Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1386–1402
Website:http://www.jaac-online.com
DOI:10.11948/20210145

SINGULAR DISCONTINUOUS HAMILTONIAN SYSTEMS

Bilender P. Allahverdiev[1] and Hüseyin Tuna[2,†]

Abstract We study a discontinuous linear Hamiltonian system in the singular case. For these systems, the Titchmarsh-Weyl theory is established.

Keywords Hamiltonian system, singular point, transmission conditions, Titchmar-sh–Weyl theory.

MSC(2010) 34B37, 34B20.

1. Introduction

In recent years a great deal of research has been devoted to the study of discontinuous problems. Since these problems describe processes that experience a sudden change of their state at certain moments, they arise in the theory of the mass and heat transfer, control theory, population dynamics, medicine, and radio science (see [1, 5–7, 9–13, 21–33]).

The Hamiltonian systems frequently occur in mathematical modeling of various physical systems, for example, in the study of electromechanical, electrical, and complex network systems with negligible dissipation (see [34]). Many researchers have paid more attention to the Hamiltonian systems, for instance see ([4,8,12–20]). But there are few studies about discontinuous Hamiltonian systems [2,8,12,13]. This paper deals with these systems. For these systems, the Titchmarsh–Weyl theory is established. In the analysis that follows, we will largely follow the development of the theory in [3, 18, 19].

2. Discontinuous linear Hamiltonian system

Consider the following discontinuous linear Hamiltonian system:

$$\Gamma(\mathcal{Z}) := J\mathcal{Z}'(x) - B_2(x)\mathcal{Z}(x) = \lambda B_1(x)\mathcal{Z}(x),\ x \in [a,c) \cup (c,b), \tag{2.1}$$

where $-\infty < a < c < b \leq +\infty$, $\lambda \in \mathbb{C}$; $B_1(.)$ and $B_2(.)$ are $2n \times 2n$ complex Hermitian matrix-valued functions, defined on $[a,c) \cup (c,b)$, and entries of this matrices are Lebesgue measurable and locally integrable functions on $[a,c) \cup (c,b)$;

†The corresponding author. Email address:hustuna@gmail.com(H. Tuna)
[1]Department of Mathematics, Süleyman Demirel University, 32260 Isparta, Turkey
[2]Department of Mathematics, Mehmet Akif Ersoy University, 15030 Burdur, Turkey

$B_1(x)$ is nonnegative-definite and

$$J=\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

where I_n is the identity matrix in $\mathbb{C}^n$.

We assume that the points a, c are regular and b is a singular for the differential expression Γ (see [4, 18]).

Let

$$\begin{aligned} &L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right] \\ &=\left\{\mathcal{Z}:\int_a^c (B_1\mathcal{Z},\mathcal{Z})_{\mathbb{C}^{2n}}\,dx+\int_c^b (B_1\mathcal{Z},\mathcal{Z})_{\mathbb{C}^{2n}}\,dx<\infty\right\}\end{aligned}$$

with the inner product

$$\begin{aligned}(\mathcal{Z},\mathcal{Y}) &:= \int_a^c (B_1\mathcal{Z},\mathcal{Y})_{\mathbb{C}^{2n}}\,dx+\int_c^b (B_1\mathcal{Z},\mathcal{Y})_{\mathbb{C}^{2n}}\,dx \\ &= \int_a^c \mathcal{Y}^* B_1\mathcal{Z}dx+\int_c^b \mathcal{Y}^* B_1\mathcal{Z}dx.\end{aligned}$$

We assume that if $\Gamma(\mathcal{Z})=B_1F$ and $B_1\mathcal{Z}=0$, then $\mathcal{Z}=0$.

Let T is the $2n\times 2n$ matrix with entries from $\mathbb{R}$ such that $TJT^*=J$ and let $\sigma_1,\sigma_2,\xi_1,\xi_2$ are matrices satisfying

$$\begin{aligned} &\sigma_1\sigma_1^*+\sigma_2\sigma_2^*=I_n, && (2.2)\\ &\sigma_1\sigma_2^*-\sigma_2\sigma_1^*=0, \\ &\xi_1\xi_1^*+\xi_2\xi_2^*=I_n, && (2.3)\\ &\xi_1\xi_2^*-\xi_2\xi_1^*=0, \end{aligned}$$

and $rank\begin{pmatrix}\sigma_1 & \sigma_2\end{pmatrix}=rank\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}=n$.

Now, we will impose the following boundary conditions:

$$\Sigma\mathcal{Z}(a)=0,\ \ \mathcal{Z}(c+)=T\mathcal{Z}(c-), \tag{2.4}$$
$$\Xi\mathcal{Z}(b_1)=0, \tag{2.5}$$

where

$$\Sigma=\begin{pmatrix}\sigma_1 & \sigma_2\\ 0 & 0\end{pmatrix},\ \ \Xi=\begin{pmatrix}0 & 0\\ \xi_1 & \xi_2\end{pmatrix}$$

and

$$-\infty<a<c<b_1<b\le+\infty.$$

It follows from (2.4) that

$$\Sigma J\Sigma^*=0,$$

and

$$\Xi J\Xi^*=0.$$

One can prove that Eq. (2.1) with conditions (2.4), (2.5) and $\Xi\mathcal{Z}(b_1)=0$ defines a regular, self-adjoint problem.

Let

$$Z=\begin{pmatrix}\varphi & \psi\end{pmatrix}=\begin{pmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\end{pmatrix} \tag{2.6}$$

be the fundamental matrix for $\Gamma(\mathcal{Z})=\lambda B_1\mathcal{Z}$ satisfying

$$Z(a)=E=\begin{pmatrix}\sigma_1^* & -\sigma_2^*\\ \sigma_2^* & \sigma_1^*\end{pmatrix}$$

and $\varphi(c+)=T\varphi(c-)$, $\psi(c+)=T\psi(c-)$. Then we have

$$\begin{pmatrix}\sigma_1 & \sigma_2\end{pmatrix}\varphi(a)=I_n,$$

and

$$\begin{pmatrix}\sigma_1 & \sigma_2\end{pmatrix}\psi(a)=0.$$

Note that

$$Z^*\left(x,\overline{\lambda}\right)JZ(x,\lambda)=J. \tag{2.7}$$

3. Titchmarsh–Weyl functions and circles

In this section, we introduce Titchmarsh–Weyl functions and circles for the system (2.1), (2.4).

