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REGIME SHIFTS BETWEEN OSCILLATORY PERSISTENCE AND EXTINCTION IN A STOCHASTIC CHEMOSTAT MODEL WITH PERIODIC PARAMETERS*

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Abstract In this paper, we mechanistically formulate a type of stochastic chemostat model with two complementary nutrients, which is affected by seasonal variations and flocculation effect. The phase transition properties of the model are investigated by theoretical analysis and numerical simulation. The well-posedness of the model is considered. Further, by utilizing Khasminskii's theory, sufficient conditions for the existence of the stochastic nontrivial positive periodic solution are obtained. The existence of the stochastic nontrivial positive periodic solution implies periodic change of microorganism's density. Some sufficient conditions for the global attractivity of the boundary periodic solution of the model are also derived. At last, numerical simulations are performed to illustrate our theoretical results. It is found numerically that the stable positive periodic solution and a stable boundary periodic solution of the model may coexist. Especially, for appropriate random perturbations, the population of the microorganisms changes from an endangered state to an oscillatory persistence state in some regions.

Keywords Stochastic chemostat model, periodic parameter, periodic solution, global attractivity, regime shift.

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1. Introduction

In recent years, the flocculation method has received extensive attention in the fields of wastewater treatment and microorganisms harvesting [6,21]. Flocculants can directly combine with suspended microorganism cells through adsorption-bridging or netting sweep to form flocculent sediments and then settle at the bottom of the water body [21]. Based on the mechanism of the flocculation method, Wang et al. [16] proposed a chemostat model with two perfectly complementary nutrients and flocculation effect. The model exhibits backward bifurcation which implies that the coexistence of the stable boundary equilibrium and a stable positive equi-

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librium. Further, following the work in [16], Liu et al. [7] considered the corresponding stochastic chemostat model. By constructing appropriate stochastic Lyapunov functions, some sufficient conditions for the existence of an ergodic stationary distribution and persistence of the stochastic model are given. The noise-induced persistent growth of microorganisms is investigated in details.

In ecology, the chemostat model can be regarded as a simple aquatic system model or a fermentation process model [10]. The model with periodic coefficients plays a crucial role in advancing our understanding of how aquatic systems respond to seasonal variations. Seasonal variations in the dissolved oxygen concentration and pH value can cause the release of nutrients (such as phosphorus) from the sediments, resulting in periodic fluctuations in the concentration of nutrients. The washout rate (dilution rate) of the aquatic system also varies periodically as the seasons vary. As a result, a large number of scholars have studied chemostat models with periodic nutrient input or periodic dilution rate [1–4, 12–15, 17, 19, 20]. In order to explain the changing patterns of the watershed with seasonal variations, Xu [4] analyzed a chemostat model with periodic nutrient input. Waltman et al. [1] considered a chemostat model with periodic washout rate and gave the conditions for competitive exclusion. They gave some conditions under which the competing populations coexist in the form of continuous oscillation. Zhao and Yuan [20] formulated a single-species stochastic chemostat model with periodic coefficients. A modified break-even concentration is determined which can completely determine whether the microorganisms will continue to exist or not.

The purpose of the paper is further to consider the influences of periodic variations of the nutrients input, the flocculants input and dilution rate on the dynamic behavior of the following stochastic chemostat model with two complementary nutrients and flocculation effect,

$$\begin{cases} dC(t) = [D(t)(C^0(t) - C(t)) - \frac{r_1}{\delta_1}\varphi_1(C(t))\varphi_2(N(t))X(t)]dt + \beta_1(t)C(t)dB_1(t), \\ dN(t) = [D(t)(N^0(t) - N(t)) - \frac{r_2}{\delta_2}\varphi_1(C(t))\varphi_2(N(t))X(t)]dt + \beta_2(t)N(t)dB_2(t), \\ dX(t) = [(r\varphi_1(C(t))\varphi_2(N(t)) - D(t) - m_1P(t))X(t)]\,dt + \beta_3(t)X(t)dB_3(t), \\ dP(t) = [D(t)(P^0(t) - P(t)) - m_2X(t)P(t)]dt + \beta_4(t)P(t)dB_4(t), \end{cases} \tag{1.1}$$

where $B_i(t)$ $(i = 1, 2, 3, 4)$ are mutually independent standard Brownian motions defined on a complete probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), $\beta_i^2(t) > 0$ $(i = 1, 2, 3, 4)$ denote the intensities of random perturbations. The concentrations of two complementary nutrients here are denoted as $C(t)$ and $N(t)$, respectively. $X(t)$ denotes the concentration of microorganisms. $P(t)$ denotes the concentration of flocculants, which are used for harvesting microorganisms. $\beta_i^2(t)$ $(i = 1, 2, 3, 4)$ and the parameter functions $D(t)$, $C^0(t)$, $N^0(t)$, $P^0(t)$ are positive, bounded and continuous functions of period T. The term $r\varphi_1(C(t))\varphi_2(N(t))$ is the growth rate of microorganisms, and the terms $r_1\varphi_1(C(t))\varphi_2(N(t)$ and $r_2\varphi_1(C(t))\varphi_2(N(t))$ represent the quantity of the decreasing of the carbon source and nitrogen source, respectively. The functions $\varphi_1(C(t))$

and $\varphi_2(N(t))$ are chosen as Monod-type functions, i.e.,

$$\varphi_1(C(t)) = \frac{C(t)}{K_1 + C(t)},\ \varphi_2(N(t)) = \frac{N(t)}{K_2 + N(t)}.$$

The flocculation rate of microorganisms $m_1X(t)P(t)$ and the consumption rate of flocculant $m_2X(t)P(t)$ are assumed to be bilinear mass-action function response. For detailed descriptions and dimensions of the remaining parameters of model (1.1), please refer to [7].

This paper is organized as follows. In Section 2, we present basic preliminaries and the well-posedness of model (1.1). Further, we obtain sufficient conditions for the existence of the nontrivial positive periodic solution and the boundary periodic solution for model (1.1). Moreover, it will be shown that the boundary periodic solution is globally attractive under some conditions. Finally, we give some numerical simulations to illustrate theoretical results and a brief discussion about the ecological interpretation of the obtained findings.

2. Theoretical results

2.1. The well-posedness of model (1.1)

For convenience, we use the following notations throughout this paper. Let $\mathbb{R}^4_+ = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i > 0, i = 1, 2, 3, 4\}$ and $\overline{\mathbb{R}}^4_+ = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i \geq 0, i = 1, 2, 3, 4\}$. If $f(t)$ is a continuous bounded function on $[0, \infty)$, define $f^l = \min_{t\in[0,\infty)} f(t)$, $f^u = \max_{t\in[0,\infty)} f(t)$. $\langle f\rangle_T$ denotes the mean value of function $f(t)$ on $[0, T]$, i.e., $\langle f\rangle_T = \frac{1}{T}\int_0^T f(s)ds$.

