f^\ell_k | \Lambda q^\ell_k), \tag{2.3}$$

$$\frac{1}{2}\frac{d}{dt}\|\Lambda q^\ell_k\|^2_{L^2} = (\Delta a^\ell_k | \Lambda q^\ell_k). \tag{2.4}$$

It follows from (2.2)-(2.4) that

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_k^2+\|(\Lambda a_k^\ell,\Lambda q_k^\ell)\|_{L^2}^2=2(f_k^\ell|a_k^\ell)-(f_k^\ell|q_k^\ell)$$

with $\mathcal{L}_k^2\triangleq 2\big(\|a_k^\ell\|_{L^2}^2+\|q_k^\ell\|_{L^2}^2\big)+\|\Lambda q_k^\ell\|_{L^2}^2-2(a_k^\ell|\Lambda q_k^\ell)$. Due to the low-frequency cut-off, we get from Hölder and Young inequalities that $\mathcal{L}_k^2\approx\|(a_k^\ell,q_k^\ell,\Lambda q_k^\ell)\|_{L^2}^2\approx\|(a_k^\ell,q_k^\ell)\|_{L^2}^2$. Consequently, we get

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_k^2+2^{2k}\mathcal{L}_k^2\lesssim\|f_k^\ell\|_{L^2}\mathcal{L}_k,$$

which leads to

$$\frac{d}{dt}\|(a_k^\ell,q_k^\ell)\|_{L^2}+2^{2k}\|(a_k^\ell,q_k^\ell)\|_{L^2}\lesssim\|f_k^\ell\|_{L^2}.$$

Then, multiplying both sides by $2^{k(d/2-2)}$ and summing up on $k\in\mathbb{Z}$ give (2.1). This completes the proof of Proposition 2.1. □

2.1.2. High-frequency estimates

Let us utilize the L^p energy argument in terms of the useful auxiliary function:

$$b=a-\Lambda^{-1}q. \tag{2.5}$$

Indeed, if (1.7) is written in light of b and q, then, up to low order terms, b fulfills a heat equation and q satisfies a damped transport equation. According to this structure of the system, we can establish the high-frequency estimates.

Proposition 2.2. *Let k_0 be chosen suitably large. It holds that for all $t\geq 0$*

$$\frac{d}{dt}\|(a,\Lambda q)\|^h_{\dot{B}^{d/p-2}_{p,1}}+\big(\|a\|^h_{\dot{B}^{d/p}_{p,1}}+\|\Lambda q\|^h_{\dot{B}^{d/p-2}_{p,1}}\big)\lesssim\|f\|^h_{\dot{B}^{d/p-2}_{p,1}}. \tag{2.6}$$

Proof. From (1.7) and (2.5), we observe that (b,q) satisfies

$$\begin{cases}\partial_t b-\Delta b=f+b+\Lambda^{-1}q,\\ \partial_t\Lambda q+\Lambda q=-\Lambda^2 b.\end{cases} \tag{2.7}$$

Applying the operator $\dot{\Delta}_k$ to the first equation of (2.7) gives for all $k\in\mathbb{Z}$,

$$\partial_t b_k-\Delta b_k=f_k+b_k+\Lambda^{-1}q_k\quad\text{with } b_k\triangleq\dot{\Delta}_k b,\ q_k\triangleq\dot{\Delta}_k q\text{ and }\ f_k\triangleq\dot{\Delta}_k f. \tag{2.8}$$

Then, multiplying each component of (2.8) by $|b_k|^{p-2}b_k$ and integrating over $\mathbb{R}^d$ yields

$$\frac{1}{p}\frac{d}{dt}\|b_k\|_{L^p}^p-\int_{\mathbb{R}^d}\Delta b_k|b_k|^{p-2}b_k dx=\int_{\mathbb{R}^d}\big(f_k+b_k+\Lambda^{-1}q_k\big)|b_k|^{p-2}b_k dx.$$

The key observation is that the second term of the l.h.s., although not spectrally localized, may be bounded from below as if it were (see Lemma 3.1). Consequently, we conclude that there exists some constant c_p depending only p so that

$$\frac{d}{dt}\|b_k\|_{L^p}+c_p2^{2k}\|b_k\|_{L^p}\leq\|f_k\|_{L^p}+\|b_k\|_{L^p}+\|\Lambda^{-1}q_k\|_{L^p}. \tag{2.9}$$

Applying the operator $\dot{\Delta}_k$ to the second equation of (2.7) gives for all $k \in \mathbb{Z}$,

$$\partial_t \Lambda q_k + \Lambda q_k = -\Lambda^2 b_k.$$

Multiplying by $|\Lambda q_k|^{p-2}\Lambda q_k$ and integrating on $\mathbb{R}^d$, we arrive at

$$\frac{d}{dt}\|\Lambda q_k\|_{L^p} + \|\Lambda q_k\|_{L^p} \le C2^{2k}\|b_k\|_{L^p}. \tag{2.10}$$

