Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1403–1417
Website:http://www.jaac-online.com
DOI:10.11948/20210166

RELAXED MODULUS-BASED SYNCHRONOUS MULTISPLITTING MULTI-PARAMETERS TOR (TWO-PARAMETERS OVER-RELAXATION) METHODS FOR LINEAR COMPLEMENTARITY PROBLEMS

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Abstract In 2013, Bai and Zhang [*Numerical Linear Algebra with Applications*, 20(2013), 425–439] constructed modulus-based synchronous multisplitting methods by an equivalent reformulation of the linear complementarity problems into a system of fixed-point equations and studied the convergence of them. In 2014, Zhang and Li [*Computers and Mathematics with Application*, 67(2014), 1954–1959] analyzed and obtained the weaker convergence results for linear complementarity problems. In 2008, Zhang et.al. [*International Journal of Computer Mathematics*, 85(2), 2008, 211–224] presented global relaxed non-stationary multisplitting multi-parameter method by introducing some relaxed parameters. In this paper, we generalize Bai and Zhang's methods and study relaxed modulus-based synchronous multisplitting multi-parameters TOR (two-parameters over-relaxation, abbreviated as TOR) methods for linear complementarity problems. Furthermore, the convergence results of our new method in this paper are given when the system matrix is an H_+-matrix.

Keywords Modulus-based method, linear complementarity, successive relaxation, H_+-matrix.

MSC(2010) 65F10, 65F50, 90C33, 65G40.

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1. Introduction

Consider the linear complementarity problems, abbreviated as LCP(q, A), for finding a pair of real vectors r and $z \in R^n$ such that

$$r := Az + q \geq 0, z \geq 0 \text{ and } z^T(Az + q) = 0, \tag{1.1}$$

where $A = (a_{ij}) \in R^{n\times n}$ is a given large, sparse and real matrix and $q = (q_1, q_2, ..., q_n)^T \in R^n$ is a given real vector. Here, z^T and $\geq$ denote the transpose of the vector z and the componentwise partial ordering between two vectors, respectively.

Many problems in scientific computing and engineering applications may lead to solutions of LCPs of the form (1.1). For example, the linear complementarity problems may arise from application problems such as the convex quadratic programming, the Nash equilibrium point of the bimatrix game, the free boundary problems of fluid dynamics etc. (e.g. see [12, 18, 23] and the references therein). Some solvers for LCP(q, A) with a special matrix A were proposed [2–8, 13–15, 17, 20, 31]. Recently, many people have focused the solvers of LCP(q, A) with an algebra equation [7, 8, 17, 20, 31, 35]. Bai introduced modulus-based matrix splitting methods [7], and Bai and Zhang proposed modulus-based matrix multisplitting methods for linear complementarity problems [1,8] . Zhang and Ren [35] extended the condition of a compatible $H-$splitting to that of an H-splitting. Li [21] extended the modulus-based matrix splitting iteration method to more general cases. Zhang et al. [37–39] gave the weaker convergence results when the system matrix is an H_+-matrix. Dai et al. [16] provided a comparison theorem between preconditioned two-step modulus-based Gauss-Seidel (PTMGS) iteration method and two-step modulus-based Gauss-Seidel (TMGS) iteration method, which shows that PTMGS method improves the convergence rate of original TMGS method for linear complementarity problem. Mezzadri et al. [22] analyzed the solution of horizontal linear complementarity problems arising from finite-difference discretizations of differential problems. Ren et al. [26, 27] considered the preconditioned general two-step modulus-based matrix splitting iteration method and the general two-sweep modulus-based matrix splitting iteration method for linear complementarity problems of H_+-matrices. Wen et al. [30] studied the relaxation modulus-based matrix splitting iteration method for solving linear complementarity problems of positive definite matrices. Wu et al. [32] provided the preconditioned general modulus-based matrix splitting iteration method for linear complementarity problems of H-matrices. Zhang et al. [34] studied the modified modulus-based multigrid method for linear complementarity problems arising from free boundary problems.

The rest of this paper is organized as follows: In section 2, we give some notations and lemmas. In section 3, we propose relaxed modulus-based synchronous multisplitting multi-parameters TOR (two-parameters over-relaxation, abbreviated as TOR) method for solving LCP(q, A). For more information about the TOR method, readers can refer to the literature [21]. In section 4, we give the convergence analysis for the proposed method.

2. Notations and Lemmas

A matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of matrix $A = (a_{ij})$ is defined by: $\alpha_{ij} = |a_{ij}|$, if $i = j; \alpha_{ij} = -|a_{ij}|$, if $i \neq j$. A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix and is called an *H_+-matrix* if it is an H-matrix with positive diagonal entries [2, 11, 30]. Let $\rho(A)$ denote the spectral radius of A. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. Let A and B be M-matrices. If $A \leq B$, then $A^{-1} \geq B^{-1}$. Let A be an H-matrix, and $A = D - B, D = \text{diag}(A)$, then $\rho(|D|^{-1}|B|) < 1$. Moreover, D is nonsingular. Finally, we define by $R^n_+ = \{x|x \geq 0, x \in R^n\}$.

Lemma 2.1.([19]) *Let A be an H-matrix. Then A is nonsingular, and $|A^{-1}| \leq \langle A\rangle^{-1}$.*

Lemma 2.2.([29]) *Let $H^{(1)}, H^{(2)}, ..., H^{(l)}$...be a sequence of nonnegative matrices in $R^{n\times n}$. If there exist a real number $0 \leq \theta < 1$, and a vector $\nu > 0$ in R^n, such that*

$$H^{(l)}\nu \leq \theta\nu,\ \ l = 1, 2, ...$$

then $\rho(K_l) \leq \theta^l < 1$, where $K_l = H^{(l)}H^{(l-1)}...H^{(1)}$, and therefore $\lim\limits_{l\to\infty} K_l = 0$.