Theorem 2.1. *For any initial value $(C(0), N(0), X(0), P(0)) \in \mathbb{R}^4_+$, there is a unique solution $(C(t), N(t), X(t), P(t))$ of model (1.1) for $t \geq 0$, and the solution will remain in $\mathbb{R}^4_+$ with probability one, i.e., $(C(t), N(t), X(t), P(t)) \in \mathbb{R}^4_+$ for all $t \geq 0$ almost surely (a.s.).*

Proof. Let us use the following C^2-function $V : \mathbb{R}^4_+ \to \overline{\mathbb{R}}_+$ which is more concise than that in [7],

$$\begin{aligned} V(C, N, X, P) = &\left(C - a - a\ln\frac{C}{a}\right) + \left(N - b - b\ln\frac{N}{b}\right) + \frac{1}{r}\left(\frac{r_1}{\delta_1} + \frac{r_2}{\delta_2}\right) \\ &\times (X - 1 - \ln X) + \frac{m_1}{rD^l}\left(\frac{r_1}{\delta_1} + \frac{r_2}{\delta_2}\right)\left(P - d - d\ln\frac{P}{d}\right). \end{aligned}$$

Choose $a = \frac{D^lK_1}{2r}$, $b = \frac{D^lK_2}{2r}$ and $d = \frac{(D^l)^2}{2m_1m_2}$ such that

$$\left[\frac{ar_1}{\delta_1K_1} + \frac{br_2}{\delta_2K_2} + \frac{dm_1m_2}{rD^l}\left(\frac{r_1}{\delta_1} + \frac{r_2}{\delta_2}\right) - \frac{D^l}{r}\left(\frac{r_1}{\delta_1} + \frac{r_2}{\delta_2}\right)\right] = 0.$$

Hence, we obtain

$$\mathcal{L}V(C, N, X, P) \leq \mathcal{H},$$

where the positive $\mathcal{H}$ is given by

$$\begin{aligned}\mathcal{H}:=&(DC^0)^u+aD^u+(DN^0)^u+bD^u+\frac{D^u}{r}\left(\frac{r_1}{\delta_1}+\frac{r_2}{\delta_2}\right)+\frac{m_1(DP^0)^u}{rD^l}\left(\frac{r_1}{\delta_1}+\frac{r_2}{\delta_2}\right)\\&+\frac{dD^um_1}{rD^l}\left(\frac{r_1}{\delta_1}+\frac{r_2}{\delta_2}\right)+\frac{a(\beta_1^2)^u}{2}+\frac{b(\beta_2^2)^u}{2}+\frac{(\beta_3^2)^u}{2r}\left(\frac{r_1}{\delta_1}+\frac{r_2}{\delta_2}\right)\\&+\frac{dm_1(\beta_4^2)^u}{2rD^l}\left(\frac{r_1}{\delta_1}+\frac{r_2}{\delta_2}\right).\end{aligned}$$

By using the standard method as that in [9,11], we can easily derive the result. □

2.2. Existence of the nontrivial positive periodic solution

In this subsection, we discuss the existence of a nontrivial positive periodic solution for model (1.1). First of all, some preliminaries and notations are presented which will be needed later.

Consider the following equation

$$x(t)=x(t_0)+\int_{t_0}^t b(s,x(s))ds+\sum_{r=1}^k\int_{t_0}^t\beta_r(s,x(s))dB_r(s),\ x\in\mathbb{R}^d, \tag{2.1}$$

where the vectors $b(s,x),\beta_1(s,x),\dots,\beta_k(s,x)$ are continuous functions of (s,x) and satisfy the following conditions

$$\begin{aligned}&|b(s,x)-b(s,y)|+\sum_{r=1}^k|\beta_r(s,x)-\beta_r(s,y)|\le\vartheta\,|x-y|\,,\\&|b(s,x)|+\sum_{r=1}^k|\beta_r(s,x)|\le\vartheta\,(1+|x|)\,,\end{aligned} \tag{2.2}$$

where ϑ is a positive constant. Let $\mathcal{M}$ be a given subset in $\overline{\mathbb{R}}_+$ and $\mathcal{Y}$ be a given subset in $\mathbb{R}^d$. Let C^2 denote the class of functions on $\mathcal{M}\times\mathbb{R}^d$ which are twice continuously differentiable with respect to $x_1,\dots,x_d$ and continuously differentiable with respect to t.

Lemma 2.1 ([5]). *Suppose that the coefficients of (2.1) are T-periodic in t and satisfy the condition (2.2) in every cylinder $\mathcal{M}\times\mathcal{Y}$, and suppose further that there exists a function $V(t,x)\in C^2$ in $\mathcal{M}\times\mathcal{Y}$ which is T-periodic in t, and satisfies the following conditions:*

$$\inf_{|x|>\kappa}V(t,x)\to\infty\ \text{as}\ \kappa\to\infty, \tag{2.3}$$

$$\mathcal{L}V(t,x)\le-1\ \text{outside some compact set}, \tag{2.4}$$

then there exists a solution of (2.1) which is a $T-$periodic Markov process.

Remark 2.1. According to the proof of Lemma 2.1, the condition (2.2) is only used to guarantee the existence and uniqueness of the solution of (2.1).

Let us define

$$[R_0^s]=\frac{r\left\langle(D^4C^0N^0)^{\frac15}\right\rangle_T^5}{\langle D+\frac12\beta_1^2\rangle_T\langle D+\frac12\beta_2^2\rangle_T\langle D(C^0+K_1)\rangle_T\langle D(N^0+K_2)\rangle_T\big(\langle D+hDP^0+\frac12\beta_3^2\rangle_T\big)},$$

where $h(t) = \frac{\int_t^{t+T} \exp\{\int_s^t D(\tau)d\tau\} m_1 ds}{1-\exp\{-\int_0^T D(\tau)d\tau\}}$ and $t \geq 0$.

Theorem 2.2. *Assume that $[R_0^s] > 1$, then there exists a nontrivial positive T-periodic solution of model (1.1).*