Furthermore, adding up (2.10) (multiplying by γc_p for some $\gamma > 0$) to (2.9) yields

$$\begin{aligned}&\frac{d}{dt}\big(\|b_k\|_{L^p} + \gamma c_p\|\Lambda q_k\|_{L^p}\big) + c_p 2^{2k}\|b_k\|_{L^p} + \gamma c_p\|\Lambda q_k\|_{L^p}\\ &\le \|(f_k, b_k, \Lambda^{-1}\, q_k)\|_{L^p} + C\gamma c_p 2^{2k}\|b_k\|_{L^p}.\end{aligned}$$

Noticing that Λ^{-1} is a homogeneous Fourier multiplier of degree -1, we get

$$\|\Lambda^{-1}\, q_k\|_{L^p} \lesssim 2^{-2k}\|\Lambda\, q_k\|_{L^p} \lesssim 2^{-2k_0}\|\Lambda q_k\|_{L^p} \quad \text{for all} \quad k \ge k_0 - 1.$$

Choosing k_0 suitably large and γ small enough, we conclude that there exists a constant $c_0 > 0$ such that for all $k \ge k_0 - 1$,

$$\frac{d}{dt}\big(\|b_k\|_{L^p} + \|\Lambda q_k\|_{L^p}\big) + c_0\big(2^{2k}\|b_k\|_{L^p} + \|\Lambda q_k\|_{L^p}\big) \lesssim \|f_k\|_{L^p}.$$

Remembering that $a = b - \Lambda^{-1}\, q$, it follows that

$$\frac{d}{dt}\|(a_k, \Lambda q_k)\|_{L^p} + c_0\|(2^{2k}a_k, \Lambda q_k)\|_{L^p} \lesssim \|f_k\|_{L^p} \quad \text{for all} \quad k \ge k_0 - 1.$$

Hence, multiplying both sides by $2^{k(d/p-2)}$, and summing up over $k \ge k_0 - 1$, we get (2.6). This completes the proof of Proposition 2.2. $\square$

2.1.3. Nonlinear estimates

From Propositions 2.1 and 2.2, we have

$$\begin{aligned}&\frac{d}{dt}\big(\|(a,q)^\ell\|_{\dot{B}^{d/2-2}_{2,1}} + \|(a,\Lambda q)\|^h_{\dot{B}^{d/p-2}_{p,1}}\big) + \|(a,q)^\ell\|_{\dot{B}^{d/2}_{2,1}} + \|(\Lambda a, q)\|^h_{\dot{B}^{d/p-1}_{p,1}}\\ &\lesssim \|f\|^\ell_{\dot{B}^{d/2-2}_{2,1}} + \|f\|^h_{\dot{B}^{d/p-2}_{p,1}}.\end{aligned} \tag{2.11}$$

According to Lemma 3.5 and (3.6), it follows that

$$\begin{aligned}\|f\|^h_{\dot{B}^{d/p-2}_{p,1}} &\lesssim \|a\nabla\Lambda^{-1}q\|^h_{\dot{B}^{d/p-1}_{p,1}} \lesssim \|\nabla\Lambda^{-1}q\|_{\dot{B}^{d/p-1}_{p,1}}\|a\|_{\dot{B}^{d/p}_{p,1}}\\ &\lesssim \|q\|_{\dot{B}^{d/p-1}_{p,1}}\|a\|_{\dot{B}^{d/p}_{p,1}} \lesssim \mathcal{E}_p(t)\big(\|a^\ell\|_{\dot{B}^{d/2}_{2,1}} + \|a\|^h_{\dot{B}^{d/p}_{p,1}}\big).\end{aligned} \tag{2.12}$$

To estimate the norm $\|f\|^\ell_{\dot{B}^{d/2-2}_{2,1}}$, we need the following so-called Bony decomposition for the product of two-tempered distributions f and g:

$$fg = T_f g + R(f,g) + T_g f \tag{2.13}$$

with

$$T_f g \triangleq \sum_j \dot{S}_{j-1} f \dot{\Delta}_j g \quad \text{and} \quad R(f,g) \triangleq \sum_j \sum_{|j'-j|\le 1} \dot{\Delta}_j f \dot{\Delta}_{j'} g,$$

where the operator T is called "paraproduct" whereas R is called "remainder". The decomposition (2.13) naturally leads to the following three inequalities (see [1,5]):

$$\|T_f g\|_{\dot{B}^{s-1+d/2-d/p}_{2,1}} \lesssim \|f\|_{\dot{B}^{d/p-1}_{p,1}} \|g\|_{\dot{B}^{s}_{p,1}} \tag{2.14}$$

if $d \geq 2$ and $1 \leq p \leq \min(4, d^*)$,

$$\|T_f g\|_{\dot{B}^{s}_{p,1}} \lesssim \|f\|_{L^{p_1}} \|g\|_{\dot{B}^{s}_{p_2,1}} \tag{2.15}$$

if $s \in \mathbb{R}$ and $1 \leq p, p_1, p_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$,

$$\|R(f,g)\|_{\dot{B}^{s-1+d/2-d/p}_{2,1}} \lesssim \|f\|_{\dot{B}^{d/p-1}_{p,1}} \|g\|_{\dot{B}^{s}_{p,1}} \tag{2.16}$$

if $s > 1 - \min(d/p, d/p')$ and $1 \leq p \leq 4$, where $d^* = 2d/(d-2)$ and $1/p + 1/p' = 1$. It follows from (2.14) that