Definition 3.1. Let

$$Y_{b_1}(x,\lambda)=Z(x,\lambda)\begin{pmatrix}I_n\\ M_{b_1}(\lambda)\end{pmatrix},$$

where $\operatorname{Im}\lambda\neq 0$ and $M_{b_1}(\lambda)$ is a $n\times n$ matrix-valued function $M_{b_1}(\lambda)$ is said to be the Titchmarsh–Weyl function for the boundary value problem (2.1), (2.4), (2.5).

Then we have the following theorem.

Theorem 3.1. *Let*

$$\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}Y_{b_1}(b_1,\lambda)=0. \tag{3.1}$$

Then, we have

$$M_{b_1}(\lambda)=-\left(\xi_1\psi_1(b_1)+\xi_2\psi_2(b_1)\right)^{-1}\left(\xi_1\varphi_1(b_1)+\xi_2\varphi_2(b_1)\right),$$

and

$$Y_{b_1}^*(b_1,\lambda)JY_{b_1}(b_1,\lambda)=0,$$

where ξ_1 and ξ_2 are defined in (2.3).

Conversely, if Y_{b_1} satisfies

$$Y_{b_1}^*(b_1,\lambda)JY_{b_1}(b_1,\lambda)=0,$$

then there exists ξ_1, ξ_2 *satisfying (2.3) such that*

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} Y_{b_1}(b_1, \lambda) = 0,$$

and

$$M_{b_1}(\lambda) = -\left(\xi_1 \psi_1(b_1) + \xi_2 \psi_2(b_1)\right)^{-1} \left(\xi_1 \varphi_1(b_1) + \xi_2 \varphi_2(b_1)\right).$$

Proof. Let

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} Y_{b_1}(b_1, \lambda) = 0.$$

Then we have

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \end{pmatrix} \begin{pmatrix} I_n \\ M_{b_1}(\lambda) \end{pmatrix} = 0.$$

Therefore, we conclude that

$$\left[\xi_1 \psi_1(b_1) + \xi_2 \psi_2(b_1)\right] M_{b_1}(\lambda) + \left(\xi_1 \varphi_1(b_1) + \xi_2 \varphi_2(b_1)\right) = 0,$$

and

$$M_{b_1}(\lambda) = -\left(\xi_1 \psi_1(b_1) + \xi_2 \psi_2(b_1)\right)^{-1} \left(\xi_1 \varphi_1(b_1) + \xi_2 \varphi_2(b_1)\right).$$

The inverse of the matrix $\xi_1 \psi_1(b_1) + \xi_2 \psi_2(b_1)$ exists because $\operatorname{Im} \lambda \neq 0$, i.e., λ is not an eigenvalue of the self-adjoint problem on $(a, c) \cup (c, b_1)$. It follows from (3.1) that

$$Y_{b_1}(b_1, \lambda) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} K,$$

for

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} K = 0.$$

Then we get

$$\begin{pmatrix} I_n & M_{b_1}^*(\lambda) \end{pmatrix} Z^*(b_1, \lambda) J Z(b_1, \lambda) \begin{pmatrix} I_n \\ M_{b_1}(\lambda) \end{pmatrix} = 0,$$

i.e.,

$$Y_{b_1}^*(b_1, \lambda) J Y_{b_1}(b_1, \lambda) = 0.$$

Conversely, let

$$Y_{b_1}^*(b_1, \lambda) J Y_{b_1}(b_1, \lambda) = 0,$$

i.e.,

$$\begin{pmatrix} I_n & M_{b_1}^*(\lambda) \end{pmatrix} Z^*(b_1, \lambda) J Z(b_1, \lambda) \begin{pmatrix} I_n \\ M_{b_1}(\lambda) \end{pmatrix} = 0,$$

for some M. If we set

$$\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} = \begin{pmatrix} I_n & M_{b_1}^*(\lambda) \end{pmatrix} Z^*(b_1, \lambda) J,$$

then we get the desired results. □

Now, we introduce Titchmarsh–Weyl circles.

Definition 3.2. Let

$$C_{TW}(b_1,\lambda)=\begin{pmatrix} I_n & M_{b_1}^*(\lambda)\end{pmatrix}\begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3\end{pmatrix}\begin{pmatrix} I_n \\ M_{b_1}(\lambda)\end{pmatrix}=0, \tag{3.2}$$

where

$$\begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3\end{pmatrix}=-sgn\,(\operatorname{Im}\lambda)\,Z^*(b_1,\lambda)\,(J/i)\,Z(b_1,\lambda), \tag{3.3}$$

and C_m are $n\times n$ matrices for $m=1,2,3$. Then $C_{TW}(b_1,\lambda)$ is said to be the Titchmarsh–Weyl circle for the boundary value problem (2.1), (2.4), (2.5).

From Definition 3, we have

$$\begin{aligned} C_{TW}(b_1,\lambda) &= \left(M_{b_1}+C_3^{-1}C_2\right)^* C_4\left(M_{b_1}+C_3^{-1}C_2\right)+C_1-C_2^*C_3^{-1}C_2 \\ &= \left(M_{b_1}-C_4\right)K_1^{-2}\left(M_{b_1}-C_4\right)-K_2^2=0, \end{aligned}$$

where $C_4=-C_3^{-1}C_2,\ K_1^{-2}=C_3^{-1}$, and $K_2^2=C_2^*C_3^{-1}C_2-C_1$.

Lemma 3.1. $C_3>0$.