Proof. According to Lemma 2.1, in order to prove Theorem 2.2, it suffices to find a C^2-function $V(t,x)$ which is T-periodic in t and a closed set $\mathcal{E}_\epsilon \subset \mathbb{R}_+^4$ such that the conditions (2.3) and (2.4) are held. Define a C^2-function V: $[0,+\infty) \times \mathbb{R}_+^4 \to \mathbb{R}$ as follows

$$\begin{aligned} V(t,C,N,X,P) =& \chi(-\ln X - \varsigma_1 \ln C - \varsigma_2 \ln N + \varsigma_3 C + \varsigma_4 N + \omega(t) + h(t)P + \omega_1(t)) \\ & - \ln C - \ln N - \ln P + \frac{1}{\theta+1}\left(\frac{r\delta_1}{r_1}C + N + X + P\right)^{\theta+1}, \end{aligned}$$

where $\theta > 0$ satisfies $\frac{\theta\rho}{2} < D^l$ $\left(\rho = (\beta_1^u)^2 \vee (\beta_2^u)^2 \vee (\beta_3^u)^2 \vee (\beta_4^u)^2\right)$,

$$\begin{aligned} \varsigma_1 =& \frac{r\left\langle \sqrt[5]{D^4C^0N^0}\right\rangle_T^5}{\left\langle D+\frac{1}{2}\beta_1^2\right\rangle_T^2 \left\langle D+\frac{1}{2}\beta_2^2\right\rangle_T \langle D(C^0+K_1)\rangle_T \langle D(N^0+K_2)\rangle_T}, \\ \varsigma_2 =& \frac{r\left\langle \sqrt[5]{D^4C^0N^0}\right\rangle_T^5}{\left\langle D+\frac{1}{2}\beta_1^2\right\rangle_T \left\langle D+\frac{1}{2}\beta_2^2\right\rangle_T^2 \langle D(C^0+K_1)\rangle_T \langle D(N^0+K_2)\rangle_T}, \\ \varsigma_3 =& \frac{r\left\langle \sqrt[5]{D^4C^0N^0}\right\rangle_T^5}{\left\langle D+\frac{1}{2}\beta_1^2\right\rangle_T \left\langle D+\frac{1}{2}\beta_2^2\right\rangle_T \langle D(C^0+K_1)\rangle_T^2 \langle D(N^0+K_2)\rangle_T}, \\ \varsigma_4 =& \frac{r\left\langle \sqrt[5]{D^4C^0N^0}\right\rangle_T^5}{\left\langle D+\frac{1}{2}\beta_1^2\right\rangle_T \left\langle D+\frac{1}{2}\beta_2^2\right\rangle_T \langle D(C^0+K_1)\rangle_T \langle D(N^0+K_2)\rangle_T^2}. \end{aligned}$$

χ is a sufficiently large number satisfying the following condition:

$$-\chi\langle\lambda\rangle_T + \Phi \leq -2,$$

where

$$\langle\lambda\rangle_T := ([R_0^s]-1) \times \left(\langle D\rangle_T + \left\langle hDP^0\right\rangle_T + \frac{1}{2}\left\langle\beta_3^2\right\rangle_T\right)$$

and

$$\begin{aligned} \Phi = \sup_{(C,N,X,P)\in\mathbb{R}_+^4} \Bigg\{ & -\frac{1}{2}\left(D^l - \frac{\theta}{2}\rho\right)\left(\frac{r^{\theta+1}\delta_1^{\theta+1}}{r_1^{\theta+1}}C^{\theta+1} + N^{\theta+1} + X^{\theta+1} + P^{\theta+1}\right) \\ & + m_2X + 3D^u + \frac{1}{2}(\beta_1^2)^u + \frac{1}{2}(\beta_2^2)^u + \frac{1}{2}(\beta_4^2)^u + \hat{\phi}\Bigg\}. \end{aligned}$$

$\omega(t)$ is a T-periodic function which satisfies

$$\omega'(t) = 5\sqrt[5]{rD^4(t)C^0(t)N^0(t)\varsigma_1\varsigma_2\varsigma_3\varsigma_4} - \varsigma_1\left(D(t) + \frac{1}{2}\beta_1^2(t)\right) - \varsigma_2\left(D(t) + \frac{1}{2}\beta_2^2(t)\right)$$

$$
\begin{aligned}
&- \varsigma_3 D(t)(C^0(t)+K_1) - \varsigma_4 D(t)(N^0(t)+K_2) \\
&- 5\sqrt[5]{r\left\langle (D^4C^0N^0)^{\frac{1}{5}}\right\rangle_T^5 \varsigma_1\varsigma_2\varsigma_3\varsigma_4} + \varsigma_1\left\langle D+\frac{1}{2}\beta_1^2\right\rangle_T + \varsigma_2\left\langle D+\frac{1}{2}\beta_2^2\right\rangle_T \\
&+ \varsigma_3\left\langle D(C^0+K_1)\right\rangle_T + \varsigma_4\left\langle D(N^0+K_2)\right\rangle_T.
\end{aligned}
$$

$h(t)$ is the unique T-periodic solution of the equation $h'(t) = h(t)D(t) - m_1$. In fact, we can further have

$$
h(t) = \frac{\int_t^{t+T}\exp\{\int_s^t D(\tau)d\tau\}m_1 ds}{1-\exp\left\{-\int_0^T D(s)ds\right\}}, \quad \langle h'\rangle_T = 0.
$$

$\omega_1(t)$ is also a T-periodic function defined on $[0,\infty)$ satisfying

$$
\omega_1'(t) = \lambda(t) - \langle\lambda\rangle_T.
$$

Therefore, $V(t,C,N,X,P)$ is T-periodic in t and satisfies

$$
\liminf_{\kappa\to\infty,(C,N,X,P)\in\mathbb{R}_+^4\setminus E_\kappa} V(t,C,N,X,P) = \infty,
$$

where $E_\kappa = \left(\frac{1}{\kappa},\kappa\right)\times\left(\frac{1}{\kappa},\kappa\right)\times\left(\frac{1}{\kappa},\kappa\right)\times\left(\frac{1}{\kappa},\kappa\right)$. Applying Itô's formula to V, we can get that

$$
\begin{aligned}
\mathcal{L}V(t,C,N,X,P) \le & -\chi\langle\lambda\rangle_T + \frac{(\chi c_1+1)r_1XN}{\delta_1(K_1+C)(K_2+N)} + \frac{(\chi c_2+1)r_2XC}{\delta_2(K_1+C)(K_2+N)} \\
&+ m_2X - \frac{D(t)C^0(t)}{C} - \frac{D(t)N^0(t)}{N} - \frac{D(t)P^0(t)}{P} + 3D(t) \\
&- \frac{1}{2}\left(D^l - \frac{\theta}{2}\rho\right)\left(\frac{r^{\theta+1}\delta_1^{\theta+1}}{r_1^{\theta+1}}C^{\theta+1} + N^{\theta+1} + X^{\theta+1} + P^{\theta+1}\right) \\
&+ \frac{1}{2}\beta_1^2(t) + \frac{1}{2}\beta_2^2(t) + \frac{1}{2}\beta_4^2(t) + \hat{\phi},
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\phi} &= \sup_{(C,N,X,P)\in\mathbb{R}_+^4}\left\{-\frac{1}{2}\left(D^l-\frac{\theta}{2}\rho\right)\hat{v}^{\theta+1} + \left(\frac{r\delta_1}{r_1}DC^0 + DN^0 + DP^0\right)^u \hat{v}^\theta\right\}, \\
\hat{v} &= \frac{r\delta_1}{r_1}C + N + X + P.
\end{aligned}
$$