Lemma 2.3. ([34]) *Let $A = (a_{ij}) \in Z^{n\times n}$ which has all positive diagonal entries. A is an M-matrix if and only if $\rho(B) < 1$, where $B = D^{-1}C,\ D = \text{diag}(A),\ A = D-C$.*

Lemma 2.4. ([4, 6, 9]) *$A \in R^{n\times n}$ be an H_+-matrix. Then, the LCP(q, A) has a unique solution for any $q \in R^n$.*

Lemma 2.5. ([7]) *Let $A = M - N$ be a splitting of the matrix $A \in R^{n\times n}, \Omega$ be a positive diagonal matrix, and γ a positive constant. Then, for the LCP(q, A) the following statements hold true:*

(i) if (z, r) is a solution of the LCP(q, A), then $x = \frac{1}{2}\gamma(z - \Omega^{-1}r)$ satisfies the implicit fixed-point equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q; \tag{2.1}$$

(ii) if x satisfies the implicit fixed-point equation (2), then

$$z = \gamma^{-1}(|x| + x) \text{ and } \mathrm{r} = \gamma^{-1}\Omega(|\mathrm{x}| - \mathrm{x}) \tag{2.2}$$

is a solution of the LCP(q, A).

3. RMSMMTOR methods

To suit computational requirements of the modern high-speed multiprocessor environments, in this paper, we further present synchronous parallel multisplitting multi-parameters TOR for the modulus-based splitting iteration methods in [7] by making use of multiple splittings of the system matrix A [10, 26]. This class of modulus-based synchronous multisplitting (MSM) iteration methods is advantageous over the synchronous multisplitting iteration methods discussed by Machida et al. [25] and by Bai [2], as at each iteration step, it only needs to solve subsystems of linear equations rather than linear complementarity subproblems (see also [6, 9]).

At first, we introduce the concept of multisplitting method and the detailed process of parallel iterative method.

$\{M_k, N_k, E_k\}_{k=1}^{l}$ is a *multisplitting* of A if
1) $A = M_k - N_k$ is a splitting for $k = 1, 2, ..., l$;
2) $E_k \geq 0$ is a nonnegative diagonal matrix, called weighting matrix;
3) $\sum_{k=1}^{l} E_k = I$, where I is the identity matrix.

Given a positive diagonal matrix Ω and a positive constant γ, from Lemma 2.5, we know that if x satisfies either of the implicit fixed-point equations

$$(\Omega + M_k)x = N_k x + (\Omega - A)|x| - \gamma q, k = 1, 2, ..., l, \tag{3.1}$$

then

$$z = \gamma^{-1}(|x| + x) \text{ and } \mathrm{r} = \gamma^{-1}\Omega(|\mathrm{x}| - \mathrm{x}) \tag{3.2}$$

is a solution of the LCP(q, A).

Let

$$A = D - L_k - F_k - U_k,\ k = 1, 2, ..., l,$$

where $D = \mathrm{diag}(A), L_k$ and F_k are strictly lower triangular, and U_k are such that $A = D - L_k - F_k - U_k$, then $(D - L_k - F_k, U_k, E_k)$ is a multisplitting of A. With the equivalent reformulations (3.1), (3.2) and TOR method of the LCP(q, A), we can establish the following relaxed modulus-based synchronous multisplitting multi-parameters TOR method (RMSMMTOR), which is similar to Method 3.1 in [1].

Method 3.1. (The RMSMMTOR method for LCP(q, A))
Let $(M_k, N_k, E_k)(k = 1, 2, ...l)$ be a multisplitting of the system matrix $A \in R^{n\times n}$. Given an initial vector $x^{(0)} \in R^n$ for $m = 0, 1, ...$ until the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty} \subset R^n_+$ is convergent, compute $z^{(m+1)} \in R^n_+$ by

$$z^{(m+1)} = \frac{1}{\gamma}(|x^{(m+1)}| + x^{(m+1)})$$

and $x^{(m+1)} \in R^n$ according to

$$x^{(m+1)} = \omega \sum_{k=1}^{l} E_k x^{(m,k)} + (1 - \omega)x^{(m)},$$

where $x^{(m,k)}, k = 1, 2, ..., l$, are obtained by solving the linear systems

$$\begin{aligned}&[\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k)]x^{(m,k)}\\ =&[(1 - \alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k + \alpha_k U_k]x^{(m)} + \alpha_k[(\Omega - A)|x^{(m)}| - \gamma q],\\ &k = 1, 2, ..., l,\end{aligned} \tag{3.3}$$

respectively.

Remark 3.1. In Method 3.1, when $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = \gamma, \omega = 1$, the RMSMM-TOR method reduces to the MSMTOR method; when $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = \gamma$, the RMSMMTOR method reduces to the RMSMTOR method; When $\gamma_k = 0, \omega = 1$, the RMSMMTOR method reduces to the MSMMAOR method; When $\gamma_k = 0$, the RMSMMTOR method reduces to the RMSMMAOR method; When $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = 0, \omega = 1$, the RMSMMTOR method reduces to the MSMAOR method; When $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = 0$, the RMSMMTOR method reduces to the RMS-MAOR method.