Proof. From (2.6) and (3.3), we have

$$\begin{aligned} \begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3\end{pmatrix} &= -sgn\,(\operatorname{Im}\lambda)\begin{pmatrix} \varphi_1^* & \varphi_2^* \\ \psi_1^* & \psi_2^*\end{pmatrix}\begin{pmatrix} 0 & iI_n \\ -iI_n & 0\end{pmatrix}\begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2\end{pmatrix} \\ &= -sgn\,(\operatorname{Im}\lambda)\begin{pmatrix} \varphi^*(J/i)\,\varphi & \varphi^*(J/i)\,\psi \\ i\psi^*(J/i)\,\varphi & \psi^*(J/i)\,\psi\end{pmatrix}. \end{aligned}$$

Thus, we get

$$C_3=-sgn\,(\operatorname{Im}\lambda)\,\psi^*(J/i)\,\psi.$$

A direct calculation gives

$$\begin{aligned} &2\operatorname{Im}\lambda\left(\int_a^c \psi^* B_1\psi dx+\int_c^{b_1}\psi^* B_1\psi dx\right) \\ =&\psi^*(J/i)\,\psi(b_1)-\psi^*(J/i)\,\psi(c+)+\psi^*(J/i)\,\psi(c-)-\psi^*(J/i)\,\psi(a). \end{aligned}$$

Since $\psi^*(J/i)\,\psi(a)=0$ and

$$\psi^*(J/i)\,\psi(c+)=\psi^*(J/i)\,\psi(c-),$$

we get the desired result. □

Lemma 3.2. $C_2^*C_3^{-1}C_2-C_1=\overline{C_3}^{-1}>0$, *where* $\overline{C_3}^{-1}=C_3^{-1}\left(\overline{\lambda}\right)$.

Proof. Using (2.7), one may get

$$Z(x,\lambda)\,JZ^*\left(x,\overline{\lambda}\right)=J.$$

Then we obtain

$$J = -\left[Z^*(x,\lambda)(J/i)Z(x,\lambda)\right] J \left[-Z^*\left(x,\overline{\lambda}\right)(J/i)Z\left(x,\overline{\lambda}\right)\right],$$
$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = -\begin{pmatrix} C_1 & C_2^* \\ C_2 & C_3 \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} \overline{C_1} & \overline{C_2}^* \\ \overline{C_2} & \overline{C_3} \end{pmatrix},$$

because there is a sign change in the matrix when λ replaces $\overline{\lambda}$. Thus

$$0 = C_1\overline{C_2} - C_2^*\overline{C_1}, \quad -I_n = C_1\overline{C_3} - C_2^*\overline{C_2},$$
$$I_n = C_2\overline{C_2} - C_3\overline{C_1}, \quad 0 = C_2\overline{C_3} - C_3\overline{C_2^*}.$$

The last and second show that

$$\overline{C_3}^{-1} = C_2^*C_3^{-1}C_2 - C_1.$$

□

Corollary 3.1. $K_2 = \overline{K_1}$

Theorem 3.2. *As b_1 increases, C_3, K_1 and K_2 decrease.*

Proof. Since

$$C_3 = 2\left|\operatorname{Im}\lambda\right|\left(\int_a^c \psi^* B_1 \psi dx + \int_c^{b_1} \psi^* B_1 \psi dx\right),$$

we get the desired results. □

Corollary 3.2. *The following limits exist*

$$\lim_{b_1\to b} K_1(b_1,\lambda) = K_0, \ \lim_{b_1\to b} K_2(b_1,\lambda) = \overline{K_0},$$

where $K_0 \geq 0$ and $\overline{K_0} \geq 0$.

Theorem 3.3. *As $b_1 \to b$, the circles $C_{TW}(b_1,\lambda) = 0$ are nested.*

Proof. The interior of the circle is

$$-sgn(\operatorname{Im}\lambda)\begin{pmatrix} I_n & M_{b_1}^*(\lambda)\end{pmatrix} Z^*(b_1,\lambda)(J/i)Z(b_1,\lambda)\begin{pmatrix} I_n \\ M_{b_1}(\lambda)\end{pmatrix} \leq 0.$$

From (3.2), we have

$$\begin{aligned} &C_{TW}(b_1,\lambda) \\ =&2\left|\operatorname{Im}\lambda\right|\left(\int_a^c Y_{b_1}^* B_1 Y_{b_1} dx + \int_c^{b_1} Y_{b_1}^* B_1 Y_{b_1} dx\right) \pm \frac{1}{i}\left(M_{b_1}^* - M_{b_1}\right). \end{aligned}$$

Thus, if M_{b_1} is in the circle at $b_2 > b_1$, then $C_{TW}(b_1,\lambda) \leq 0$ at the point b_2. At the point b_2, $C_{TW}(b_1,\lambda)$ is certainly smaller, and so $C_{TW}(b_1,\lambda)$ is in the circle at the point b_2 as well. $C_{TW}(b_1,\lambda) = 0$ are nested as $b_1 \to b$. □

Theorem 3.4. *The following limit exists*

$$\lim_{b_1 \to b} C_{TW}\left(b_1, \lambda\right) = C_{TW}^0.$$

Proof. From (3.2), we get

$$C_{TW}\left(b_1, \lambda\right) = \left(M_{b_1}\left(\lambda\right) - D\right)^* K_1^{-2}\left(M_{b_1}\left(\lambda\right) - D\right) - K_2^2 = 0.$$

Then we have

$$\left[K_1^{-1}\left(M_{b_1}\left(\lambda\right) - D\right)\overline{K_1^{-1}}\right]^* \left[K_1^{-1}\left(M_{b_1}\left(\lambda\right) - D\right)\overline{K_1^{-1}}\right] = I_n. \tag{3.4}$$

It follows from (3.4) that

$$U = K_1^{-1}\left(M_{b_1}\left(\lambda\right) - D\right)\overline{K_1^{-1}},$$

where U is a unitary matrix, i.e., $U^*U = I_n$. Hence

$$M_{b_1}\left(\lambda\right) = D + K_1 U \overline{K_1}. \tag{3.5}$$

As U varies over the $n \times n$ unit sphere, $M_{b_1}\left(\lambda\right)$ varies over a circle with center D.

