Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1434–1442
Website:http://www.jaac-online.com
DOI:10.11948/20210266

SOME COMMENTS ON BEST PROXIMITY POINTS FOR ORDERED PROXIMAL CONTRACTIONS*

Moosa Gabeleh[1,†], Jack Markin[2] and Manuel De La Sen[3]

Abstract In this article, we focus on the existence of an optimal approximate solution, designated as a best proximity point for non-self mappings which are ordered proximal contractions in the setting of partially ordered metric spaces and prove that these results are particular cases of existing fixed point theorems in the literature.

Keywords Best proximity point, fixed point, proximal contraction, integral type contraction.

MSC(2010) 54H25, 47H10.

1. Introduction

In 2005, an interesting extension of the *Banach contraction principle* was presented by Nieto and Rodriguez-Lopez; see the paper [14], where the authors established some applications to ordinary differential equations as well. Before stating an extended version of the fixed point problem due to Nieto and Rodriguez-Lopez, we recall that if $(X, \preceq)$ is a partially ordered set, then a self mapping $T : X \to X$ is said to be *monotone nondecreasing* provided that $T(x) \preceq T(y)$ whenever $x, y \in X, x \preceq y$. Also, the *orbit* of $x \in X$ is defined by

$$\mathcal{O}(x) := \{x, Tx, T^2x, \dots\}.$$

Definition 1.1 ([3]). Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let $T : X \to X$ be a self mapping. Then X is said to be a nondecreasing T-orbitally complete provided that every nondecreasing Cauchy sequence which is contained in $\mathcal{O}(x)$ for $x \in X$, converges to a point in X.

Remark 1.1. It is worth noticing that in [3], the notion of T-orbitally completeness of a set was presented in metric spaces without partially ordered relation. So, the assumption of the Cauchy sequence to be non-decreasing, considered in Definition 1.1 was not considered by Ćirić in [3].

†The corresponding author.
Email: gab.moo@gmail.com, Gabeleh@abru.ac.ir(M. Gabeleh)
1Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
21440 8th St. Golden, Co 80401, USA
3Institute of Research and Development of Processes, University of the Basque Country, 48940 Leioa, Bizkaia, Spain
*The third author is thankful for the support of Basque Government (No. 1207-19).

Theorem 1.1 ([1,5]). *Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a self mapping which is monotone nondecreasing. Assume that there is a metric d on X such that X is a nondecreasing T-orbitally complete and satisfies the condition*

$$\text{if a nondecreasing sequence } \{x_n\} \to x \in X, \text{ then } x_n \preceq x, \ \forall n. \tag{1.1}$$

Suppose that there exists $\alpha \in [0, 1[$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$ with $x \preceq y$. If there exists $x_0 \in X$ with $x_0 \preceq T(x_0)$, then T has a fixed point. Moreover, if we define $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to a fixed point of T.

Recent activities of fixed point theory in partially ordered metric spaces can be found in [4,6,10,11,13].

Throughout this article, Φ denotes the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ which are Lebesgue-integrable on each compact subset of $[0, \infty)$ and $\int_0^\varepsilon \varphi(t) > 0$ for all $\varepsilon > 0$.

Here, we state another extension of the Banach contraction principle.

Theorem 1.2 ([2]). *Let A be a nonempty and closed subset of a complete metric space (X, d) equipped with a partial order relation "$\preceq$". Let $T : A \to A$ be a self-mapping such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{d(x,y)} \varphi(t)dt,$$

for all $x, y \in A$ with $x \preceq y$, where $\varphi \in \Phi$ and $\alpha \in (0, 1)$. Assume there exists an element $x_0 \in A$ for which $x_0 \preceq Tx_0$. If T is continuous, then T has a fixed point.

Theorem 1.2 can be reformulated by considering another type of contractions in the sense of Kannan ([12]) as follows.