$$\|T_f g\|^{\ell}_{\dot{B}^{d/2-1}_{2,1}} \lesssim \|T_f g\|^{\ell}_{\dot{B}^{d/2-1-m}_{2,1}} \lesssim \|T_f g\|_{\dot{B}^{d/2-1-m}_{2,1}} \lesssim \|f\|_{\dot{B}^{d/p-1}_{p,1}} \|g\|_{\dot{B}^{d/p-m}_{p,1}} \tag{2.17}$$

with $m \geq 0$ and $1 \leq p \leq \min(4, d^*)$. Taking advantage of Bernstein inequality gives

$$\|f\|^{\ell}_{\dot{B}^{d/2-2}_{2,1}} = \|\mathrm{div}(a\nabla\Lambda^{-1}q)\|^{\ell}_{\dot{B}^{d/2-2}_{2,1}} \lesssim \|a\nabla\Lambda^{-1}q\|^{\ell}_{\dot{B}^{d/2-1}_{2,1}}.$$

For the term with $a\nabla\Lambda^{-1}q$, using Bony's paraproduct decomposition, one has that

$$a\nabla\Lambda^{-1}q = T_{\nabla\Lambda^{-1}q} a + R(\nabla\Lambda^{-1}q, a) + T_a \nabla\Lambda^{-1}q^{\ell} + T_a \nabla\Lambda^{-1}q^{h}.$$

With the aid of (2.14) and (2.16) with $s = d/p$, we arrive at

$$\begin{aligned}
&\|T_{\nabla\Lambda^{-1}q} a\|_{\dot{B}^{d/2-1}_{2,1}} + \|R(\nabla\Lambda^{-1}q, a)\|_{\dot{B}^{d/2-1}_{2,1}} \\
&\lesssim \|\nabla\Lambda^{-1}q\|_{\dot{B}^{d/p-1}_{p,1}} \|a\|_{\dot{B}^{d/p}_{p,1}} \lesssim \|q\|_{\dot{B}^{d/p-1}_{p,1}} \|a\|_{\dot{B}^{d/p}_{p,1}} \\
&\lesssim \mathcal{E}_p(t)\big(\|a^{\ell}\|_{\dot{B}^{d/2}_{2,1}} + \|a\|^{h}_{\dot{B}^{d/p}_{p,1}}\big),
\end{aligned}$$

where $\mathcal{E}_p(t)$ has been defined in (1.6). To bound the term with $T_a\nabla\Lambda^{-1}q^{\ell}$, we note that, owing to (2.15) and the embedding $\dot{B}^{d/p}_{p,1} \hookrightarrow L^{\infty}$, we have

$$\begin{aligned}
\|T_a \nabla\Lambda^{-1}q^{\ell}\|_{\dot{B}^{d/2-1}_{2,1}} &\lesssim \|a\|_{L^{\infty}} \|\nabla\Lambda^{-1}q^{\ell}\|_{\dot{B}^{d/2-1}_{2,1}} \lesssim \|a\|_{\dot{B}^{d/p}_{p,1}} \|q^{\ell}\|_{\dot{B}^{d/2-1}_{2,1}} \\
&\lesssim \mathcal{E}_p(t)\big(\|a^{\ell}\|_{\dot{B}^{d/2}_{2,1}} + \|a\|^{h}_{\dot{B}^{d/p}_{p,1}}\big).
\end{aligned} \tag{2.18}$$

For the term with $T_a\nabla\Lambda^{-1}q^{h}$, we note that, owing to (2.17) with $m = 1$,

$$\begin{aligned}
\|T_a \nabla\Lambda^{-1}q^{h}\|^{\ell}_{\dot{B}^{d/2-1}_{2,1}} &\lesssim \|T_a \nabla\Lambda^{-1}q^{h}\|^{\ell}_{\dot{B}^{d/2-2}_{2,1}} \lesssim \|a\|_{\dot{B}^{d/p-1}_{p,1}} \|\nabla\Lambda^{-1}q^{h}\|_{\dot{B}^{d/p-1}_{p,1}} \\
&\lesssim \|a\|_{\dot{B}^{d/p-1}_{p,1}} \|q\|^{h}_{\dot{B}^{d/p-1}_{p,1}} \lesssim \mathcal{E}_p(t) \|(\Lambda a, q)\|^{h}_{\dot{B}^{d/p-1}_{p,1}}.
\end{aligned}$$

Hence, combing the above estimates gives

$$\|f\|^{\ell}_{\dot{B}^{d/2-2}_{2,1}} \lesssim \mathcal{E}_p(t)\big(\|a^{\ell}\|_{\dot{B}^{d/2}_{2,1}} + \|(\Lambda a, q)\|^{h}_{\dot{B}^{d/p-1}_{p,1}}\big). \tag{2.19}$$

Inserting both (2.12) and (2.19) into (2.11) and performing the fact that $\mathcal{E}_p(t) \lesssim \mathcal{E}_{p,0} \ll 1$ for all $t \geq 0$ guaranteed by Theorem 1.1, we end up with

$$\frac{d}{dt}\big(\|(a,q)^{\ell}\|_{\dot{B}^{d/2-2}_{2,1}} + \|(a, \Lambda q)\|^{h}_{\dot{B}^{d/p-2}_{p,1}}\big) + \|(a,q)^{\ell}\|_{\dot{B}^{d/2}_{2,1}} + \|(\Lambda a, q)\|^{h}_{\dot{B}^{d/p-1}_{p,1}} \leq 0. \tag{2.20}$$

2.2. The evolution of negative Besov norm

In this section, we give the evolution of Besov norms at low frequencies, which is the main ingredient in the proof of Theorem 1.2.