Let D_1 be the center at b_1', D_2 be the center at b_1'', where $b_1'' < b_1'$. From (3.5), we have

$$M_{b_1'}\left(\lambda\right) = D_1 + K_1\left(b_1'\right) U_1 \overline{K_1\left(b_1'\right)},$$

and

$$M_{b_1''}\left(\lambda\right) = D_2 + K_1\left(b_1''\right) U_2 \overline{K_1\left(b_1''\right)}. \tag{3.6}$$

Since $C_{TW}\left(b_1'', \lambda\right) \subset C_{TW}\left(b_1', \lambda\right)$, we infer that

$$M_{b_1''}\left(\lambda\right) = D_1 + K_1\left(b_1'\right) V_1 \overline{K_1\left(b_1'\right)}, \tag{3.7}$$

where V_1 is a contraction. Subtracting (3.6) from (3.7) gives

$$D_1 - D_2 = K_1\left(b_1''\right) U_2 \overline{K_1\left(b_1''\right)} - K_1\left(b_1'\right) V_1 \overline{K_1\left(b_1'\right)}.$$

Hence we get

$$V_1 = \left[D_1 - D_2 + K_1\left(b_1'\right) V_1 \overline{K_1\left(b_1'\right)}\right].$$

Let us define a mapping Υ by the formula

$$\Upsilon\left(U_2\right) = V_1.$$

Υ is a continuous mapping from the unit ball into itself. Hence, it has a unique fixed point. Letting U_2 and V_1 be replaced by U, we conclude that

$$\begin{aligned} \left\|D_1 - D_2\right\| &= \left\|K_1\left(b_1''\right) U \overline{K_1\left(b_1''\right)} - K_1\left(b_1'\right) U \overline{K_1\left(b_1'\right)}\right\| \\ &\leq \left\|K_1\left(b_1''\right)\right\| \left\|\overline{K_1\left(b_1''\right)} - \overline{K_1\left(b_1'\right)}\right\| + \left\|K_1\left(b_1''\right) - K_1\left(b_1'\right)\right\| \left\|\overline{K_1\left(b_1'\right)}\right\|. \end{aligned}$$

As b_1' and b_1'' approach b_1, K_1 and $\overline{K_1}$ have limits. The centers form a Cauchy sequence and converge.

A direct calculation gives

$$C_2 = \pm \left[2 \operatorname{Im} \lambda \left(\int_a^c \psi^* B_1 \varphi dx + \int_c^{b_1'} \psi^* B_1 \varphi dx \right) - i I_n \right].$$

Thus at b_1', the center

$$\begin{aligned} D &= - C_3^{-1} C_2 \\ &= - \left[2 \operatorname{Im} \lambda \left(\int_a^c \psi^* B_1 \psi dx + \int_c^{b_1'} \psi^* B_1 \psi dx \right) \right]^{-1} \\ &\quad \times \left[2 \operatorname{Im} \lambda \left(\int_a^c \psi^* B_1 \psi dx + \int_c^{b_1'} \psi^* B_1 \psi dx \right) - i I_n \right]. \end{aligned}$$

Consequently, the limit $\lim_{b_1' \to b_1} C_{TW} (b_1', \lambda) = C_{TW}^0$. □

It is clear that

$$M_{b_1} (\lambda) = D + K_1 U \overline{K_1}$$

is well defined. As U varies over the unit circle in $n \times n$ space, the limit circle or point C_{TW}^0 is covered.

4. Square integrable solutions

In this section, we study the number of square-integrable solutions of the discontinuous Hamilton system.

Theorem 4.1. *Let M be a point inside $C_{TW}^0 \leq 0$. Let $\chi = \varphi + \psi M$. Then we have $\chi \in L_{B_1}^2 \left[(a, c) \cup (c, b) ; \mathbb{C}^{2n} \right]$.*

Proof. Since

$$2 \left| \operatorname{Im} \lambda \right| \left(\int_a^c \chi^* B_1 \chi dx + \int_c^{b_1} \chi^* B_1 \chi dx \right) \pm \frac{1}{i} \left[M - M^* \right] = C_{TW} (b_1, \lambda) \leq 0,$$

we get

$$0 \leq \int_a^c \chi^* B_1 \chi dx + \int_c^{b_1} \chi^* B_1 \chi dx \leq \frac{1}{2i \left| \operatorname{Im} \lambda \right|} \left[M - M^* \right].$$

As $b_1 \to b$, the upper bound is fixed. □

Lemma 4.1. *Let rank$\overline{K_1}$ and $S(U) = K_1 U \overline{K_1}$, where U is unitary. Then we have the following relations:*
i) rank$S(U) \leq r$,
ii) $\sup_U$ rank$S(U) = r$.

Proof. This is clear from the matrix theory. □

Theorem 4.2. *Let $m = n + r$. For $\operatorname{Im} \lambda \neq 0$, there exists at least m square integrable solutions of Eq. (2.1), $n \leq m \leq 2n$.*

Proof. $\varphi + D\psi$ consists n solutions in the space $L_{B_1}^2 \left[(a, c) \cup (c, b) ; \mathbb{C}^{2n} \right]$. As U varies, $\psi \left(K_1 U \overline{K_1} \right)$ gives an additional linearly independent $m - n$ solutions. By the reflection principles, the number of solutions is the same for $\operatorname{Im} \lambda < 0$ or $\operatorname{Im} \lambda > 0$. □

Theorem 4.3. *Let* $\nu_1\left(b_1^{'}\right) \leq ... \leq \nu_n\left(b_1^{'}\right)$ *be the eigenvalues of* $C_3\left(b_1^{'},\lambda\right)$. *Let there be* m, $n \leq m \leq 2n$, *square-integrable solutions of Eq. (2.1),* $\operatorname{Im}\lambda \neq 0$. *Then* $\nu_1\left(b_1^{'}\right) \leq ... \leq \nu_{m-n}\left(b_1^{'}\right)$ *remain finite and* $\nu_{m-n+1}\left(b_1^{'}\right) \leq ... \leq \nu_{2n}\left(b_1^{'}\right)$ *approach infinity as* $b_1^{'} \to b$.

Proof. Assume $\nu\left(b_1^{'}\right) < C_2$ for all $b_1^{'}$. Let $\mu_{b_1^{'}}$ be a unit eigenvector of $C_3\left(b_1^{'},\lambda\right)$ and let $\chi_{b_1^{'}} = \psi\mu_{b_1^{'}}$. Then we get

$$\begin{aligned}&2i\operatorname{Im}\lambda\left(\int_a^c \chi_{b_1^{'}}^{*} B_1 \chi_{b_1^{'}}\,dx + \int_c^{b_1^{'}} \chi_{b_1^{'}}^{*} B_1 \chi_{b_1^{'}}\,dx\right)\\ =&\mu_{b_1^{'}}^{*}\psi^{*}J\psi\mu_{b_1^{'}}\left(b_1^{'}\right) - \mu_{b_1^{'}}^{*}\psi^{*}J\psi\mu_{b_1^{'}}(a)\\ =&isgn\left(\operatorname{Im}\lambda\right)\mu_{b_1^{'}}.\end{aligned}$$