4. Convergence analysis

Based relaxed modulus-based synchronous multisplitting multi-parameter TOR method, we will present a weaker convergence results of the multisplitting methods for the linear complementarity problem for H_+-matrix, which is as follows:

Theorem 4.1. *Let $A \in R^{n\times n}$ be an H_+-matrix, with $D = diag(A)$ and $B = D - A$, and let $(M_k, N_k, E_k)(k = 1, 2, ..., l)$ and $(D - L_k - F_k, U_k, E_k)(k = 1, 2, ..., l)$ be a multisplitting and a triangular multisplitting of the matrix A, respectively. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If $A = D - L_k - F_k - U_k(k = 1, 2, ..., l)$ satisfies $\langle A\rangle = D - |L_k| - |F_k| - |U_k|(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the RMSMMTOR iteration method converges to the unique solution z_* of $LCP(q, A)$ for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α_k and β_k, ω satisfy*

$$\begin{aligned} &0 < \beta_k,\ \gamma_k \leq \alpha_k \leq 1,\ 0 < \omega < \frac{2}{1+\rho'} \\ \text{or}\quad &0 < \beta_k,\ \gamma_k < \frac{1}{\rho(D^{-1}|B|)},\ 1 < \alpha_k < \frac{1}{\rho(D^{-1}|B|)},\ 0 < \omega < \frac{2}{1+\rho'}, \end{aligned} \tag{4.1}$$

where $\rho = \rho(J), J = D^{-1}|B|$, $\rho' = \max_{1\leq k\leq l}\{1 - 2\alpha_k + 2\alpha_k\rho_\epsilon, 2\delta_k\rho_\epsilon - 1, 2\alpha_k\rho_\epsilon - 1\}$, $\delta_k = \max\{\beta_k, \gamma_k\}$. Moreover, β_k, γ_k should be greater than or less than α_k at once.

Proof. From Lemma 2.3 and (3.3), for the RMSMMTOR method, it holds that

$$\begin{aligned} &(\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))x_* \\ =&[(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k + \alpha_k U_k]x_* + \alpha_k[(\Omega - A)|x_*| - \gamma q], \\ &k = 1, 2, ..., l, \end{aligned} \tag{4.2}$$

By subtracting (4.2) from (3.3), we have

$$\begin{aligned} x^{(m,k)} - x_* =&(\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1} \\ &\times [(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k + \alpha_k U_k](x^{(m)} - x_*) \\ &+ (\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1}\alpha_k(\Omega - A)(|x^{(m)}| - |x_*|), \\ k = 1, 2, ..., l,& \end{aligned}$$

which immediately results in the error about the RMSMMTOR method as follows:

$$\begin{aligned} x^{(m+1)} - x_* =&\omega\sum_{k=1}^{l} E_k(\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1} \\ &\times [(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k + \alpha_k U_k](x^{(m)} - x_*) \\ &+ \omega\sum_{k=1}^{l} E_k(\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1} \\ &\times \alpha_k(\Omega - A)(|x^{(m)}| - |x_*|) + (1-\omega)(x^{(m)} - x_*). \end{aligned}$$

The above error relationship is the base for proving the convergence of RMSMMTOR method. By taking absolute values on both sides of the above equality, making use of Lemma 2.1 and estimate $||x^{(m)}|-|x_*|| \leq |x^{(m)}-x_*|$, defining $\epsilon^{(m)} = x^{(m)}-x_*$ and arranging similar terms together, we can obtain

$$|\epsilon^{(m)}| = |x^{(m+1)} - x_*| \leq \mathcal{T}_{RMSMMTOR}|x^{(m)} - x_*|, \tag{4.3}$$

where

$$\begin{aligned}\mathcal{T}_{RMSMMTOR} =&\omega \sum_{k=1}^{l} E_k H_{RMSMMTOR} + |1-\omega| I \\ =&\omega \sum_{k=1}^{l} E_k (\alpha_k \Omega + D - (\beta_k |L_k| + \gamma_k |F_k|))^{-1} \\ &\times [|1-\alpha_k| D + |\alpha_k - \beta_k||L_k| + |\alpha_k - \gamma_k||F_k| \\ &+ \alpha_k |U_k| + \alpha_k |\Omega - A|] + |1-\omega| I.\end{aligned} \tag{4.4}$$

Case 1: Let $0 < \beta_k, \gamma_k \leq \alpha_k \leq 1, 0 < \omega < \frac{2}{1+\rho'}$. For this subcase, the matrix M_k, N_k read as

$$\begin{aligned}M_k &= \alpha_k \Omega + D - (\beta_k |L_k| + \gamma_k |F_k|), \\ N_k &= (1-\alpha_k) D + (\alpha_k - \beta_k)|L_k| + (\alpha_k - \gamma_k)|F_k| + \alpha_k |U_k| + \alpha_k |\Omega - A|.\end{aligned} \tag{4.5}$$

By (4.5) and $|\Omega - A| = (\Omega - D) + |B|, |B| = |L_k| + |F_k| + |U_k|, k = 1, 2, ..., l$, we have $N_k = M_k - 2\alpha_k D + 2\alpha_k |B|$. So

$$H_{RMSMMTOR} = M_k^{-1} N_k = M_k^{-1}(M_k - 2\alpha_k D + 2\alpha_k |B|) = I_k - 2\alpha_k M_k^{-1}(D - |B|).$$

So

$$\begin{aligned}|H_{RMSMMTOR}| &\leq M_k^{-1}[M_k - 2\alpha_k (D - |B|)] \\ &\leq I - 2\alpha_k M_k^{-1} D(I - D^{-1}|B|).\end{aligned}$$

Let e denote the vector $e = (1, 1, ..., 1)^T \in R^n$. Since J is nonnegative, the matrix $J + \epsilon e e^T$ has only positive entries and irreducible for any $\epsilon > 0$. By the Perron-Frobenius theorem for any $\epsilon > 0$, there is a vector $x_\epsilon > 0$ such that

$$(J + \epsilon e e^T) x_\epsilon = \rho_\epsilon x_\epsilon,$$

where $\rho_\epsilon = \rho(J + \epsilon e e^T) = \rho(J_\epsilon)$. Moreover, if $\epsilon > 0$ is small enough, we have $\rho_\epsilon < 1$ by continuity of the spectral radius. Because of $0 < \alpha_k \leq 1$, we also have $1 - 2\alpha_k + 2\alpha_k \rho < 1$, and $1 - 2\alpha_k + 2\alpha_k \rho_\epsilon < 1$. So

$$\begin{aligned}|H_{RMSMMTOR}| &\leq I - 2\alpha_k M_k^{-1} D[I - (D^{-1}|B| + \epsilon e e^T)] \\ &= I - 2\alpha_k M_k^{-1} D[I - J_\epsilon].\end{aligned}$$