Proposition 2.3. *If $2-\frac{d}{2}<\sigma_1\leq\sigma_0$, then it holds that for all $t\geq 0$,*

$$\|(a,q)(t)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}\leq C_0, \tag{2.21}$$

where $C_0>0$ depends on the norm $\|(a_0,q_0)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}$.

Proof. We apply the operator $\dot\Delta_k$ to (1.7). Following the procedure leading to (2.2), we get

$$\frac12\frac{d}{dt}\big(\|a_k\|^2_{L^2}+\|q_k\|^2_{L^2}\big)+\|\Lambda a_k\|^2_{L^2}\leq\|f_k\|_{L^2}\|a_k\|_{L^2}.$$

By performing a routine procedure, we deduce that

$$\|(a,q)(t)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}\lesssim\|(a_0,q_0)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}+\int_0^t\|f\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}d\tau\quad\text{with}\quad 2-\frac{d}{2}<\sigma_1\leq\sigma_0. \tag{2.22}$$

In what follows, let us bound the norm $\|f\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}$. To this end, we write that

$$f=-\mathrm{div}\,(a\nabla\Lambda^{-1}q)=-\mathrm{div}\,(a\nabla\Lambda^{-1}q^{\ell})-\mathrm{div}\,(a\nabla\Lambda^{-1}q^{h}).$$

To bound the term with $\mathrm{div}\,(a\nabla\Lambda^{-1}q^{\ell})$, we shall take advantage of the following inequality whose proof has been shown by [22].

$$\|FG\|_{\dot B^{-s_1}_{2,\infty}}\lesssim\|F\|_{\dot B^{d/p}_{p,1}}\|G\|_{\dot B^{-s_1}_{2,\infty}}\quad\text{for all}\quad 1-d/2<s_1\leq s_0\triangleq 2d/p-d/2. \tag{2.23}$$

Thanks to $2-d/2<\sigma_1\leq\sigma_0=2d/p-d/2+1$, we have $1-d/2<\sigma_1-1\leq\sigma_0-1=s_0$. Then it follows from Bernstein inequality and (2.23) that

$$\begin{aligned}\|\mathrm{div}\,(a\nabla\Lambda^{-1}q^{\ell})\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}&\lesssim\|a\nabla\Lambda^{-1}q^{\ell}\|^{\ell}_{\dot B^{-\sigma_1+1}_{2,\infty}}\lesssim\|a\|_{\dot B^{d/p}_{p,1}}\|q^{\ell}\|_{\dot B^{-\sigma_1+1}_{2,\infty}}\\&\lesssim\|a\|_{\dot B^{d/p}_{p,1}}\|q\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}},\end{aligned} \tag{2.24}$$

where $\nabla\Lambda^{-1}$ is an homogeneous Fourier multiplier of degree 0. For the term with $\mathrm{div}\,(a\nabla\Lambda^{-1}q^{h})$, we shall use the following two inequalities (see [20,22]):

$$\|FG^h\|^{\ell}_{\dot B^{-s_1}_{2,\infty}}\lesssim\|FG^h\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\lesssim\|F\|_{\dot B^{d/p-1}_{p,1}}\|G^h\|_{\dot B^{d/p-1}_{p,1}}\quad\text{if}\quad 2\leq p\leq d, \tag{2.25}$$

$$\|FG^h\|^{\ell}_{\dot B^{-s_1}_{2,\infty}}\lesssim\|FG^h\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\lesssim\big(\|F^{\ell}\|_{\dot B^{d/2-1}_{2,1}}+\|F^h\|_{\dot B^{d/p}_{p,1}}\big)\|G^h\|_{\dot B^{d/p-1}_{p,1}}\ \text{if}\ p>d, \tag{2.26}$$

where $1-d/2<s_1\leq s_0$. Noticing that $1-d/2<\sigma_1-1\leq\sigma_0-1=s_0$, we conclude that, due to (2.25) and embedding $\dot B^{d/2-1}_{2,1}\hookrightarrow\dot B^{d/p-1}_{p,1}$,

$$\begin{aligned}\|\mathrm{div}\,(a\nabla\Lambda^{-1}q^{h})\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}&\lesssim\|a\nabla\Lambda^{-1}q^{h}\|^{\ell}_{\dot B^{-\sigma_1+1}_{2,\infty}}\\&\lesssim\big(\|a\|^{\ell}_{\dot B^{d/2-2}_{2,1}}+\|a\|^{h}_{\dot B^{d/p}_{p,1}}\big)\|q\|^{h}_{\dot B^{d/p-1}_{p,1}},\end{aligned} \tag{2.27}$$

for all $2 \le p \le d$, and also that, owing to (2.26) and Bernstein inequality,

$$\|\mathrm{div}\,(a\nabla\Lambda^{-1}q^h)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}} \lesssim \big\|a\nabla\Lambda^{-1}q^h\big\|^{\ell}_{\dot B^{-\sigma_1+1}_{2,\infty}}$$
$$\lesssim \big(\|a\|^{\ell}_{\dot B^{\frac{d}{2}-2}_{2,1}} + \|a\|^{h}_{\dot B^{d/p}_{p,1}}\big)\|q\|^{h}_{\dot B^{d/p-1}_{p,1}} \text{ for all } p > d. \quad (2.28)$$