Hence

$$\int_a^c \chi_{b_1^{'}}^{*} B_1 \chi_{b_1^{'}}\,dx + \int_c^{b_1^{'}} \chi_{b_1^{'}}^{*} B_1 \chi_{b_1^{'}}\,dx = \frac{\mu_{b_1^{'}}}{|\operatorname{Im}\lambda|} \leq \frac{C_2}{|\operatorname{Im}\lambda|}.$$

If we choose a subsequence $\left(\mu_{b_1^{'}}\right)$'s that converge, then we get a square-integrable solution $\chi = \psi\mu$. Sine there are only m square-integrable solutions and $\chi = \varphi + M\psi$ comprises n of these, there can only be $m-n$ such χ 's and only $m-n$ finite ν 's. □

5. Boundary conditions in the singular case

Let

$$\mathcal{D}_{\max} := \left\{\begin{array}{l} \mathcal{Z} \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]: \\ \mathcal{Z} \text{ is locally absolutely continuous on } [a,c)\cup(c,b), \\ \text{one-sided limits } \mathcal{Z}(c\pm) \text{ exist and finite}, \\ J\mathcal{Z}'(x) - B_2(x)\mathcal{Z}(x) = B_1(x)F(x) \text{ exists in } (a,c)\cup(c,b), \\ F \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right],\ \mathcal{Z}(c+) = T\mathcal{Z}(c-),\ TJT^{*} = J, \end{array}\right\}$$

We define the maximal operator $L_{\max}$ by the formula

$$L_{\max}\mathcal{Z} = F,$$

for all $\mathcal{Z} \in \mathcal{D}_{\max}$.

Theorem 5.1. *Let* $\mathcal{Z}_j$ *be a solution of the equation*

$$J\mathcal{Z}_j'(x) = \left(\overline{\lambda_0}B_1 + B_2\right)\mathcal{Z}_j,$$

where $\operatorname{Im}\lambda_0 \neq 0$. *Then for all* $\mathcal{Z} \in \mathcal{D}_{\max}$, *the following limit*

$$A_{b_j}(\mathcal{Z}) = \lim_{x\to b}\mathcal{Z}_j^{*}J\mathcal{Z}$$

exists if and only if $\mathcal{Z}_j \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$.

Proof. From the equations

$$J\mathcal{Z}'(x) - B_2(x)\,\mathcal{Z}(x) = B_1(x)\,F(x),$$

and

$$J\mathcal{Z}_j'(x) - B_2(x)\,\mathcal{Z}_j(x) = \overline{\lambda} B_1(x)\,\mathcal{Z}_j(x),$$

we get

$$\left(\mathcal{Z}_j^* J\mathcal{Z}\right)'(x) = \mathcal{Z}_j^*(x)\,B_1(x)\left[F(x) - \lambda\mathcal{Z}(x)\right].$$

Integrating, we obtain

$$\begin{aligned}\left(\mathcal{Z}_j^* J\mathcal{Z}\right)(x) = {} & \left(\mathcal{Z}_j^* J\mathcal{Z}\right)(c+) - \left(\mathcal{Z}_j^* J\mathcal{Z}\right)(c-) + \left(\mathcal{Z}_j^* J\mathcal{Z}\right)(a) \\ & + \int_a^c \mathcal{Z}_j^*(x)\,B_1(x)\left[F(x) - \lambda\mathcal{Z}(x)\right]dx \\ & + \int_c^x \mathcal{Z}_j^*(x)\,B_1(x)\left[F(x) - \lambda\mathcal{Z}(x)\right]dx. \end{aligned} \tag{5.1}$$

If $\mathcal{Z}_j \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$, then as $x \to b$, the integral in (5.1) converges, and the limit $\lim_{x\to b}(\mathcal{Z}_j^* J\mathcal{Z})(x)$ exists. Conversely, suppose that the integral in (5.1) converges for all $\mathcal{Z}, F \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$. From the Hahn–Banach theorem and the Riesz representation theorem, we conclude that $\mathcal{Z}_j \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$. □

Suppose that λ_0 is fixed, where $\operatorname{Im}\lambda_0 \neq 0$.

Definition 5.1. Let $M_{b_1}\left(\overline{\lambda}\right) = \overline{D} + \overline{K_1} U K_1$ be on the limit circle. Let $\chi\left(x,\overline{\lambda_0}\right) = \varphi\left(x,\overline{\lambda_0}\right) + \psi\left(x,\overline{\lambda_0}\right) M\left(\overline{\lambda_0}\right) \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$ and let $\chi\left(x,\overline{\lambda_0}\right)$ satisfies the equation $J\mathcal{Z}'(x) = \left(\lambda_0 B_1(x) + B_2(x)\right)\mathcal{Z}(x)$. Then we define $A_{\lambda_0}(\mathcal{Z})$ by the formula

$$A_{\lambda_0}(\mathcal{Z}) = \lim_{x\to b} \chi\left(x,\overline{\lambda_0}\right) J\mathcal{Z}(x)$$

for all $\mathcal{Z} \in \mathcal{D}_{\max}$.

6. A self-adjoint operator

In this section, we shall define a self-adjoint operator. We assume that the number of solutions of Eq. (2.1) is m.

Let

$$\mathcal{D} := \left\{ \begin{array}{l} \mathcal{Z} \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]: \\ \mathcal{Z} \text{ is locally absolutely continuous on } [a,c)\cup(c,b), \\ \text{one-sided limits } \mathcal{Z}(c\pm) \text{ exist and finite}, \\ J\mathcal{Z}'(x) - B_2(x)\,\mathcal{Z}(x) = B_1(x)\,F(x) \text{ exists in } (a,c)\cup(c,b), \\ F \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right], \\ \Sigma\mathcal{Z}(a) = 0, \\ \mathcal{Z}(c+) = T\mathcal{Z}(c-),\ TJT^* = J, \\ \text{and } A_{\lambda_0}(\mathcal{Z}) = 0,\ \operatorname{Im}\lambda_0 \neq 0. \end{array} \right\}.$$

The operator L is defined by

$$\begin{aligned} &L : \mathcal{D} \rightarrow L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right] \\ &\mathcal{Z} \rightarrow L\mathcal{Z} = F \text{ if and only if } \Gamma(\mathcal{Z}) = B_1F. \end{aligned}$$

Now, we calculate the operator $(L-I)^{-1}$. Let us consider the following nonhomogeneous equation

$$J\mathcal{Z}'(x) - B_2(x)\mathcal{Z}(x) = \lambda_0 B_1(x)\mathcal{Z}(x) + B_1(x)F(x).$$

By using the method of variation of parameters, the substitution of $\mathcal{Z} = ZC$, where Z is the fundamental matrix, we obtain

$$JZK = B_1F.$$

Hence

$$C' = -JZ^*\left(x,\overline{\lambda_0}\right)B_1F.$$

Integrating, we get

$$\mathcal{Z} = -Z(x,\lambda_0)\begin{bmatrix} \int_a^c JZ^*\left(t,\overline{\lambda_0}\right)B_1(t)F(t)\,dt \\ \\ +\int_c^x JZ^*\left(t,\overline{\lambda_0}\right)B_1(t)F(t)\,dt \end{bmatrix} + Z(x,\lambda_0)K,$$

where K is constant.