Multiplying x_ϵ in two sides of the above inequality, and $M_k^{-1} \geq D^{-1}$, we can obtain

$$\begin{aligned}|H_{RMSMMTOR}| x_\epsilon &\leq x_\epsilon - 2\alpha_k M_k^{-1} D[1 - \rho(J_\epsilon)] x_\epsilon \\ &\leq x_\epsilon - 2\alpha_k D^{-1} D[1 - \rho(J_\epsilon)] x_\epsilon \\ &= (1 - 2\alpha_k + 2\alpha_k \rho(J_\epsilon)) x_\epsilon.\end{aligned}$$

By (4.1), we have

$$\begin{aligned}|\mathcal{T}_{RMSMMTOR}|x_\epsilon &\le \omega\Sigma_{k=1}^{l}E_k(1-2\alpha_k+2\alpha_k\rho(J_\epsilon))x_\epsilon+|1-\omega|x_\epsilon\\ &\le \omega(1-2\alpha_k+2\alpha_k\rho_\epsilon)x_\epsilon+|1-\omega|x_\epsilon\\ &=(\omega\rho_1+|1-\omega|)x_\epsilon\\ &=\theta_1x_\epsilon(\epsilon\to 0),\end{aligned}$$

where $\theta_1=\omega\rho_1+|1-\omega|<1$.

So, the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the RMSMMTOR iteration method converges to the unique solution z_* of LCP(q,A) for any initial vector $z^{(0)}\in R_+^n$, provided the relaxation parameters α_k and β_k,ω satisfy $0<\beta_k,\gamma_k\le\alpha_k\le 1, 0<\omega<\frac{2}{1+\rho'}$.

Case 2: $0<\beta_k,\gamma_k<\frac{1}{\rho(D^{-1}|B|)}, 1<\alpha_k<\frac{1}{\rho(D^{-1}|B|)}, 0<\omega<\frac{2}{1+\rho'}$.

Subcase 1: $\alpha_k\ge\beta_k$ and $\alpha_k\ge\gamma_k$. For this subcase, the matrix M_k,N_k read as

$$\begin{aligned}M_k&=\alpha_k\Omega+D-(\beta_k|L_k|+\gamma_k|F_k|),\\ N_k&=(\alpha_k-1)D+(\alpha_k-\beta_k)|L_k|+(\alpha_k-\gamma_k)|F_k|+\alpha_k|U_k|+\alpha_k|\Omega-A|\\ &=M_k-2D+2\alpha_k|B|.\end{aligned}\tag{4.6}$$

So

$$\begin{aligned}|H_{RMSMMTOR}|&\le M_k^{-1}[M_k-2(D-\alpha_k|B|)]\\ &\le I-2M_k^{-1}D(I-\alpha_kD^{-1}|B|).\end{aligned}$$

Similar to the *Case 1*, let e denote the vector $e=(1,1,...,1)^T\in R^n$, and $x_\epsilon>0$ such that $J_\epsilon x_\epsilon=(J+\epsilon ee^T)x_\epsilon=\rho(J_\epsilon)x_\epsilon$. Moreover, if $\epsilon>0$ is small enough, we have $\rho_\epsilon<1$ by continuity of the spectral radius. Because of $1<\alpha_k<\frac{1}{\rho(D^{-1}|B|)}$, we also have

$$2\alpha_k\rho-1<1 \text{ and } 2\alpha_k\rho_\epsilon-1<1.$$

So

$$\begin{aligned}|H_{RMSMMTOR}|&\le I-2M_k^{-1}D[I-\alpha_k(D^{-1}|B|+\epsilon ee^T)]\\ &=I-2M_k^{-1}D[I-\alpha_kJ_\epsilon].\end{aligned}$$

Multiplying x_ϵ in two sides of the above inequality, and $M_k^{-1}\ge D^{-1}$, we can obtain

$$\begin{aligned}|H_{RMSMMAOR}|x_\epsilon&\le x_\epsilon-2M_k^{-1}D[1-\alpha_k\rho(J_\epsilon)]x_\epsilon\\ &\le x_\epsilon-2(1-\alpha_k\rho(J_\epsilon))]x_\epsilon\\ &=(2\alpha_k\rho(J_\epsilon)-1)x_\epsilon.\end{aligned}$$

By (4.1), we have

$$\begin{aligned}|\mathcal{T}_{RMSMMTOR}|x_\epsilon&\le\omega\Sigma_{k=1}^{l}E_k(2\alpha_k\rho(J_\epsilon)-1)x_\epsilon+|1-\omega|x_\epsilon\\ &\le\omega(2\alpha_k\rho_\epsilon-1)x_\epsilon+|1-\omega|x_\epsilon\\ &=(\omega\rho_2+|1-\omega|)x_\epsilon\\ &=\theta_2x_\epsilon(\epsilon\to 0),\end{aligned}$$

where $\theta_2 = \omega\rho_2 + |1-\omega| < 1$.

So, the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the RMSMMTOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R_+^n$, provided the relaxation parameters $\alpha_k \geq \beta_k$ and $\alpha_k \geq \gamma_k$.