By inserting (2.24), (2.27) and (2.28) into (2.22) gives

$$\|(a,q)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}} \lesssim \|(a_0,q_0)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}} + \int_0^t \|a\|_{\dot B^{d/p}_{p,1}}\|(a,q)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}d\tau$$
$$+ \int_0^t \big(\|a\|^{\ell}_{\dot B^{d/2-2}_{2,1}} + \|a\|^{h}_{\dot B^{d/p}_{p,1}}\big)\|q\|^{h}_{\dot B^{d/p-1}_{p,1}}d\tau.$$

The global existence result (see Theorem 1.1) ensures that $\mathcal{E}_p(t) \lesssim \mathcal{E}_{p,0} \ll 1$, we have

$$\int_0^t \|a\|_{\dot B^{d/p}_{p,1}}d\tau \le C\mathcal{E}_{p,0} \text{ and } \int_0^t \big(\|a\|^{\ell}_{\dot B^{d/2-2}_{2,1}} + \|a\|^{h}_{\dot B^{d/p}_{p,1}}\big)\|q\|^{h}_{\dot B^{d/p-1}_{p,1}}d\tau \le C\mathcal{E}_{p,0}. \quad (2.29)$$

Finally, combining (2.29), one can make use of nonlinear generalisations of the Gronwall's inequality (see for example, Page 360 of [18]) and get (2.21). This completes the proof of Proposition 2.3. □

2.3. Lyapunov-type inequality for energy norms and decay estimates

In this section, we establish the Lyapunov-type inequality in time for energy norms, which leads to the time-decay estimates. Thanks to $-\sigma_1 < d/2-2 < d/2$, we get from Lemma 3.3 that

$$\|(a,q)^{\ell}\|_{\dot B^{d/2-2}_{2,1}} \lesssim \big(\|(a,q)\|^{\ell}_{\dot B^{-\sigma_1}_{2,\infty}}\big)^{\theta_0}\big(\|(a,q)\|^{\ell}_{\dot B^{d/2}_{2,\infty}}\big)^{1-\theta_0} \text{ with } \theta_0 = \frac{2}{d/2+\sigma_1} \in (0,1).$$

According to (2.21), we infer that

$$\|(a,q)\|^{\ell}_{\dot B^{d/2}_{2,1}} \ge c_0\big(\|(a,q)^{\ell}\|_{\dot B^{d/2-2}_{2,1}}\big)^{\frac{1}{1-\theta_0}} \quad \text{with} \quad c_0 = C^{-\frac{1}{1-\theta_0}}C_0^{-\frac{\theta_0}{1-\theta_0}}. \quad (2.30)$$

In addition, it follows from the fact $\|(a,\Lambda q)\|^{h}_{\dot B^{d/p-2}_{p,1}} \lesssim \mathcal{E}_{p,0} \ll 1$ for all $t \ge 0$ that

$$\|(\Lambda a,q)\|^{h}_{\dot B^{d/p-1}_{p,1}} \gtrsim \big(\|(a,\Lambda q)\|^{h}_{\dot B^{d/p-2}_{p,1}}\big)^{\frac{1}{1-\theta_0}}. \quad (2.31)$$

Putting both (2.30) and (2.31) into (2.20), we conclude that there exists a constant $\widetilde{c}_0 > 0$ such that the following Lyapunov-type inequality in time holds.

$$\frac{d}{dt}\big(\|(a,q)^{\ell}\|_{\dot B^{d/2-2}_{2,1}} + \|(a,\Lambda q)\|^{h}_{\dot B^{d/p-2}_{p,1}}\big)$$
$$+ \widetilde{c}_0\big(\|(a,q)^{\ell}\|_{\dot B^{d/2-2}_{2,1}} + \|(a,\Lambda q)\|^{h}_{\dot B^{d/p-2}_{p,1}}\big)^{1+\frac{2}{d/2-2+\sigma_1}} \le 0. \quad (2.32)$$

Solving (2.32) directly yields

$$\|(a,q)^\ell\|_{\dot B^{d/2-2}_{2,1}}+\|(a,\Lambda q)\|^h_{\dot B^{d/p-2}_{p,1}}\le\Big(\mathcal{E}_{p,0}^{-\frac{2}{d/2-2+\sigma_1}}+\frac{2\widetilde{c}_0t}{d/2-2+\sigma_1}\Big)^{-\frac{d/2-2+\sigma_1}{2}}$$
$$\lesssim(1+t)^{-\frac{d/2-2+\sigma_1}{2}}\quad\text{for all}\quad t\ge0. \tag{2.33}$$