From the condition $\Sigma\mathcal{Z}(a) = 0$, we conclude that

$$\begin{pmatrix} \sigma_1 & \sigma_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_1^* & -\sigma_2^* \\ \sigma_2^* & \sigma_1^* \end{pmatrix}K = 0,$$

or

$$\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}K = 0. \tag{6.1}$$

Further,

$$\begin{aligned} &\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix}Z^*\left(x,\overline{\lambda_0}\right)JZ(x) \\ =& -\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix}Z^*\left(x,\overline{\lambda_0}\right)JZ(x,\lambda)\begin{bmatrix} \int_a^c JZ^*\left(t,\overline{\lambda_0}\right)B_1(t)F(t)\,dt \\ +\int_c^x JZ^*\left(t,\overline{\lambda_0}\right)B_1(t)F(t)\,dt \end{bmatrix} \\ &+\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix}Z^*\left(x,\overline{\lambda_0}\right)JZ(x,\lambda)K. \end{aligned}$$

Since $Z^*\left(x,\overline{\lambda_0}\right)JZ\left(x,\lambda\right)=J$ for all x, we get

$$\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(x,\overline{\lambda_0}\right)JZ\left(x\right)=\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix}\begin{bmatrix} \int_a^c Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt \\ \\ +\int_c^x Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt \end{bmatrix}$$
$$+\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}K.$$

Hence

$$\begin{aligned} A_{\lambda_0}\left(\mathcal{Z}\right) &= \lim_{x\to b}\chi\left(x,\overline{\lambda_0}\right)J\mathcal{Z}\left(x\right) \\ &= \lim_{x\to b}\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(x,\overline{\lambda_0}\right)JZ\left(x\right) \\ &= \begin{bmatrix} \int_a^c \begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt \\ \\ +\int_c^b \begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt \end{bmatrix}+\begin{pmatrix} 0 & 0 \\ M^* & -I \end{pmatrix}K=0. \end{aligned}$$

By (6.1), we conclude that

$$\int_a^c \begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt+\int_c^b \begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt$$
$$+\begin{pmatrix} I_n & 0 \\ M^* & -I_n \end{pmatrix}K=0.$$

Since the matrix $\begin{pmatrix} I_n & 0 \\ M^* & -I_n \end{pmatrix}$ is its own inverse, we obtain

$$\begin{aligned} K=&\int_a^c \begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt \\ &+\int_c^b \begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt \end{aligned}$$

and

$$\mathcal{Z} = Z\left(x,\lambda_0\right)\int_a^c \begin{pmatrix} 0 & I_n \\ 0 & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right)B_1\left(t\right)F\left(t\right)dt$$

$$+ Z(x,\lambda_0)\int_c^x \begin{pmatrix} 0 & I_n \\ 0 & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right) B_1(t) F(t)\,dt$$

$$+ Z(x,\lambda_0)\int_x^b \begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right) B_1(t) F(t)\,dt.$$

Since $M(\lambda_0) = M^*\left(\overline{\lambda_0}\right)$ and

$$\chi(x,\lambda_0) = \varphi(x,\lambda_0) + \psi(x,\lambda_0) M(\lambda_0) \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right],$$

we have

$$Z(x,\lambda_0)\begin{pmatrix} 0 & I_n \\ 0 & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right) = \chi(x,\lambda_0)\,\psi^*(t,\lambda_0),$$

and

$$Z(x,\lambda_0)\begin{pmatrix} 0 & 0 \\ I_n & M^* \end{pmatrix} Z^*\left(t,\overline{\lambda_0}\right) = \psi(x,\lambda_0)\,\chi^*(t,\lambda_0).$$

Hence

$$\mathcal{Z} = \int_a^c G(\lambda_0,x,t) B_1(t) F(t)\,dt + \int_c^b G(\lambda_0,x,t) B_1(t) F(t)\,dt,$$

where

$$G(\lambda_0,x,t) = \begin{cases} \chi(x,\lambda_0)\,\psi^*(t,\lambda_0), & a\le t\le x\le b,\ x\neq c,\ t\neq c \\ \psi(x,\lambda_0)\,\chi^*(t,\lambda_0), & a\le x\le t\le b,\ x\neq c,\ t\neq c. \end{cases}$$

Thus we obtain the following theorem.

Theorem 6.1. *The resolvent operator of L is given by the formula*

$$(L-\lambda I)^{-1} = \int_a^c G(\lambda,x,t) B_1(t) F(t)\,dt + \int_c^b G(\lambda,x,t) B_1(t) F(t)\,dt,$$

where $\operatorname{Im}\lambda \neq 0$ *and*

$$G(\lambda,x,t) = \begin{cases} \chi(x,\lambda)\,\psi^*(t,\lambda), & a\le t\le x\le b,\ x\neq c,\ t\neq c \\ \psi(x,\lambda)\,\chi^*(t,\lambda), & a\le x\le t\le b,\ x\neq c,\ t\neq c. \end{cases}$$

Theorem 6.2. *If $J\mathcal{Z}'(x) - B_2(x)\mathcal{Z}(x) = B_1(x)F(x)$, $B_1\mathcal{Z} = 0$ implies $\mathcal{Z} = 0$, then the set $\mathcal{D}$ is dense in $L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$.*