Subcase 2: $\alpha_k \leq \beta_k$ and $\alpha_k \leq \gamma_k$. For this subcase, the matrix M_k, N_k read as

$$\begin{aligned} M_k &= \alpha_k\Omega + D - (\beta_k|L_k| + \gamma_k|F_k|),\\ N_k &= (\alpha_k - 1)D + (\beta_k - \alpha_k)|L_k| + (\gamma_k - \alpha_k)|F_k| + \alpha_k|U_k| + \alpha_k|\Omega - A|\\ &= M_k - 2D + 2\beta_k|L_k| + 2\gamma_k|F_k| + 2\alpha_k|U_k|\\ &\leq M_k - 2D + 2\delta_k|B|. \end{aligned} \tag{4.7}$$

where $\delta_k = \max\{\beta_k, \gamma_k\}$. So

$$\begin{aligned} |H_{RMSMMTOR}| &\leq M_k^{-1}[M_k - 2(D - \delta_k|B|)]\\ &\leq I - 2M_k^{-1}D(I - \delta_k D^{-1}|B|). \end{aligned}$$

Similar to the *Case 1*, let e denote the vector $e = (1, 1, ..., 1)^T \in R^n$, and $x_\epsilon > 0$ such that $J_\epsilon x_\epsilon = (J + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon$. Moreover, if $\epsilon > 0$ is small enough, we have $\rho_\epsilon < 1$ by continuity of the spectral radius. Because of $0 < \beta_k, \gamma_k < \frac{1}{\rho(D^{-1}|B|)}$, we also have

$$2\delta_k\rho - 1 < 1 \text{ and } 2\delta_k\rho_\epsilon - 1 < 1.$$

So

$$\begin{aligned} |H_{RMSMMTOR}| &\leq I - 2M_k^{-1}D[I - \delta_k(D^{-1}|B| + \epsilon ee^T)]\\ &= I - 2M_k^{-1}D[I - \delta_k J_\epsilon]. \end{aligned}$$

Multiplying x_ϵ in two sides of the above inequality, and $M_k^{-1} \geq D^{-1}$, we can obtain

$$|H_{RMSMMTOR}|x_\epsilon \leq x_\epsilon - 2(1 - \delta_k\rho(J_\epsilon))]x_\epsilon = (2\delta_k\rho(J_\epsilon) - 1)x_\epsilon$$

By (4.1), we have

$$\begin{aligned} |\mathcal{T}_{RMSMMTOR}|x_\epsilon &\leq \omega\Sigma_{k=1}^{l} E_k(2\delta_k\rho(J_\epsilon) - 1)x_\epsilon + |1-\omega|x_\epsilon\\ &\leq \omega(2\delta_k\rho_\epsilon - 1)x_\epsilon + |1-\omega|x_\epsilon\\ &= (\omega\rho_3 + |1-\omega|)x_\epsilon\\ &= \theta_3 x_\epsilon(\epsilon \to 0), \end{aligned}$$

where $\theta_3 = \omega\rho_3 + |1-\omega| < 1$.

So, the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the RMSMMTOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R_+^n$, provided the relaxation parameters $\alpha_k \leq \beta_k$ and $\alpha_k \leq \gamma_k$.

Remark 4.1. Obviously, we can find that the conditions of Theorem 4.4 in this paper are weaker than those of Theorem 2.3 in [41]. Moreover, the parameters can be adjusted suitably so that the convergence property of method can be substantially improved. That is to say, we have more choices for the splitting $A = B - C$ which makes the multisplitting iteration methods converge. Therefore, our convergence theories extend the scope of multisplitting iteration methods in applications.

Remark 4.2. In this paper, RMSMMTOR method is also the generalization of MSMAOR method in [1] and MSMMAOR method in [38]. The readers can refer to the following three theorems.

Theorem 4.2. ([1]) *Let $A \in R^{n\times n}$ be an H_+-matrix, with $D =$diag(A) and $B = D - A$, and let $(M_k, N_k, E_k)(k = 1, 2, ..., l)$ and $(D - L_k, U_k, E_k)(k = 1, 2, ..., l)$ be a multisplitting and a triangular multisplitting of the matrix A, respectively. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If $A = D - L_k - U_k(k = 1, 2, ..., l)$ satisfies $\langle A\rangle = D - |L_k| - |U_k|(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the MSMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α and β satisfy*

$$0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}.$$

Theorem 4.3. ([38]) *Let $A \in R^{n\times n}$ be an H_+-matrix, with $D =$diag(A) and $B = D - A$, and let $(M_k, N_k, E_k)(k = 1, 2, ..., l)$ and $(D - L_k, U_k, E_k)(k = 1, 2, ..., l)$ be a multisplitting and a triangular multisplitting of the matrix A, respectively. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If $A = D - L_k - U_k(k = 1, 2, ..., l)$ satisfies $\langle A\rangle = D - |L_k| - |U_k|(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the MSMMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α_k and β_k satisfy*

$$0 < \beta_k \leq \alpha_k \leq 1 \text{ or } 0 < \beta_k < \frac{1}{\rho(D^{-1}|B|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}|B|)}.$$

Theorem 4.4. ([38]) *Let A be an H-matrix, and for $k = 1, 2, ..., l$, L_k and F_k be strictly lower triangular matrices. Define the matrix $U_k, k = 1, 2, ..., l$, such that $A = D - L_k - F_k - U_k$. Assume that we have $\langle A\rangle = |D| - |L_k| - |F_k| - |U_k| = |D| - |B|$. If*

$$0 \leq \beta_k \leq \gamma_k, 0 \leq \alpha_k \leq \gamma_k, 0 < \gamma_k < \frac{2}{1+\rho}, 0 < \omega < \frac{2}{1+\rho_{\gamma_k}},$$

then GRNMMTOR method converges for any initial vector $x^{(0)}$, where $\rho = \rho(J), J = |D|^{-1}|B|, \rho_{\gamma_k} = \max_{1\leq k\leq\alpha} \{|1 - \gamma_k| + \gamma_k\rho_\epsilon\}, q(m, k) \geq 1, m = 0, 1, ..., k = 1, 2, ..., l$.

Remark 4.3. From Table 1, we obviously see that the MSMMAOR method in [1] uses the same parameters α, β in different processors, but the RMSMMTOR method in this paper uses different parameters $\alpha_k, \beta_k, \gamma_k(k = 1, 2, ..., l)$ in different processors. In RMSMMTOR method, we may choose proper E_k to balance the load of each processor and avoid synchronization.