Theorem 1.3 ([18]). *Let A be a nonempty and closed subset of a complete metric space (X, d) equipped with a partial order relation " $\preceq$ ". Let $T : A \to A$ be a self-mapping such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{d(x,Tx)} \varphi(t)dt + \beta \int_0^{d(y,Ty)} \varphi(t)dt,$$

for all $x, y \in A$ with $x \preceq y$, where $\varphi \in \Phi$ and $\alpha, \beta \in (0, \frac{1}{2})$. Assume there exists an element $x_0 \in A$ for which $x_0 \preceq Tx_0$. If T is continuous, then T has a fixed point.

We also refer to [15,19], where in [19], working on reflexive Banach space and using notions of continuity, the authors obtain a scheme which converges strongly to a common zero of monotone mappings. The similar result is used to approximate solutions of certain convex optimization problems. In [15], the authors focus on Hilbert spaces and obtain a shrinking projection method of an inertial type with self-adaptive step size. The developed method is used to approximate a common element of the set of solutions of split generalized equilibrium problems

In this paper we show that some of generalizations of Theorem 1.1, Theorem 1.2 and Theorem 1.3 cannot be considered as real extensions. We refer to the recent papers [8,9] which explain that the existence of some best proximity points for various classes of proximal non-self mappings can be concluded from the corresponding fixed point results.

2. Preliminaries

Let (X, d) be a metric space equipped with a partial order relation "$\preceq$" and (A, B) be a pair of nonempty subsets of X. Throughout of this article, we use the following notions and notations:

$$\begin{aligned}
&\operatorname{dist}(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\},\\
&A_0 := \{x \in A : d(x,y) = \operatorname{dist}(A,B),\ \text{for some } y \in B\},\\
&B_0 := \{y \in B : d(x,y) = \operatorname{dist}(A,B),\ \text{for some } x \in A\},\\
&d^*(a,b) = d(a,b) - \operatorname{dist}(A,B), \quad \forall (a,b) \in A \times B.
\end{aligned}$$

We recall that a point $x^\star \in A$ is said to be a *best proximity point* for a non-self mapping $T : A \to B$ provided that

$$d(x^\star, Tx^\star) = \operatorname{dist}(A,B).$$

Definition 2.1 ([17]). A non-self mapping $T : A \to B$ is said to be proximally increasing if it satisfies the condition that

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = \operatorname{dist}(A,B), \\ d(u_2, Tx_2) = \operatorname{dist}(A,B), \end{cases} \implies u_1 \preceq u_2,$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.2 ([17]). A non-self mapping $T : A \to B$ is said to be an ordered proximal contraction of the first kind if there exists a non-negative real number $\alpha < 1$ such that

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = \operatorname{dist}(A,B), \\ d(u_2, Tx_2) = \operatorname{dist}(A,B), \end{cases} \implies d(u_1, u_2) \leq \alpha d(x_1, x_2),$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.3 ([17]). A non-self mapping $T : A \to B$ is said to be an ordered proximal contraction of the second kind if there exists a non-negative real number $\alpha < 1$ such that

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = \operatorname{dist}(A,B), \\ d(u_2, Tx_2) = \operatorname{dist}(A,B), \end{cases} \implies d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2),$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.4 ([17]). Given a mapping $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to preserve isometric distance with respect to g if

$$d(T(gx_1), T(gx_2)) = d(Tx_1, Tx_2),$$

for all $x_1, x_2 \in A$.

We are now ready to state the main existence result of [17].

Theorem 2.1 (see Theorem 3.1 of [17])**.** *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Let A and B be non-void closed subsets of the metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(i) T is a proximally increasing and ordered proximal contraction of the first and second kinds such that $T(A_0) \subseteq B_0$;

(ii) g is a surjective isometry, its inverse is an increasing mapping, and A_0 is contained in $g(A_0)$;

(iii) there exist elements x_0 and x_1 in A_0 such that

$$x_0 \preceq x_1, \quad d(gx_1, Tx_0) = \mathrm{dist}(A, B),$$

(iv) T preserves isometric distance with respect to g;

(v) If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$ for all n.