Proof. Suppose that the set $\mathcal{D}$ is not dense in $L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$. Then there exists a $G \in L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$ such that G is orthogonal to the set $\mathcal{D}$. Let $\mathcal{Y}$ satisfy $\mathcal{Y}\in\mathcal{D}$, $J\mathcal{Y}'(x) - B_2(x)\mathcal{Y}(x) = \overline{\lambda_0}B_1(x)\mathcal{Y}(x) + B_1(x)G(x)$ for $\operatorname{Im}\lambda_0 \neq 0$. Then for $\mathcal{Z}\in\mathcal{D}$, we have

$$0 = (\mathcal{Z},G) = \int_a^c G^* B_1 \mathcal{Z}\,dx + \int_c^b G^* B_1 \mathcal{Z}\,dx$$

$$
\begin{aligned}
&= \int_a^c \left[J\mathcal{Y}'(x) - B_2(x)\mathcal{Y}(x) - \overline{\lambda_0}B_1(x)\mathcal{Y}(x)\right]^* \mathcal{Z}dx \\
&\quad + \int_c^b \left[J\mathcal{Y}'(x) - B_2(x)\mathcal{Y}(x) - \overline{\lambda_0}B_1(x)\mathcal{Y}(x)\right]^* \mathcal{Z}dx \\
&= \int_a^c \mathcal{Y}^* \left[J\mathcal{Z}'(x) - B_2(x)\mathcal{Z}(x) - \lambda_0 B_1(x)\mathcal{Z}(x)\right] dx \\
&\quad + \int_c^b \mathcal{Y}^* \left[J\mathcal{Z}'(x) - B_2(x)\mathcal{Z}(x) - \lambda_0 B_1(x)\mathcal{Z}(x)\right] dx.
\end{aligned}
$$

Let $J\mathcal{Z}'(x) - B_2(x)\mathcal{Z}(x) - \lambda_0 B_1(x)\mathcal{Z}(x) = B_1(x)F(x)$. Then we get

$$
0 = (F, \mathcal{Y}) = \int_a^c \mathcal{Y}^* B_1 F dx + \int_c^b \mathcal{Y}^* B_1 F dx. \tag{6.2}
$$

Since F is arbitrary, we take $F = \mathcal{Y}$. From (6.2), we find that $\mathcal{Y} = 0$ which yields $B_1 G = 0$ and $G = 0$ in $L^2_{B_1}\left[(a,c)\cup(c,b);\mathbb{C}^{2n}\right]$. □

Theorem 6.3. *L is a self-adjoint operator.*

Proof. Let $L\mathcal{Z} - \lambda_0\mathcal{Z} = F$ and $L^*\mathcal{Z} - \overline{\lambda_0}\mathcal{Z} = H$ ($\operatorname{Im}\lambda_0 \neq 0$). Hence, we have

$$
\begin{aligned}
&\left((L-\lambda_0 I)^{-1} F, H\right) \\
&= \int_a^c H^*(x) B_1(x) \left[\int_a^c G(\lambda_0, x, t) B_1(t) F(t) dt\right] dx \\
&\quad + \int_c^b H^*(x) B_1(x) \left[\int_c^b G(\lambda_0, x, t) B_1(t) F(t) dt\right] dx \\
&= \int_a^c \left[\int_a^c G^*(\lambda_0, x, t) B_1(x) H(x) dx\right]^* B_1(t) F(t) dt \\
&\quad + \int_c^b \left[\int_c^b G^*(\lambda_0, x, t) B_1(x) H(x) dx\right]^* B_1(t) F(t) dt \\
&= \int_a^c \left[\int_a^c G\left(\overline{\lambda_0}, x, t\right) B_1(t) H(t) dt\right]^* B_1(x) F(x) dx \\
&\quad + \int_c^b \left[\int_c^b G\left(\overline{\lambda_0}, x, t\right) B_1(t) H(t) dt\right]^* B_1(x) F(x) dx \\
&= \left(F, \left(L - \overline{\lambda_0} I\right)^{-1} H\right),
\end{aligned}
$$

due to $G(\lambda_0, x, t) = G^*\left(\overline{\lambda_0}, x, t\right)$.

By using the formula

$$
\left((L-\lambda_0 I)^{-1} F, H\right) = \left(F, \left(L^* - \overline{\lambda_0} I\right)^{-1} H\right),
$$

we get $\left(L - \overline{\lambda_0} I\right)^{-1} = \left(L^* - \overline{\lambda_0} I\right)^{-1}$. This implies that $L = L^*$. □

Theorem 6.4. $(L-\lambda_0 I)^{-1}$ $(\operatorname{Im}\lambda_0 \neq 0)$ *is a bounded operator and*

$$
\left\|(L-\lambda_0 I)^{-1}\right\| \leq \frac{1}{|\operatorname{Im}\lambda_0|}.
$$

Proof. Let $(L-\lambda_0 I)\,\mathcal{Z} = F$. Then

$$(\mathcal{Z},F)-(F,\mathcal{Z})=\left(\lambda_0-\overline{\lambda_0}\right)(\mathcal{Z},\mathcal{Z})\,.$$

Using Schwartz's inequality, we get

$$2\left|\operatorname{Im}\lambda_0\right|\left\|\mathcal{Z}\right\|^2\leq 2\left\|\mathcal{Z}\right\|\left\|F\right\|.$$

Hence

$$\left\|(L-\lambda_0 I)^{-1}F\right\|\leq\frac{1}{\left|\operatorname{Im}\lambda_0\right|}\left\|F\right\|$$

yields the result. □

Theorem 6.5. *Let* $\chi(x,\lambda_0)=\varphi(x,\lambda_0)+\psi(x,\lambda_0)\,M(\lambda_0)$ $(\operatorname{Im}\lambda_0\neq 0)$. *Then we have*

$$\lim_{x\to b}\chi^*\left(x,\overline{\lambda_0}\right)J\chi(x,\lambda_0)=0.$$

Proof. Since

$$\begin{aligned}\chi^*\left(x,\overline{\lambda_0}\right)J\chi(x,\lambda_0)&=\begin{pmatrix}I_n & M^*(\lambda_0)\end{pmatrix}\mathcal{Z}^*\left(x,\overline{\lambda_0}\right)J\mathcal{Z}(x,\lambda)\begin{pmatrix}I_n\\ M(\lambda_0)\end{pmatrix}\\&=\begin{pmatrix}I_n & M^*(\lambda_0)\end{pmatrix}J\begin{pmatrix}I_n\\ M(\lambda_0)\end{pmatrix}=0,\end{aligned}$$

we get the desired result. □

References

[1] V. Ala and K. R. Mamedov, *Basisness of eigenfunctions of a discontinuous Sturm–Liouville operator*, J. Adv. Math. Stud., 2020, 13(1), 81–87.