Based on the similar method in [15], define a compact subset as follows

$$
\mathcal{E}_\epsilon = \left\{(C,N,X,P)\in\mathbb{R}_+^4 : \epsilon\le C\le\frac{1}{\epsilon},\ \epsilon\le N\le\frac{1}{\epsilon},\ \epsilon\le X\le\frac{1}{\epsilon},\ \epsilon\le P\le\frac{1}{\epsilon}\right\},
$$

we can take ϵ sufficiently small such that

$$
LV \le -1 \text{ for any } (C,N,X,P)\in\mathbb{R}_+^4\setminus\mathcal{E}_\epsilon.
$$

Therefore, the conditions in the Lemma 2.1 hold. □

2.3. Global attractivity of boundary periodic solution

Lemma 2.2 ([7]). *Suppose that the functions $f(s)$ and $g(s)$ are nonnegative and integrable on $[0,t]$ $(t \geq 0)$, and satisfy $\int_0^t f(s)ds > 0$, $\int_0^t g(s)ds > 0$. Then it has*

$$\frac{1}{t}\int_0^t \frac{f(s)}{K_1+f(s)}\frac{g(s)}{K_2+g(s)}ds \leq \frac{\langle f\rangle_t\langle g\rangle_t\langle fg\rangle_t}{K_1K_2\langle f\rangle_t\langle g\rangle_t+(K_1\langle g\rangle_t+K_2\langle f\rangle_t+\langle f\rangle_t\langle g\rangle_t)\langle fg\rangle_t}.$$

Lemma 2.3. *Consider the following first order linear stochastic differential equation*

$$dS(t) = D(t)(S^0(t) - S(t))dt + \beta(t)S(t)dB(t) \tag{2.5}$$

with any initial value $S(0) \in \mathbb{R}_+$, where $S^0(t)$, $D(t)$ and $\beta(t)$ are positive and continuous T -periodic functions defined on $[0,\infty)$. Then the equation (2.5) has a unique positive periodic solution $S^\star(t)$ which is globally attractive, i.e., attracts all other positive solutions of the equation (2.5).

For the detailed proof of Lemma 2.3, please refer to Theorem 5.1 in [8].

Based on Lemma 4.2 in [14], we can easily obtain the following Lemma 2.4.

Lemma 2.4. *Let $S(t)$ be the solution of the equation (2.5) with any initial value $S(0) \in \mathbb{R}_+$. If $\left\langle D - \frac{1}{2}\beta^2\right\rangle_T > 0$, then*

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0,\ \ \lim_{t\to\infty}\frac{1}{t}\int_0^t \beta(s)S(s)dB(s) = 0,\ \ \lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)ds \leq (S^0)^u.$$

Let us define

$$\overline{R_0} = \frac{rD^u\left(C^0\right)^u\left(N^0\right)^u}{\left(K_1K_2D^l + K_1D^u\left(N^0\right)^u + K_2D^u\left(C^0\right)^u + D^u\left(C^0\right)^u\left(N^0\right)^u\right)\left\langle D + \frac{1}{2}\beta_3^2\right\rangle_T}.$$

Theorem 2.3. *Let $(C(t), N(t), X(t), P(t))$ be the solution of model (1.1) with any initial value $(C(0), N(0), X(0), P(0)) \in \mathbb{R}_+^4$. If $\langle D\rangle_T > \max\left\{\frac{1}{2}\left\langle\beta_1^2\right\rangle_T, \frac{1}{2}\left\langle\beta_2^2\right\rangle_T\right\}$ and $\overline{R_0} < 1$, there exists a boundary periodic solution $(C^\star(t), N^\star(t), 0, P^\star(t))$ of model (1.1) which is globally attractive.*

Proof. Consider the following auxiliary equations,

$$d\widetilde{C}(t) = D(t)(C^0(t) - \widetilde{C}(t))dt + \beta_1(t)\widetilde{C}(t)dB_1(t), \tag{2.6}$$
$$d\widetilde{N}(t) = D(t)(N^0(t) - \widetilde{N}(t))dt + \beta_2(t)\widetilde{N}(t)dB_2(t), \tag{2.7}$$
$$d\widetilde{P}(t) = D(t)(P^0(t) - \widetilde{P}(t))dt + \beta_4(t)\widetilde{P}(t)dB_4(t). \tag{2.8}$$

Let $\widetilde{C}(t)$, $\widetilde{N}(t)$ and $\widetilde{P}(t)$ be the solutions of above the equations with the initial values $\widetilde{C}(0) = C(0)$, $\widetilde{N}(0) = N(0)$ and $\widetilde{P}(0) = P(0)$, respectively. By the comparison theorem of stochastic differential equation, we have

$$C(t) \leq \widetilde{C}(t),\ N(t) \leq \widetilde{N}(t),\ P(t) \leq \widetilde{P}(t),\ t \in [0, +\infty)\,,\ a.s.,$$

where $(C(t), N(t), X(t), P(t))$ is the solution of model (1.1) with the initial value $(C(0), N(0), X(0), P(0))$. From Lemma 2.4, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t C(s)ds \leq \limsup_{t\to\infty}\frac{1}{t}\int_0^t \widetilde{C}(s)ds \leq \left(C^0\right)^u, \tag{2.9}$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t N(s)ds \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t \widetilde{N}(s)ds \le \left(N^0\right)^u. \tag{2.10}$$

Furthermore, by Itô's formula, we have

$$\begin{aligned}d(\widetilde{C}(t)\widetilde{N}(t)) =&[\widetilde{N}(t)D(t)(C^0(t)-\widetilde{C}(t))+\widetilde{C}(t)D(t)(N^0(t)-\widetilde{N}(t))]dt\\&+\beta_1(t)\widetilde{C}(t)\widetilde{N}(t)dB_1(t)+\beta_2(t)\widetilde{C}(t)\widetilde{N}(t)dB_2(t).\end{aligned}$$

Let $W(\widetilde{C},\widetilde{N})=(1+\widetilde{C}\widetilde{N})^\eta$, where η is a positive constant such that

$$2<\eta<\frac{2\left\langle D\right\rangle_T}{\frac{1}{2}\left\langle\beta_1^2\right\rangle_T+\frac{1}{2}\left\langle\beta_2^2\right\rangle_T}+1.$$

Let $\varpi(t)$ be the T-periodic function on $[0,\infty)$ satisfying the equation

$$\varpi'(t)=\eta\left(2D(t)-\frac{\eta-1}{2}(\beta_1^2(t)+\beta_2^2(t))\right)-\eta\left\langle 2D-\frac{\eta-1}{2}(\beta_1^2+\beta_2^2)\right\rangle_T.$$