Then there exists an element $x^\star \in A$ such that

$$d(gx^\star, Tx^\star) = \mathrm{dist}(A, B).$$

Further the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Tx_n) = \mathrm{dist}(A, B), \quad \forall n \in \mathbb{N} \cup \{0\},$$

converges to the element $x^\star$.

Definition 2.5 ([7])**.** A non-self mapping $T : A \to B$ is said to be an integral Banach type contraction if

(1) T is an ordered proximal;

(2) There exist $\varphi \in \Phi$ and $\alpha \in (0, 1)$ such that

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{d(x,y)} \varphi(t)dt,$$

for all $x, y \in A$ with $x \preceq y$.

Definition 2.6 ([7])**.** A non-self mapping $T : A \to B$ is said to be an integral Kannan type contraction if

(1) T is an ordered proximal;

(2) There exist $\varphi \in \Phi$ and $\alpha, \beta \in (0, \frac{1}{2})$ such that

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{d^*(x,Tx)} \varphi(t)dt + \beta \int_0^{d^*(y,Ty)} \varphi(t)dt,$$

for all $x, y \in A$ with $x \preceq y$.

Definition 2.7 ([16]). The pair (A,B) is said to have P-property if and only if

$$\begin{cases} d(x_1,y_1)=\operatorname{dist}(A,B), \\ d(x_2,y_2)=\operatorname{dist}(A,B), \end{cases} \Longrightarrow d(x_1,x_2)=d(y_1,y_2),$$

where $x_1,x_2\in A_0$ and $y_1,y_2\in B_0$.

We now state the next best proximity point theorems which are the main results of [7].

Theorem 2.2 (see Theorem 1 of [7]). *Let A and B be two nonempty closed subsets of X such that A_0 is nonempty and (A,B) has the P-property. Assume that $T:A\to B$ is an integral Banach type contraction mapping which is continuous and $T(A_0)\subseteq B_0$. If moreover there exist $x_0,x_1\in A_0$ with $x_0\preceq x_1$ such that $d(x_1,Tx_0)=\operatorname{dist}(A,B)$, then T has a best proximity point.*

Theorem 2.3 (see Theorem 2 of [7]). *Let A and B be two nonempty closed subsets of X such that A_0 is nonempty and (A,B) has the P-property. Assume that $T:A\to B$ is an integral Kannan type contraction mapping which is continuous and $T(A_0)\subseteq B_0$. If moreover there exist $x_0,x_1\in A_0$ with $x_0\preceq x_1$ such that $d(x_1,Tx_0)=\operatorname{dist}(A,B)$, then T has a best proximity point.*

3. Main results

Theorem 3.1. *Theorem 2.1 is a straightforward consequence of Theorem 1.1.*

Proof. Let $x\in A_0$. Since $Tx\in B_0$, there exists an element $u\in A_0$ such that $d(u,Tx)=\operatorname{dist}(A,B)$. If there is another point $v\in A_0$ for which $d(v,Tx)=\operatorname{dist}(A,B)$, then by the fact that T is an ordered proximal contraction of the first kind, there exists $\alpha\in[0,1)$ such that $d(u,v)\le\alpha d(x,x)=0$ which implies that $u=v$. Thus for any $x\in A_0$ there is a unique point $u\in A_0$ such that $d(u,Tx)=\operatorname{dist}(A,B)$. So we can define the self mapping $\Upsilon:A_0\to A_0$ such that

$$d(\Upsilon x,Tx)=\operatorname{dist}(A,B),\quad \forall x\in A_0.$$

Now consider the self mapping $g^{-1}\Upsilon:A_0\to A_0$. In what follows we check the assumptions of Theorem 1.2 on the self mapping $g^{-1}\Upsilon$.