5. Numerical experiments

In this section, numerical examples are used to illustrate the feasibility and effectiveness of the relaxed modulus-based synchronous multisplitting multi-parameter

Table 1. The relaxed modulus-based synchronous multisplitting multi-parameter method and corresponding convergence results.

Method	$\alpha_k, \beta_k, \omega$	Description	Ref
MSMJ	$\alpha_k = 1, \beta_k = 0, \omega = 1$	Modulus-based synchronous multisplitting Jacobi method	[1]
MSMGS	$\alpha_k = \beta_k = 1, \omega = 1$	Modulus-based synchronous multisplitting Gauss-Seidel method	[1]
MSMSOR	$0 < \alpha(\alpha_k) = \beta(\beta_k) < \frac{1}{\rho(D^{-1}\|B\|)}, \omega = 1$	Modulus-based synchronous multisplitting SOR method	[1]
MSMAOR	$0 < \beta(\beta_k) \le \alpha(\alpha_k) < \frac{1}{\rho(D^{-1}\|B\|)}, \omega = 1$	Modulus-based synchronous multisplitting AOR method	[1]
MSMAOR	$0 < \beta(\beta_k) \le \alpha(\alpha_k) < \frac{1}{\rho(D^{-1}\|B\|)}$	Modulus-based synchronous multisplitting AOR method	[1]
MSMMAOR	$\omega = 1, 0 < \beta_k \le \alpha_k \le 1$ or $0 < \beta_k < \frac{1}{\rho(D^{-1}\|B\|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}\|B\|)}$	Modulus-based synchronous multisplitting multi-parameters AOR method	[34]
RMSMMTOR	$0 < \beta_k, \gamma_k \le \alpha_k \le 1, 0 < \omega < \frac{2}{1+\rho'}$ or $0 < \beta_k, \gamma_k < \frac{1}{\rho(D^{-1}\|B\|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}\|B\|)}$ $0 < \omega < \frac{2}{1+\rho'}$ where $\rho' = \max_{1\le k\le l}\{1 - 2\alpha_k + 2\alpha_k\rho_\epsilon,$ $2\delta_k\rho_\epsilon - 1, 2\alpha_k\rho_\epsilon - 1\}, \delta_k = \max\{\beta_k, \gamma_k\}$	Relaxed modulus-based synchronous multisplitting multi-parameter TOR method	this paper

AOR methods (RMSMMAOR) ($F = U$) in terms of iteration count (denoted by 'IT') and computing time (denoted by 'CPU'), and norm of absolute residual vectors (denoted by 'RES'). Here, 'RES' is defined as

$$\mathrm{RES}(z^{(k)}) := \|\min(Az^{(k)} + q, z^{(k)})\|_2$$

where $z^{(k)}$ is the kth approximate solution to the LCP(q, A) and the minimum is taken componentwise in [10].

All initial vectors are chosen to be

$$x^{(0)} = (1, 0, 1, 0, \cdots, 1, 0, \cdots)^T \in R^n.$$

In the table, α, β denote the iteration parameters in the relaxed modulus-based synchronous multisplitting multi-parameter AOR methods (RMSMMAOR) and the modulus-based synchronous multisplitting multi-parameter methods (MSMAOR). In addition, we take $\Omega = \frac{1}{2\alpha}D$ in [11] for RMSMMAOR and MSMAOR methods.

Let m be a prescribed positive integer and $n = m^2$. Consider the LCP(q, A), in which $A \in R^{n\times n}$ is given by $A = \hat{A} + \mu I$ and $q \in R^n$ is given by $q = -Mz^*$ where

$$\hat{A} = \mathrm{tridiag}(-rI, S, -tI) = \begin{pmatrix} S & -tI & 0 & \cdots & 0 & 0 \\ -rI & S & -tI & \cdots & 0 & 0 \\ 0 & -rI & S & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & S & -tI \\ 0 & 0 & \cdots & \cdots & -rI & S \end{pmatrix} \in R^{n\times n} \tag{5.1}$$

is a block-tridiagonal matrix,

$$S = \text{tridiag}(-1,4,-1) = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 4 & -1 \\ 0 & 0 & \cdots & \cdots & -1 & 4 \end{pmatrix} \in R^{n\times n} \tag{5.2}$$

is a tridianonal matrix, and

$$z^* = (1,2,1,2,\cdots,1,2,\cdots)^T \in R^n$$

is the unique solution of the LCP(q, A), one can see [11] for more details.

For symmetric case, we take $r = t = 1$, which is considered in [11]. In this case, the system matrix $A \in R^{n\times n}$ is symmetric positive definite for $\mu \geq 0$. Obviously, the LCP(q, A) has a unique solution.

In Table 2, the iteration steps, the CPU times, and the residual norms of RMSMMAOR and MSMAOR methods for the symmetric case are listed for different parameters and different problem sizes of m. When both RMSMMAOR and MSMAOR methods are applied to solve the LCP(q, A), the iteration parameters α, β about MSMAOR method satisfy Theorem 4.1 in [33] and Theorem 4.2 in this paper, but the iteration parameters α, β about RMSMMAOR method only satisfy Theorem 4.2 in this paper and don't satisfy Theorem 4.1 in [33].