Then, it has

$$\begin{aligned}&d(e^{\varpi(t)}W(\widetilde{C}(t),\widetilde{N}(t)))\\=&\mathcal{L}(e^{\varpi(t)}W(\widetilde{C}(t),\widetilde{N}(t)))dt+\eta(1+\widetilde{C}(t)\widetilde{N}(t))^{\eta-1}\beta_1(t)\widetilde{C}(t)\widetilde{N}(t)dB_1(t)\\&+\eta(1+\widetilde{C}(t)\widetilde{N}(t))^{\eta-1}\beta_2(t)\widetilde{C}(t)\widetilde{N}(t)dB_2(t),\end{aligned}$$

where

$$\begin{aligned}&\mathcal{L}(e^{\varpi(t)}W(\widetilde{C},\widetilde{N}))dt\\=&e^{\varpi(t)}[\varpi'(t)(1+\widetilde{C}\widetilde{N})^\eta+\eta(1+\widetilde{C}\widetilde{N})^{\eta-1}\widetilde{N}D(C^0-\widetilde{C})\\&+\eta(1+\widetilde{C}\widetilde{N})^{\eta-1}\widetilde{C}D(N^0-\widetilde{N})\\&+\frac{\eta(\eta-1)}{2}(\beta_1^2+\beta_2^2)(1+\widetilde{C}\widetilde{N})^{\eta-2}\widetilde{C}^2\widetilde{N}^2]\\=&\eta e^{\varpi(t)}(1+\widetilde{C}\widetilde{N})^{\eta-2}\left[\left(\frac{\varpi'(t)}{\eta}-2D+\frac{\eta-1}{2}(\beta_1^2+\beta_2^2)\right)\widetilde{C}^2\widetilde{N}^2\right.\\&+DC^0\widetilde{C}\widetilde{N}^2+DN^0\widetilde{N}\widetilde{C}^2+\frac{2\varpi'(t)}{\eta}\widetilde{C}\widetilde{N}-2D\widetilde{C}\widetilde{N}+DC^0\widetilde{N}\\&\left.+DN^0\widetilde{C}+\frac{\varpi'(t)}{\eta}\right]\\\le&\eta e^{\varpi(t)}(1+\widetilde{C}\widetilde{N})^{\eta-2}\left[-\left\langle 2D-\frac{\eta-1}{2}(\beta_1^2+\beta_2^2)\right\rangle_T\widetilde{C}^2\widetilde{N}^2\right.\\&+(DC^0)^u\widetilde{C}\widetilde{N}^2+(DN^0)^u\widetilde{N}\widetilde{C}^2+2\left(\frac{\varpi'(t)}{\eta}\right)^u\widetilde{C}\widetilde{N}-2D^l\widetilde{C}\widetilde{N}\\&\left.+(DC^0)^u\widetilde{N}+(DN^0)^u\widetilde{C}+\left(\frac{\varpi'(t)}{\eta}\right)^u\right]\\:=&\eta e^{\varpi(t)}(1+\widetilde{C}\widetilde{N})^{\eta-2}[-a_1\widetilde{C}^2\widetilde{N}^2+a_2\widetilde{C}\widetilde{N}^2+a_3\widetilde{N}\widetilde{C}^2\\&+2(a_4-a_5)\widetilde{C}\widetilde{N}+a_2\widetilde{N}+a_3\widetilde{C}+a_4].\end{aligned}$$

Obviously,

$$a_1 = \left\langle 2D - \frac{\eta - 1}{2}(\beta_1^2 + \beta_2^2) \right\rangle_T > 0.$$

Then, using the same method as in Lemma 4.2 in [14], we can get

$$\lim_{t\to\infty} \frac{\widetilde{C}(t)\widetilde{N}(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_i(s)\widetilde{C}(s)\widetilde{N}(s)dB_i(s) = 0 \ (i = 1, 2).$$

Thus, we obtain that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\widetilde{C}(t)\widetilde{N}(t)}{t} &= \lim_{t\to\infty} \frac{\widetilde{C}(0)\widetilde{N}(0)}{t} + \lim_{t\to\infty} \frac{1}{t}\int_0^t D(s)C^0(s)\widetilde{N}(s)ds \\
&\quad + \lim_{t\to\infty} \frac{1}{t}\int_0^t D(s)N^0(s)\widetilde{C}(s)ds - \lim_{t\to\infty} \frac{2}{t}\int_0^t D(s)\widetilde{C}(s)\widetilde{N}(s)ds \\
&\quad + \lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_1(s)\widetilde{C}(s)\widetilde{N}(s)dB_1(s) \\
&\quad + \lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_2(s)\widetilde{C}(s)\widetilde{N}(s)dB_2(s) \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t D(s)C^0(s)\widetilde{N}(s)ds + \lim_{t\to\infty} \frac{1}{t}\int_0^t D(s)N^0(s)\widetilde{C}(s)ds \\
&\quad - \lim_{t\to\infty} \frac{2}{t}\int_0^t D(s)\widetilde{C}(s)\widetilde{N}(s)ds \\
&= 0.
\end{aligned}$$

Further, from (2.9) and (2.10), we have

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\int_0^t C(s)N(s)ds \\
\le& \lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{C}(s)\widetilde{N}(s)ds \\
\le& \lim_{t\to\infty} \frac{1}{2D^l t}\int_0^t D(s)C^0(s)\widetilde{N}(s)ds + \lim_{t\to\infty} \frac{1}{2D^l t}\int_0^t D(s)N^0(s)\widetilde{C}(s)ds \\
\le& \frac{D^u}{D^l}\left(C^0\right)^u \left(N^0\right)^u.
\end{aligned} \tag{2.11}$$

Applying Itô's formula to $\ln X(t)$, we have

$$\begin{aligned}
d\ln X(t) &= \left[r\varphi_1(C(t))\varphi_2(N(t)) - D(t) - m_1 P(t) - \frac{\beta_3^2}{2}\right] dt + \beta_3 dB_3(t) \\
&\le \left[r\varphi_1(C(t))\varphi_2(N(t)) - D(t) - \frac{\beta_3^2}{2}\right] dt + \beta_3 dB_3(t).
\end{aligned}$$

It follows from the above inequality and Lemma 2.2 that we have

$$\begin{aligned}
\frac{\ln X(t)}{t} \le& \frac{1}{t}\int_0^t \left(r\varphi_1(C(t))\varphi_2(N(t)) - D(s) - \frac{1}{2}\beta_3^2(s)\right) ds + \frac{1}{t}\int_0^t \beta_3(s)dB_3(s) \\
&+ \frac{\ln X(0)}{t}
\end{aligned}$$

$$\leq \frac{r\langle C\rangle_t\langle N\rangle_t \times \frac{1}{t}\int_0^t C(s)N(s)ds}{K_1K_2\langle C\rangle_t\langle N\rangle_t + (K_1\langle N\rangle_t + K_2\langle C\rangle_t + \langle C\rangle_t\langle N\rangle_t) \times \frac{1}{t}\int_0^t C(s)N(s)ds}$$
$$- \frac{1}{t}\int_0^t \left(D(s) + \frac{1}{2}\beta_3^2(s)\right)ds + \frac{1}{t}\int_0^t \beta_3(s)dB_3(s) + \frac{\ln X(0)}{t}.$$