[2] B. P. Allahverdiev and H. Tuna, *Discontinuous linear Hamiltonian systems*, Filomat, 2022, 36(3), 813–827.

[3] B. P. Allahverdiev and H. Tuna, *Singular Hahn–Hamiltonian systems*, Ufa Mathematical Journal, 2022 (In Press).

[4] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Acad. Press Inc., New York, 1964.

[5] K. Aydemir and O. S. Mukhtarov, *Generalized Fourier series as Green's function expansion for multi-interval Sturm–Liouville systems*, Mediterr. J. Math., 2017, 14(100), DOI: 10.1007/s00009-017-0901-2.

[6] D. Bainov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 66, Longman Scientific & Technical, Harlow, 1993.

[7] E. Bairamov and E. Uğurlu, *On the characteristic values of the real component of a dissipative boundary value transmission problem*, Appl. Math. Comput., 2012, 218, 9657–9663.

[8] H. Behncke and D. Hinton, *Two singular point linear Hamiltonian systems with an interface condition*, Math. Nachr., 2010, 283(3), 365–378.

[9] F. A. Cetinkaya and K. R. Mamedov, *A boundary value problem with retarded argument and discontinuous coefficient in the differential equation*, Azerb. J. Math., 2017, 7(1), 135–145.

[10] R. K. George, A. K. Nandakumaran and A. Arapostathis, *A note on controllability of impulsive systems*, J. Math. Anal. Appl., 2000, 241, 276–283.

[11] Z. Guan, G. Chen and T. Ueta, *On impulsive control of a periodically forced chaotic pendulum system*, IEEE Trans. Automat. Control, 2000, 45, 1724–1727.

[12] G. S. Guseinov, *Boundary value problems for nonlinear impulsive Hamiltonian systems*, J. Comput. Appl. Math., 2014, 259, 780–789

[13] G. S. Guseinov, *On the impulsive boundary value problems for nonlinear Hamiltonian systems*, Math. Meth. Appl. Sci., 2016, 39, 4496–4503.

[14] D. B. Hinton and J. K. Shaw, *On Titchmarsh–Weyl $M(\lambda)$-functions for linear Hamiltonian systems*, J. Differ. Equat., 1981, 40(3), 316–342.

[15] D. B. Hinton and J. K. Shaw, *Titchmarsh–Weyl theory for Hamiltonian systems*, *Spectral theory of differential operators*, Birmingham, Ala. (1981), North-Holland Math. Stud., North-Holland, Amsterdam-New York, 1981, 55, 219–231.

[16] D. B. Hinton and J. K. Shaw, *Parameterization of the $M(\lambda)$ function for a Hamiltonian system of limit circle type*, Proc. Roy. Soc. Edinburgh Sect. A, 1983, 93(3-4), 349–360.

[17] D. B. Hinton and J. K. Shaw, *Hamiltonian systems of limit point or limit circle type with both endpoints singular*, J. Differ. Equat., 1983, 50, 444–464.

[18] A. M. Krall, *Hilbert Space, Boundary Value Problems and Orthogonal Polynomials*, Birkhäuser Verlag, Basel, 2002.

[19] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with one singular point*, SIAM J. Math. Anal., 1989, 20(3), 664–700.

[20] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with two singular points*, SIAM J. Math. Anal., 1989, 20(3), 701–715.

[21] F. R. Lapwood and T. Usami, *Free Oscillations of the Earth*, Cambridge University Press, Cambridge, 1981.

[22] A. Lakmeche and O. Arino, *Bifurcation of nontrivial periodic solutions of impulsive differential equations arising from chemotherapeutic treatment*, Dyn. Contin. Discrete Impuls. Syst., 2000, 7, 265–287.

[23] S. Lenci and G. Rega, *Periodic solutions and bifurcations in an impact inverted pendulum under impulsive excitation*, Chaos Solitons Fractals, 2000, 11, 2453–2472.

[24] A. V. Likov and Y. A. Mikhailov, *The Theory of Heat and Mass Transfer*, Translated from Russian by I. Shechtman, Israel Program for Scientific Translations, Jerusalem, 1965.

[25] O. N. Litvinenko and V. I. Soshnikov, *The Theory of Heterogenous Lines and their Applications in Radio Engineering*, Radio, Moscow, 1964 (in Russian).

[26] K. R. Mamedov, *Spectral expansion formula for a discontinuous Sturm–Liouville problem*, Proc. Inst. Math. Mech., Natl. Acad. Sci. Azerb., 2014, 40, 275–282 .

[27] K. R. Mamedov, *On an inverse scattering problem for a discontinuous Sturm–Liouville equation with a spectral parameter in the boundary condition*, Bound. Value Probl., 2010, Article ID: 171967, 1–17.

[28] K. R. Mamedov and N. Palamut, *On a direct problem of scattering theory for a class of Sturm–Liouville operator with discontinuous coefficient*, Proc. Jangjeon Math. Soc., 2009, 12(2), 243–251.

[29] O. S. Mukhtarov, *Discontinuous boundary-value problem with spectral parameter in boundary conditions*, Turkish J. Math., 1994, 18, 183–192.

[30] O. S. Mukhtarov and K. Aydemir, *The eigenvalue problem with interaction conditions at one interior singular point*, Filomat, 2017, 31(17), 5411–5420.

[31] O. S. Mukhtarov, H. Olğar and K. Aydemir, *Resolvent operator and spectrum of new type boundary value problems*, Filomat, 2015, 29(7), 1671–1680.

[32] S. I. Nenov, *Impulsive controllability and optimization problems in population dynamics*, Nonlinear Anal., 1999, 36, 881–890.

[33] H. Olğar and O. S. Mukhtarov, *Weak eigenfunctions of two-Interval Sturm–Liouville problems together with interaction conditions*, J. Math. Phys., 2017, 58, 042201, DOI: 10.1063/1.4979615.

[34] Y. Yalcin, L. G. Sümer and S. Kurtulan, *Discrete-time modeling of Hamiltonian systems*, Turkish J. Electric. Eng. Comput. Sci., 2015, 23(1), 149–170.