It follows from Lemma 3.4 and (2.33) that

$$\|(a,q)\|_{\dot B^{d/p-2}_{p,1}}\lesssim\|(a,q)^\ell\|_{\dot B^{d/2-2}_{2,1}}+\|(a,\Lambda q)\|^h_{\dot B^{d/p-2}_{p,1}}\lesssim(1+t)^{-\frac{d}{4}-\frac{\sigma_1-2}{2}}. \tag{2.34}$$

In addition, if $\sigma\in(-\widetilde{\sigma}_1,d/p-2)$ with $\widetilde{\sigma}_1\triangleq\sigma_1+d(1/2-1/p)$, then employing the interpolation once again implies that

$$\|(a,q)^\ell\|_{\dot B^{\sigma}_{p,1}}\lesssim\|(a,q)^\ell\|_{\dot B^{\sigma+d(1/2-1/p)}_{2,1}}\lesssim\big(\|(a,q)\|^\ell_{\dot B^{-\sigma_1}_{2,\infty}}\big)^{\theta_1}\big(\|(a,q)^\ell\|_{\dot B^{d/2-2}_{2,\infty}}\big)^{1-\theta_1}, \tag{2.35}$$

where $\theta_1=\frac{d/p-2-\sigma}{d/2-2+\sigma_1}\in(0,1)$. With the aid of (2.21), (2.33) and (2.35), one can conclude that

$$\|(a,q)^\ell(t)\|_{\dot B^{\sigma}_{p,1}}\lesssim\big((1+t)^{-\frac{d/2-2+\sigma_1}{2}}\big)^{1-\theta_1}=(1+t)^{-\frac{d}{2}(\frac12-\frac1p)-\frac{\sigma_1+\sigma}{2}}\quad\text{for all}\quad t\ge0,$$

which lead to

$$\|(a,q)(t)\|_{\dot B^{\sigma}_{p,1}}\lesssim\|(a,q)^\ell(t)\|_{\dot B^{\sigma}_{p,1}}+\|(a,q)(t)\|^h_{\dot B^{\sigma}_{p,1}}\lesssim(1+t)^{-\frac{d}{2}(\frac12-\frac1p)-\frac{\sigma_1+\sigma}{2}} \tag{2.36}$$

for $\sigma\in(-\widetilde{\sigma}_1,d/p-2)$. Owing to the embedding $\dot B^0_{p,1}\hookrightarrow L^p$, combining with (2.34) and (2.36) yields Theorem 1.2. This completes the proof of Theorem 1.2.

3. Appendix

For convenience of reader, we here give some technical results that have been used repeatedly in Section 2. In the first paragraph, we show Littlewood-Paley decomposition and Besov spaces. Next, we state some related analysis tools in Besov spaces.

3.1. Littlewood-Paley decomposition and Besov spaces

Let us briefly recall Littlewood-Paley decomposition and Besov spaces. The reader is referred to Chap. 2 and Chap. 3 of [1] for more details. We fix a smooth radial non increasing function χ with $\operatorname{Supp}\chi\subset B(0,4/3)$ and $\chi\equiv1$ on $B(0,3/4)$, then set $\varphi(\xi)=\chi(\xi/2)-\chi(\xi)$ so that

$$\sum_{k\in\mathbb{Z}}\varphi(2^{-k}\cdot)=1\ \text{ in }\ \mathbb{R}^d\setminus\{0\}\ \text{ and }\ \operatorname{Supp}\varphi\subset\big\{\xi\in\mathbb{R}^d:3/4\le|\xi|\le8/3\big\}.$$

The homogeneous dyadic blocks $\dot\Delta_k$ $(k\in\mathbb{Z})$ are defined by

$$\dot\Delta_kf\triangleq\varphi(2^{-k}D)f=\mathcal{F}^{-1}(\varphi(2^{-k}\cdot)\mathcal{F}f)=2^{kd}h(2^k\cdot)*f\quad\text{with}\quad h\triangleq\mathcal{F}^{-1}\varphi.$$

Formally, we have the unit decomposition for any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$f = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k f. \tag{3.1}$$

As it holds only modulo polynomials, it is convenient to consider the subspace of those tempered distributions f such that

$$\lim_{k\to-\infty} \|\dot{S}_k f\|_{L^\infty} = 0, \tag{3.2}$$

where $\dot{S}_k f$ stands for the low frequency cut-off defined by $\dot{S}_k f \triangleq \chi(2^{-k}D)f$. Indeed, if (3.2) is fulfilled, then (3.1) holds in $\mathcal{S}'(\mathbb{R}^d)$. For convenience, we denote by $\mathcal{S}'_0(\mathbb{R}^d)$ the subspace of tempered distributions satisfying (3.2).

With the aid of Littlewood-Paley decomposition, Besov spaces are defined as follows.

Definition 3.1. For $\sigma \in \mathbb{R}$ and $1 \le p, r \le \infty$, the homogeneous Besov spaces $\dot{B}^\sigma_{p,r}$ is defined by

$$\dot{B}^\sigma_{p,r} \triangleq \big\{ f \in \mathcal{S}'_0 : \|f\|_{\dot{B}^\sigma_{p,r}} < +\infty \big\},$$

where

$$\|f\|_{\dot{B}^\sigma_{p,r}} \triangleq \|(2^{k\sigma}\|\dot{\Delta}_k f\|_{L^p})\|_{\ell^r(\mathbb{Z})}. \tag{3.3}$$

On the other hand, a class of mixed space-time Besov spaces are also used, which was initiated by J.-Y. Chemin and N. Lerner [3] (see also [2] for the particular case of Sobolev spaces).