By using (2.9), (2.10), (2.11) and the strong law of large numbers, we can easily get

$$\limsup_{t\to\infty} \frac{\ln X(t)}{t} \leq \left\langle D + \frac{1}{2}\beta_3^2 \right\rangle_T (\overline{R_0} - 1).$$

Hence, $\lim_{t\to\infty} X(t) = 0$ *a.s.*, if $\overline{R_0} < 1$. Therefore, for arbitrary small $\varepsilon > 0$, there exist a positive constant $t_0 > 0$ and a nonempty set $\Omega_\varepsilon \subset \Omega$ such that $P(\Omega_\varepsilon) \geq 1-\varepsilon$ and $X(t) < \varepsilon$ for $t > t_0$ and $\omega \in \Omega_\varepsilon$. Now from the first equation in model (1.1), we obtain that for $t > t_0$ and $\omega \in \Omega_\varepsilon$,

$$\begin{aligned} dC(t) =& [D(t)(C^0(t) - C(t)) - \frac{r_1}{\delta_1}\varphi_1(C(t))\varphi_2(N(t))X(t)]dt + \beta_1(t)C(t)dB_1(t) \\ \geq& \left[D(t)(C^0(t) - C(t)) - \frac{r_1\varepsilon}{\delta_1}\right]dt + \beta_1(t)C(t)dB_1(t). \end{aligned}$$

Let $C_1(t)$ be the solution of the equation

$$dC_1(t) = \left[D(t)(C^0(t) - C_1(t)) - \frac{r_1\varepsilon}{\delta_1}\right]dt + \beta_1(t)C_1(t)dB_1(t)$$

with the initial value $C_1(0) = C(0)$. Then it follows from the stochastic comparison theorem that, for almost all $\omega \in \Omega_\varepsilon$,

$$C_1(t) \leq C(t) \leq \widetilde{C}(t),\ t > t_0.$$

Note that $C_1(t) \to \widetilde{C}(t)$ as $\varepsilon \to 0$, then we have

$$C(t) = \widetilde{C}(t),\ t \in [0, +\infty),\ a.s..$$

By Lemma 2.3, the equation (2.6) has a unique positive periodic solution $C^\star(t)$. Hence, combining with the global attractivity of $C^\star(t)$, we have

$$\lim_{t\to\infty} |C(t) - C^\star(t)| = 0,\ a.s..$$

Similarly, we can have $\lim_{t\to\infty} |N(t) - N^\star(t)| = 0$ and $\lim_{t\to\infty} |P(t) - P^\star(t)| = 0$ *a.s.*, where $N^\star(t)$ and $P^\star(t)$ are the unique positive periodic solutions of (2.7) and (2.8), respectively. Therefore, the boundary periodic solution $(C^\star, N^\star, 0, P^\star)$ is globally attractive. □

Remark 2.2. It follows from Theorems 2.2 and 2.3 that the parameter $[R_0^s]$ is related to the existence of a nontrivial positive periodic solution of the model (1.1) and the parameter $\overline{R_0}$ is related to the global attractivity of the boundary periodic solution of the model (1.1). Furthermore, it easily has that these two parameters $[R_0^s]$ and $\overline{R_0}$ have the relationship

$$[R_0^s] < \overline{R_0}.$$

Considering the existence of backward bifurcation in the corresponding autonomous deterministic model, it has that the above relationship between $[R_0^s]$ and $\overline{R_0}$ is reasonable. If the model (1.1) with periodic parameters degenerates to the corresponding stochastic autonomous model, it has that $[R_0^s]$ and $\overline{R_0}$ automatically becomes

$$R_0^s := \frac{rD^2C^0N^0}{\left(D+\frac{\beta_1^2}{2}\right)\left(D+\frac{\beta_2^2}{2}\right)(C^0+K_1)(N^0+K_2)\left(D+m_1P^0+\frac{\beta_3^2}{2}\right)},$$

and

$$\check{R}_0 := \frac{rC^0N^0}{(C^0+K_1)(N^0+K_2)\left(D+\frac{\beta_3^2}{2}\right)},$$

respectively. Hence, Theorem 2.2 includes Theorem 3.1 in [7] as a special case. Further, compared with the results in Theorem 4.1 in [7], we see that $\check{R}_0$ gives a better estimation.

3. Stochastic responses and discussions

In the paper, we consider the stochastic chemostat model (1.1) with two complementary nutrients, periodic parameters and flocculation effect. By using two important parameters $[R_0^s]$ and $\overline{R_0}$, we have shown in Theorems 2.2 and 2.3 that, if $[R_0^s] > 1$, there exists a stochastic nontrivial positive T-periodic solution of model (1.1); and if $\overline{R_0} < 1$, the boundary periodic solution of model (1.1) is globally attractive. The parameter $[R_0^s]$ has clear biological meaning, since the parameter $[R_0^s]$ is consistent with the reproduction number for the corresponding deterministic autonomous model. As pointed out in Remark 2.2, the relationship $[R_0^s] < \overline{R_0}$ may be reasonable for the case where the deterministic autonomous model corresponding to model (1.1) presents backward bifurcation.

Now, let use Milstein's Higher Order Method and Theorems 2.2 and 2.3 to some practical examples.

First, it follows from [18] that, for the deterministic model corresponding to model (1.1), the basic reproduction ratio can be given as follows

$$[R_0] = \left\langle \frac{rC^*N^*}{(K_1+C^*)(K_2+N^*)\left\langle D+m_1P^0\right\rangle_T} \right\rangle_T,$$

where

$$C^*(t) = \frac{\int_t^{t+T} \exp\{-\int_s^t D(\tau)d\tau\}D(s)C^0(s)ds}{\exp\{\int_0^T D(s)ds\}-1},$$
$$N^*(t) = \frac{\int_t^{t+T} \exp\{-\int_s^t D(\tau)d\tau\}D(s)N^0(s)ds}{\exp\{\int_0^T D(s)ds\}-1}.$$

To illustrate the application of Theorem 2.2, excepting the parameters β_i ($i = 1,2,3,4$), the intensities of random perturbations, let us choose all other parameters in model (1.1) as follows,

$$D(t) = 2+0.5\sin t, \quad C^0(t) = 2+0.5\sin t, \quad N^0(t) = 3+0.5\sin t, \quad P^0(t) \equiv 1,$$

$$r = 5.5, \quad r_1 = 6.5, \quad r_2 = 5.5, \quad K_1 = K_2 = 0.5, \quad \delta_1 = \delta_2 = 1, \quad m_1 = m_2 = 0.1.$$

If the effects of random perturbations are not considered in model (1.1), i.e., $\beta_i(t) = 0$ $(i = 1, 2, 3, 4)$, model (1.1) becomes the corresponding deterministic model. By direct calculations, we have that $[R_0] \approx 1.785 > 1$. Figure 1 (a) and (b) give the phase trajectories of the corresponding deterministic model, which show that, as t tends to infinity, the solution curves will converge to some positive 2π-periodic solution.