Table 2. IT, CPU and Error for RMSMMAOR and MSMAOR with different parameters in symmetric case.

		m	20	30	40	50	60
$\mu = 0.5$	RMSMMAOR	IT	22	22	22	22	22
	$\alpha = 1$	CPU	0.1560	0.7800	2.4336	5.9280	12.4957
	$\beta = 1.2$	Error	7.2225×10^{-6}	7.2598×10^{-6}	7.2970×10^{-6}	7.3390×10^{-6}	7.3707×10^{-6}
$\mu = 0.5$	MSMAOR	IT	30	30	30	31	31
	$\alpha = 1$	CPU	0.2184	1.0764	3.2916	8.3773	17.6905
	$\beta = 0.7$	Error	9.7188×10^{-6}	9.8399×10^{-6}	9.9531×10^{-6}	7.3792×10^{-6}	7.4496×10^{-6}
$\mu = 1.5$	RMSMMAOR	IT	19	19	19	19	19
	$\alpha = 1$	CPU	0.1716	0.6552	2.0748	5.0856	10.7797
	$\beta = 1.2$	Error	6.6884×10^{-6}	6.8943×10^{-6}	7.0943×10^{-6}	7.2888×10^{-6}	7.4782×10^{-6}
$\mu = 1.5$	MSMAOR	IT	23	24	24	24	24
	$\alpha = 1$	CPU	0.1716	0.8424	2.6520	6.4584	13.6657
	$\beta = 0.7$	Error	9.5945×10^{-6}	6.6969×10^{-6}	7.0677×10^{-6}	7.4200×10^{-6}	7.7563×10^{-6}
$\mu = 2.5$	RMSMMAOR	IT	17	17	17	17	17
	$\alpha = 1$	CPU	0.1404	0.6084	1.8720	4.5552	9.6565
	$\beta = 1.2$	Error	7.6793×10^{-6}	8.2513×10^{-6}	8.7861×10^{-6}	9.2902×10^{-6}	9.7683×10^{-6}
$\mu = 2.5$	MSMAOR	IT	20	20	20	21	21
	$\alpha = 1$	CPU	0.1404	0.7020	2.2932	5.6472	11.9341
	$\beta = 0.7$	Error	8.3861×10^{-6}	9.4078×10^{-6}	6.1592×10^{-6}	6.6458×10^{-6}	7.0992×10^{-6}

From Table 2, for RMSMMAOR and MSMAOR methods with $\alpha = 1, \beta = 1.2$ and $\alpha = 1, \beta = 0.7$, fixing the value of μ, it is easy to see that the iteration steps unchange as the increasing of the problem size m, however, CPU times increase as the increasing of the problem size m. Moreover, for RMSMMAOR and MSMAOR methods, fixing the value of m, it is also easy to see that the iteration steps and CPU times decrease as the increasing of the problem size μ. In our numerical experiments, we find that the iteration steps and CPU times of RMSMMAOR are less than that of MSMAOR under certain conditions.

6. Conclusions

In this paper, we establish relaxed modulus-based synchronous multisplitting multi-parameters TOR methods and analyze its convergence properties in detail when the system matrix is either a positive-definite matrix or an $H+$-matrix. Numerical experiments show that the RMSMMTOR methods are feasible under certain conditions. In future work, we can consider extending RMSMMTOR method for nonlinear complementarity problems and horizontal linear complementarity problems.

Acknowledgment

This research of this author is supported by the National Natural Science Foundation of China (Nos. 11226337, 11501525), Basic Research Projects of Key Scientific Research Projects Plan in Henan Higher Education Institutions (No. 20zx003), Henan Natural Science Foundation (No. 222300420579), Henan science and technology research program (Nos. 212102110206, 222102110404, 202102310942), Henan College Students' innovation training program (No. s202110485045) and college students' innovation training program (2021-70), Key projects of colleges and universities in Henan Province(22A880022), Sichuan Science and Technology Program (No. 2019YJ0357).

References

[1] Z. Bai and L. Zhang, *Modulus-based synchronous multisplitting iteration methods for linear complementarity problems,* Numerical Linear Algebra with Applications, 2013, 20, 425–439.

[2] Z. Bai, *On the convergence of the multisplitting methods for the linear complementarity problem,* SIAM Journal on Matrix Analysis and Applications, 1999, 21, 67–78.

[3] Z. Bai, *The convergence of parallel iteration algorithms for linear complementarity problems,* Computers and Mathematics with Applications, 1996, 32, 1–17.

[4] Z. Bai and D. J. Evans, *Matrix multisplitting relaxation methods for linear complementarity problems,* International Journal of Computer Mathematics, 1997, 63, 309–326.

[5] Z. Bai, *On the monotone convergence of matrix multisplitting relaxation methods for the linear complementarity problem,* IMA Journal of Numerical Analysis, 1998, 18, 509–518.

[6] Z. Bai and D. J. Evans, *Matrix multisplitting methods with applications to linear complementarity problems: parallel synchronous and chaotic methods,* Reseaux et systemes repartis: Calculateurs Paralleles, 2001, 13, 125–154.

[7] Z. Bai, *Modulus-based matrix splitting iteration methods for linear complementarity problems,* Numerical Linear Algebra with Applications, 2010, 17, 917–933.

[8] Z. Bai and L. Zhang, *Modulus-based synchronous two-stage multisplitting iteration methods for linear complementarity problems,* Numerical Algorithms, 2013, 62, 59–77.

[9] Z. Bai and D. J. Evans, *Matrix multisplitting methods with applications to linear complementarity problems: parallel asynchronous methods,* International Journal of Computer Mathematics, 2002, 79, 205–232.

[10] Z. Bai, J. Sun and D. Wang, *A unified framework for the construction of various matrix multisplitting iterative methods for large sparse system of linear equations,* Computers and Mathematics with Applications, 1996, 32, 51–76.

[11] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences,* Academic Press, New York, 1979.

[12] R. W. Cottle, J. Pang and R. E. Stone, *The Linear Complementarity Problem,* Academic Press, San Diego, 1992.

[13] L. Cui, X. Zhang and Y. Zheng, *A preconditioner based on a splitting-type iteration method for solving complex symmetric indefinite linear systems,* Japan Journal of Industrial and Applied Mathematics, 2021, 38, 965–978.

[14] L. Cui, Q. Hu, Y. Chen and Y. Song, *A Rayleigh quotient-gradient neural network method for computing Z-eigenpairs of general tensors,* Numerical Linear Algebra with Applications, 2021.