Definition 3.2. For $T > 0, \sigma \in \mathbb{R}, 1 \le r, \theta \le \infty$, the homogeneous Chemin-Lerner space $\widetilde{L}^\theta_T(\dot{B}^\sigma_{p,r})$ is defined by

$$\widetilde{L}^\theta_T(\dot{B}^\sigma_{p,r}) \triangleq \big\{ f \in L^\theta(0,T;\mathcal{S}'_0) : \|f\|_{\widetilde{L}^\theta_T(\dot{B}^\sigma_{p,r})} < +\infty \big\},$$

where

$$\|f\|_{\widetilde{L}^\theta_T(\dot{B}^\sigma_{p,r})} \triangleq \|(2^{k\sigma}\|\dot{\Delta}_j f\|_{L^\theta_T(L^p)})\|_{\ell^r(\mathbb{Z})}. \tag{3.4}$$

For notational simplicity, index T is omitted if $T = +\infty$. We agree with the notation

$$\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^\sigma_{p,r}) \triangleq \big\{ f \in \mathcal{C}(\mathbb{R}_+;\dot{B}^\sigma_{p,r}) \text{ s.t } \|f\|_{\widetilde{L}^\infty(\dot{B}^\sigma_{p,r})} < +\infty \big\}.$$

The Chemin-Lerner space $\widetilde{L}^\theta_T(\dot{B}^\sigma_{p,r})$ may be linked with the standard spaces $L^\theta_T(\dot{B}^\sigma_{p,r})$ by means of Minkowski's inequality.

Remark 3.1. It holds that

$$\|f\|_{\widetilde{L}^\theta_T(\dot{B}^\sigma_{p,r})} \le \|f\|_{L^\theta_T(\dot{B}^\sigma_{p,r})} \text{ if } r \ge \theta; \quad \|f\|_{\widetilde{L}^\theta_T(\dot{B}^\sigma_{p,r})} \ge \|f\|_{L^\theta_T(\dot{B}^\sigma_{p,r})} \text{ if } r \le \theta.$$

Restricting the above norms (3.3) and (3.4) to the low or high frequencies parts of distributions will be fundamental in our method. For instance, let us fix some integer k_0 (the value of which will follow from the proof of the high-frequency estimates) and put*

$$\|f\|^\ell_{\dot{B}^\sigma_{p,1}} \triangleq \sum_{k\le k_0} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^p} \text{ and } \|f\|^h_{\dot{B}^\sigma_{p,1}} \triangleq \sum_{k\ge k_0-1} 2^{j\sigma}\|\dot{\Delta}_k f\|_{L^p},$$

*Note that for technical reasons, we need a small overlap between low and high frequencies.

$$\|f\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot B^{\sigma}_{p,1})} \triangleq \sum_{k\le k_0} 2^{k\sigma}\|\dot\Delta_k f\|_{L^{\infty}_T(L^p)} \text{ and } \|f\|^{h}_{\widetilde{L}^{\infty}_T(\dot B^{\sigma}_{p,1})} \triangleq \sum_{k\ge k_0-1} 2^{j\sigma}\|\dot\Delta_k f\|_{L^{\infty}_T(L^p)}.$$

3.2. Analysis tools in Besov spaces

Let us here recall the classical *Bernstein inequality*:

$$\|D^k f\|_{L^b} \le C^{1+k}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a} \tag{3.5}$$

that holds for all function f such that $\operatorname{Supp}\mathcal{F}f \subset \left\{\xi\in\mathbb{R}^d : |\xi|\le R\lambda\right\}$ for some $R>0$ and $\lambda>0$, if $k\in\mathbb{N}$ and $1\le a\le b\le\infty$.

More generally, if we assume f to satisfy $\operatorname{Supp}\mathcal{F}f \subset \left\{\xi\in\mathbb{R}^d : R_1\lambda\le|\xi|\le R_2\lambda\right\}$ for some $0<R_1<R_2$ and $\lambda>0$, then for any smooth homogeneous of degree m function A on $\mathbb{R}^d\setminus\{0\}$ and $1\le a\le\infty$, we have (see e.g. Lemma 2.2 in [1]):

$$\|A(D)f\|_{L^a} \lesssim \lambda^m\|f\|_{L^a}. \tag{3.6}$$

An obvious consequence of (3.5) and (3.6) is that $\|D^k f\|_{\dot B^s_{p,r}} \approx \|f\|_{\dot B^{s+k}_{p,r}}$ for all $k\in\mathbb{N}$.

The following nonlinear generalization of (3.6) will be also used (see Lemma 8 in [4]).