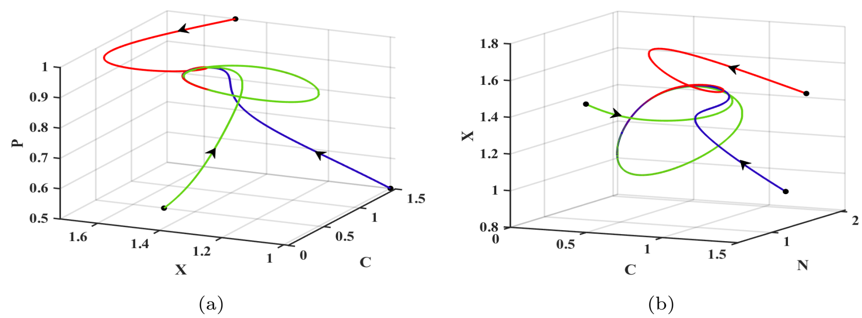


Figure 1. (a) is the phase trajectories of the corresponding deterministic model in the three-dimensional space (C, X, P). (b) is the phase trajectories of the corresponding deterministic model in the three-dimensional space (C, N, X). $[R_0] \approx 1.785 > 1$.

Let consider the case with random perturbations in model (1.1) and choose

$$\beta_1(t) = \beta_2(t) = \beta_3(t) = 0.1 + 0.05 \sin t, \quad \beta_4(t) = 0.2 + 0.05 \sin t.$$

Note that $h(t)$ is given by $h(t) = \frac{\int_t^{t+T} \exp\{\int_s^t D(\tau)d\tau\} m_1 ds}{1 - \exp\{-\int_0^T D(\tau)d\tau\}}$ and the function $P^0(t) \equiv 1$ is a constant, we easily have that $[R_0^s]$ becomes

$$[R_0^s] = \frac{r \left\langle \left(D^4 C^0 N^0\right)^{\frac{1}{5}} \right\rangle_T^5}{\left\langle D + \frac{1}{2}\beta_1^2 \right\rangle_T \left\langle D + \frac{1}{2}\beta_2^2 \right\rangle_T \left\langle D\left(C^0 + K_1\right)\right\rangle_T \left\langle D\left(N^0 + K_2\right)\right\rangle_T \left(\left\langle D + \frac{\beta_3^2}{2} \right\rangle_T + m_1 P^0\right)}.$$

By direct calculations, we have that $[R_0^s] \approx 1.749 > 1$. Hence, the condition of Theorem 2.2 is satisfied. It follows from Theorem 2.2 that model (1.1) has one stochastic positive periodic solution.

Figure 2 gives the sample paths of of $C(t)$, $N(t)$, $X(t)$ and $P(t)$ which show that the sample paths are oscillatory around the solution curves of the corresponding deterministic model. Furthermore, the circular areas in Figure 3 represent the areas through which the sample trajectories pass, where the colorbar depicts the density of the sample trajectories.

Now, let us only change the value $\beta_3(t) = 0.1 + 0.05 \sin t$ by $\beta_3(t) = 2.2 + \sin t$ and all other parameter values are the same as above. Straightforward calculation yields that $\overline{R_0} \approx 0.866 < 1$. By Theorem 2.3, we have that the boundary periodic solution of model (1.1) is globally attractive. It can be seen from Figure 4 that the microorganisms become extinct with probability one, while the corresponding deterministic model may be persistent. This implies that large environmental noise

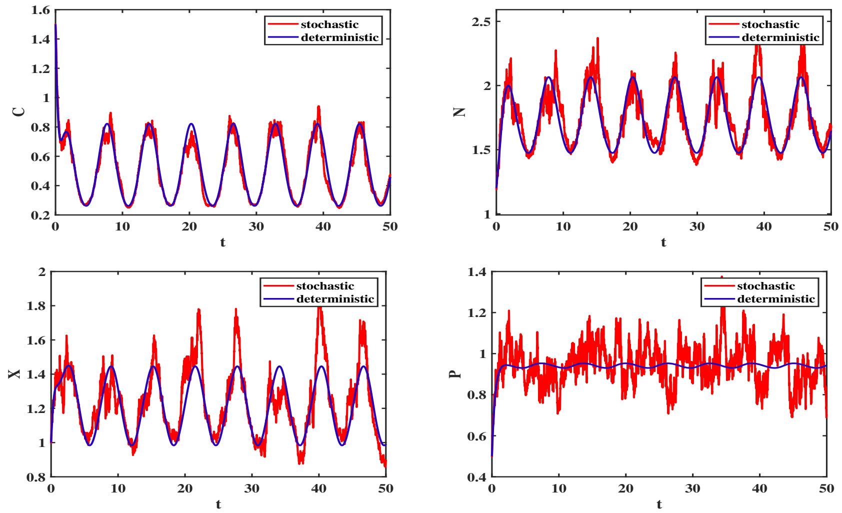


Figure 2. The sample paths of model (1.1) and the solution curves of the corresponding deterministic model. $[R_0^s] \approx 1.749 > 1$.

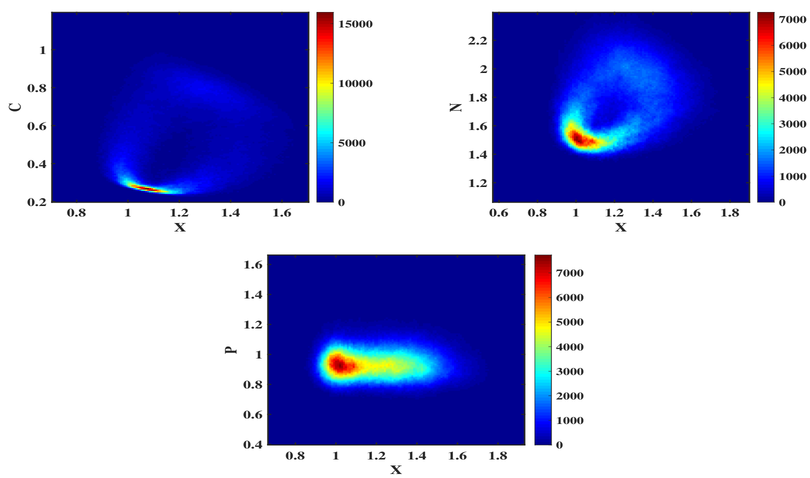


Figure 3. (a), (b), (c) are the path ranges of $(X(t), C(t))$, $(X(t), N(t))$, $(X(t), P(t))$ in model (1.1), respectively. The circular areas in the subfigures represent the areas through which the sample paths pass. $[R_0^s] \approx 1.749 > 1$.

can inhibit the growth of the microorganisms and may lead to the extinction of the microorganisms.