[15] L. Cui, Y. Fan, Y. Song and S. Wu, *The Existence and Uniqueness of Solution for Tensor Complementarity Problem and Related Systems,* Journal of Optimization Theory and Applicaions, 2022, 192, 321–334.

[16] P. Dai, J. Li, J. Bai and J. Qiu, *A preconditioned two-step modulus-based matrix splitting iteration method for linear complementarity problem,* Applied Mathematics and Computation, 2019, 348, 542–551.

[17] J. Dong and M. Jiang, *A modified modulus method for symmetric positive-definite linear complementarity problems,* Numerical Linear Algebra with Applications, 2009, 16, 129–143.

[18] M. C. Ferris and J. Pang, *Engineering and economic applications of complementarity problems,* SIAM Review, 1997, 39, 669–713.

[19] A. Frommer and G. Mayer, *Convergence of relaxed parallel multisplitting methods,* Linear Algebra and Its Applications, 1989, 119, 141–152.

[20] A. Hadjidimos and M. Tzoumas, *Nonstationary extrapolated modulus algorithms for the solution of the linear complementarity problem,* Linear Algebra and Its Applications, 2009, 431, 197–210.

[21] J. Kuang,*On the Two-parameter Overrelaxation Method for Numerical Solution of Large Linear Systems,* Journal of Shanghai Normal University, 1983, 4, 1–11.

[22] W. Li, *A general modulus-based matrix splitting method for linear complementarity problems of H-matrices,* Applied Mathematics Letters, 2013, 26, 1159–1164.

[23] F. Mezzadri and E. Galligani, *On the convergence of modulus-based matrix splitting methods for horizontal linear complementarity problems in hydrodynamic lubrication,* Mathematics and Computers in Simulation, 2020, 176, 226–242.

[24] K. G. Murty, *Linear Complementarity,* Linear and Nonlinear Programming, Internet Edition, 1997.

[25] N. Machida, M. Fukushima and T. Ibaraki, *A multisplitting method for symmetric linear complementarity problems,* Journal of Computational and Applied Mathematics, 1995, 62, 217–227.

[26] D. P. O¡Leary and R. E. White, *Multi-splittings of matrices and parallel solution of linear systems,* SIAM Journal on Algebraic and Discrete Methods, 1985, 630, 630–640.

[27] H. Ren, X. Wang, X. Tang and T. Wang, *A preconditioned general two-step modulus-based matrix splitting iteration method for linear complementarity problems of H_+-matrices,* Numerical Algorithms, 2019, 82, 969–986.

[28] H. Ren, X. Wang, X. Tang and T. Wang,*The general two-sweep modulus-based matrix splitting iteration method for solving linear complementarity problems,* Computers and Mathematics with Applications, 2019, 77, 1071–1081.

[29] F. Robert, M. Charnay and F. Musy, *Iterations chaotiques serie-parallel pour des equations non-lineaires de point fixe,* Aplikace Matematiky, 1975, 20, 1–38.

[30] R. S. Varga, *Matrix Iterative Analysis,* Springer-Verlag: Berlin and Heidelberg, 2000.

[31] W. M. G. van Bokhoven, *Piecewise-Linear Modelling and Analysis,* Proefschrift, Eindhoven, 1981.

[32] B. Wen, H. Zheng, W. Li and X. Peng, *The relaxation modulus-based matrix splitting iteration method for solving linear complementarity problems of positive definite matrices,* Applied Mathematics and Computation, 2018, 321, 349–357.

[33] X. Wu, X. Peng and W. Li, *A preconditioned general modulus-based matrix splitting iteration method for linear complementarity problems of H-matrices,* Numerical Algorithms, 2018, 79, 1131–1146.

[34] D. M. Young, *Iterative Solution of Large Linear Systems,* Academic Press, New York, 1972.

[35] L. Zhang and Z. Ren, *A modified modulus-based multigrid method for linear complementarity problems arising from free boundary problems,* Applied Numerical Mathematics, 2021, 164, 89–100.

[36] L. Zhang and Z. Ren, *Improved convergence theorems of modulus-based matrix splitting iteration methods for linear complementarity problems,* Applied Mathematics Letters, 2013, 26, 638–642.

[37] L. Zhang, T. Huang, S. Cheng and T. Gu, *The weaker convergence of nonstationary matrix multisplitting methods for almost linear systems,* Taiwanese Journal of Mathematics, 2011, 15, 1423–1436.

[38] L. Zhang and J. Li, *The weaker convergence of modulus-based synchronous multisplitting multi-parameters methods for linear complementarity problems,* Computers and Mathematics with Application, 2014, 67, 1954–1959.

[39] L. Zhang, X. Zuo, T. Gu and X. Liu, *Improved convergence theorems of multisplitting methods for the linear complementarity problem,* Applied Mathematics and Computation, 2014, 243, 982–987.

[40] L. Zhang, Y. Zhang, T. Gu, X. Liu and L. Zhang, *New convergence of modulus-based synchronous block multisplitting multi-parameters methods for linear complementarity problems,* Computational and Applied Mathematics, 2015, 1–12.

[41] L. Zhang, T. Huang and T. Gu, *Global relaxed non-stationary multisplitting multi-parameter methods,* International Journal of Computer Mathematics, 2008, 85(2), 211–224.

[42] H. Zheng, S. Vong and L. Liu, *The relaxation modulus-based matrix splitting iteration method for solving a class of nonlinear complementarity problems,* International Journal of Computer Mathematics, 2019, 96, 1648–1667.

[43] H. Zheng, W. Li and S. Vong, *An iteration method for nonlinear complementarity problems,* Journal of Computational and Applied Mathematics, 2020, 372, 112681.

[44] H. Zheng and S. Vong, *On convergence of the modulus-based matrix splitting iteration method for horizontal linear complementarity problems of H_+-matrices,* Applied Mathematics and Computation, 2020, 369, 124890.