Lemma 3.1. *If* $\operatorname{Supp}\mathcal{F}f \subset \left\{\xi\in\mathbb{R}^d : R_1\lambda\le|\xi|\le R_2\lambda\right\}$*, then there exists* c *depending only on* d*,* R_1 *and* R_2 *so that for all* $1<p<\infty$*,*

$$c\lambda^2\left(\frac{p-1}{p}\right)\int_{\mathbb{R}^d}|f|^p dx \le (p-1)\int_{\mathbb{R}^d}|\nabla f|^2|f|^{p-2}dx = -\int_{\mathbb{R}^d}\Delta f|f|^{p-2}f dx.$$

Next, let us give the classical properties (see [1]):

Lemma 3.2.

- *Scaling invariance: For any* $\sigma\in\mathbb{R}$ *and* $(p,r)\in[1,\infty]^2$*, there exists a constant* $C=C(\sigma,p,r,d)$ *such that for all* $\lambda>0$ *and* $f\in\dot B^{\sigma}_{p,r}$*, we have*

$$C^{-1}\lambda^{\sigma-d/p}\|f\|_{\dot B^{\sigma}_{p,r}} \le \left\|f(\lambda\cdot)\right\|_{\dot B^{\sigma}_{p,r}} \le C\lambda^{\sigma-d/p}\|f\|_{\dot B^{\sigma}_{p,r}}.$$

- *Completeness:* $\dot B^{\sigma}_{p,r}$ *is a Banach space whenever* $\sigma<d/p$ *or* $\sigma\le d/p$ *and* $r=1$*.*
- *Action of Fourier multipliers: If* F *is a smooth homogeneous of degree* m *function on* $\mathbb{R}^d\backslash\{0\}$ *then*

$$F(D) : \dot B^{\sigma}_{p,r} \to \dot B^{\sigma-m}_{p,r}.$$

Lemma 3.3. *Let* $1\le p,r_1,r_2,r\le\infty$*.*

- *Complex interpolation: If* $f\in\dot B^{\sigma_1}_{p,r_1}\cap\dot B^{\sigma_2}_{p,r_2}$ *and* $\sigma_1\ne\sigma_2$*, then* $f\in\dot B^{\theta\sigma_1+(1-\theta)\sigma_2}_{p,r}$ *for all* $\theta\in(0,1)$ *and*

$$\|f\|_{\dot B^{\theta\sigma_1+(1-\theta)\sigma_2}_{p,r}} \lesssim \|f\|^{\theta}_{\dot B^{\sigma_1}_{p,r_1}}\|f\|^{1-\theta}_{\dot B^{\sigma_2}_{p,r_2}} \quad \textit{with} \quad 1/r=\theta/r_1+(1-\theta)/r_2.$$

- *Real interpolation: If* $f \in \dot{B}^{\sigma_1}_{p,\infty} \cap \dot{B}^{\sigma_2}_{p,\infty}$ *and* $\sigma_1 < \sigma_2$*, then* $f \in \dot{B}^{\theta\sigma_1+(1-\theta)\sigma_2}_{p,1}$ *for all* $\theta \in (0,1)$ *and*

$$\|f\|_{\dot{B}^{\theta\sigma_1+(1-\theta)\sigma_2}_{p,1}} \lesssim \frac{C}{\theta(1-\theta)(\sigma_2-\sigma_1)}\|f\|^{\theta}_{\dot{B}^{\sigma_1}_{p,\infty}}\|f\|^{1-\theta}_{\dot{B}^{\sigma_2}_{p,\infty}}.$$

The following embedding properties will be used frequently throughout this paper.

Lemma 3.4 (Embedding for Besov spaces on $\mathbb{R}^d$)**.**

- *For any* $p \in [1,\infty]$ *we have the continuous embedding* $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$.
- *If* $\sigma \in \mathbb{R}$, $1 \le p_1 \le p_2 \le \infty$ *and* $1 \le r_1 \le r_2 \le \infty$*, then* $\dot{B}^{\sigma}_{p_1,r_1} \hookrightarrow \dot{B}^{\sigma-d\,(1/p_1-1/p_2)}_{p_2,r_2}$.
- *The space* $\dot{B}^{d/p}_{p,1}$ *is continuously embedded in the set of bounded continuous functions (going to zero at infinity if, additionally,* $p < \infty$*).*

The product estimate in Besov spaces plays a fundamental role in bounding bilinear terms of (1.7) (see for example [1,6,20,22]).

Lemma 3.5. *Let* $\sigma > 0$ *and* $1 \le p, r \le \infty$*. Then* $\dot{B}^{\sigma}_{p,r} \cap L^\infty$ *is an algebra and*

$$\|fg\|_{\dot{B}^{\sigma}_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^{\sigma}_{p,r}} + \|g\|_{L^\infty}\|f\|_{\dot{B}^{\sigma}_{p,r}}.$$

Let the real numbers σ_1, σ_2, p_1 *and* p_2 *fulfill*

$$\sigma_1+\sigma_2>0, \quad \sigma_1 \le d/p_1, \quad \sigma_2 \le d/p_2, \quad \sigma_1 \ge \sigma_2, \quad 1/p_1+1/p_2 \le 1.$$

Then we have

$$\|fg\|_{\dot{B}^{\sigma_2}_{q,1}} \lesssim \|f\|_{\dot{B}^{\sigma_1}_{p_1,1}}\|g\|_{\dot{B}^{\sigma_2}_{p_2,1}} \quad \textit{with} \quad 1/q = 1/p_1+1/p_2-\sigma_1/d.$$

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