On the other hand, as shown numerically in [7] that, due to the existence of backward bifurcation of the equilibria of the corresponding deterministic autonomous model, there exist bistable phenomenon, that is, the trajectories of the corresponding deterministic autonomous model starting from the neighborhood of the boundary equilibrium will approach the boundary equilibrium state, which means that the microorganisms will eventually become extinction. However, when there are random perturbations, the trajectories of the corresponding stochastic autonomous model will change to approach a positive stationary distribution, which implies

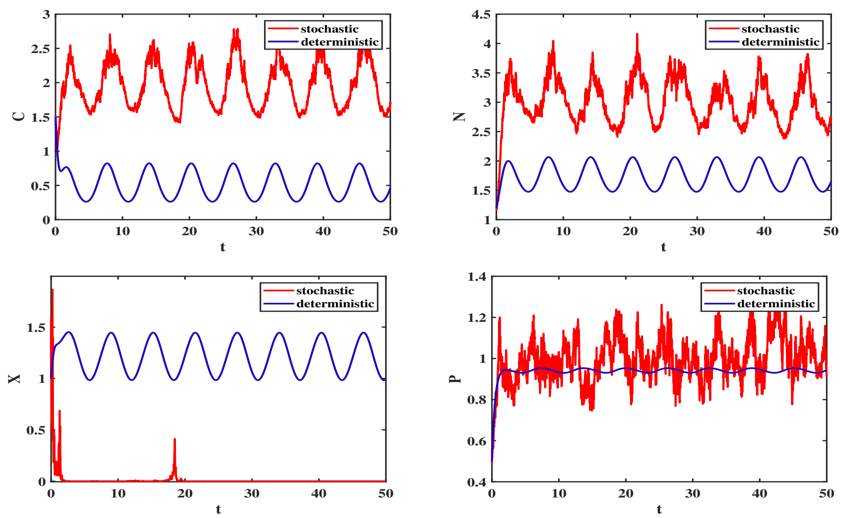
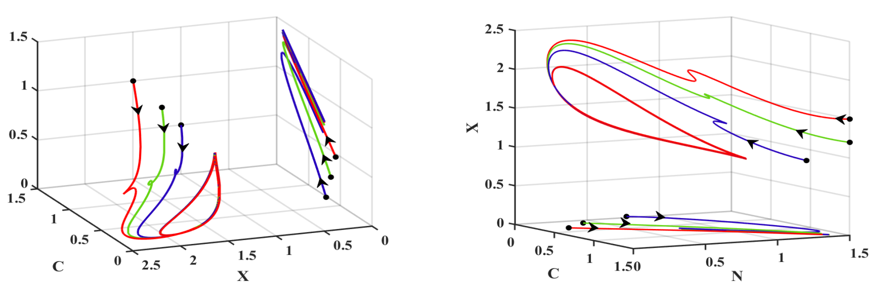


Figure 4. The sample paths of model (1.1) and the solution curves of the corresponding deterministic model. $\overline{R_0} \approx 0.866 < 1$ and $[R_0] \approx 1.785 > 1$.



(a) (b)

Figure 5. (a) is the phase trajectories of the corresponding deterministic model in the three-dimensional space (C, X, P). (b) is the phase trajectories of the corresponding deterministic model in the three-dimensional space (C, N, X). The initial values are given as $(1.5, 1.2, 1, 0.5)$, $(1.5, 1.5, 1.2, 0.7)$, $(1.5, 1.5, 1.5, 1)$, $(0.5, 0.1, 0.05, 0.5)$, $(0.5, 0.2, 0.1, 0.3)$ and $(0.5, 0.5, 0.15, 0.1)$. $[R_0] \approx 0.808 < 1$.

the survival of the microorganisms. Next, let us numerically shown that a similar phenomenon may exist for model (1.1) with period parameters.

Let us choose all the parameters in model (1.1) as follows,

$$D(t) = C^0(t) = N^0(t) = P^0(t) = 1 + 0.5\sin t, \quad r = 3.5, \ r_1 = 0.96,$$
$$r_2 = 1.0001, \ K_1 = 0.36, \ K_2 = 0.3, \ \delta_1 = \delta_2 = 1, \ m_1 = 1.5, \ m_2 = 4,$$
$$\beta_1(t) = \beta_2(t) = \beta_3(t) = 0.2 + 0.005\sin t, \quad \beta_4(t) = 0.2 + 0.05\sin t.$$

By simple computations, we have that $[R_0] \approx 0.808 < 1$, $[R_0^s] \approx 0.681 < 1$ and $\overline{R_0} \approx 2.357 > 1$. It is seen that the conditions of Theorems 2.2 and 2.3 are not satisfied. First, for the corresponding deterministic model, Figure 5 shows the existence of bistable periodic solutions. From Figure 6, we see that, for the corresponding deterministic model, the trajectory starting from $(0.83, 0.83, 0.41, 0.83)$ converges to the boundary periodic solution; for model (1.1), the sample trajectory starting from $(0.83, 0.83, 0.41, 0.83)$ converges to the stochastic nontrivial positive periodic solution. This means that the effects of random perturbations may have positive effect to the survival of the microorganisms.

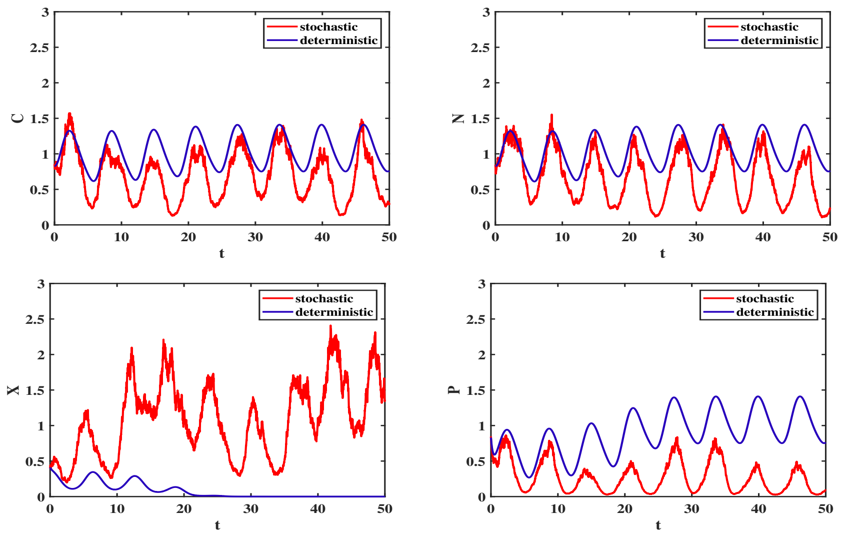


Figure 6. The sample paths of model (1.1) and the solution curves of the corresponding deterministic model. The initial value is taken as $(C(0), N(0), X(0), P(0)) = (0.83, 0.83, 0.41, 0.83)$. $[R_0] \approx 0.808 < 1$, $[R_0^s] \approx 0.681 < 1$ and $\overline{R_0} \approx 2.357 > 1$.

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