• Let $x,y\in A_0$ be such that $x\preceq y$. Since T is a proximally increasing, then

$$\begin{cases} d(\Upsilon x,Tx)=\operatorname{dist}(A,B), \\ d(\Upsilon y,Ty)=\operatorname{dist}(A,B), \end{cases} \Longrightarrow \Upsilon x\preceq\Upsilon y.$$

By the fact that g^{-1} is increasing, we must have $g^{-1}\Upsilon x\preceq g^{-1}\Upsilon y$, that is, $g^{-1}\Upsilon$ is monotone nondecreasing.

• Let $x,y\in A_0$ be such that $x\preceq y$. Since T is an ordered proximal contraction of the first kind, we obtain

$$\begin{cases} d(\Upsilon x,Tx)=\operatorname{dist}(A,B), \\ d(\Upsilon y,Ty)=\operatorname{dist}(A,B), \end{cases} \Longrightarrow d(\Upsilon x,\Upsilon y)\le\alpha d(x,y).$$

In view of the fact that the mapping g^{-1} is an isometry, we conclude that

$$d\Big((g^{-1}\Upsilon)x,(g^{-1}\Upsilon)y\Big)\le\alpha d(x,y).$$

• By the assumption (iii) of Theorem 1.3, there exist $x_0, x_1 \in A_0$ with $x_0 \preceq x_1$ such that $d(gx_1, Tx_0) = \text{dist}(A, B)$. Also, by the definition of the mapping Υ, we have $d(\Upsilon x_0, Tx_0) = \text{dist}(A, B)$. Again using this reality that T is an ordered proximal contraction of the first kind, we conclude that $gx_1 = \Upsilon x_0$ or $x_1 = (g^{-1}\Upsilon)x_0$. Thus $x_0 \preceq (g^{-1}\Upsilon)x_0$.
• Suppose that $x \in A_0$ and $\{u_n\}$ is a nondecreasing Cauchy sequence which is contained in $\mathcal{O}(x) := \{x, (g^{-1}\Upsilon)x, (g^{-1}\Upsilon)^2 x, \dots\}$. Without loss in generality, we assume that $u_n = (g^{-1}\Upsilon)^n x$ for all $n \in \mathbb{N} \cup \{0\}$, where $u_0 := x$. We now have $u_1 = (g^{-1}\Upsilon)x$ and so $gu_1 = \Upsilon x$. Also, $u_2 = (g^{-1}\Upsilon)u_1$ which implies that $gu_2 = \Upsilon u_1$. Continuing this process and by induction, we obtain

$$gu_n = \Upsilon u_{n-1}, \quad \forall n \in \mathbb{N}.$$

It follows from the definition of Υ that

$$d(gu_n, Tu_{n-1}) = d(\Upsilon u_{n-1}, Tu_{n-1}) = \text{dist}(A, B), \quad \forall n \in \mathbb{N}.$$

Because of the fact that T is an ordered proximal contraction of the second kind and that T preserves isometric distance with respect to g, we obtain

$$d(Tu_{n+1}, Tu_n) = d(T(gu_{n+1}), T(gu_n)) \leq \alpha d(Tu_n, Tu_{n-1}), \quad \forall n \in \mathbb{N}.$$

Therefore, the sequence $\{Tu_n\}$ is also Cauchy in a complete metric space X. Suppose that $Tu_n \to q \in B$ and $u_n \to p \in A$. Since g is an isometry, $\{gu_n\}$ is a Cauchy sequence in A_0 and so, there exists a point $w \in A$ such that $gu_n \to w$. Hence,

$$d(w, q) = \lim_{n\to\infty} d(gu_n, Tu_{n-1}) = \text{dist}(A, B).$$

This ensures that $w \in A_0$. Thereby, $gu_n \to w \in A_0$ and by the continuity of g^{-1}, we must have $u_n \to g^{-1}w \in A_0$, that is, $p = g^{-1}w \in A_0$. This implies that A_0 is a nondecreasing T-orbitally complete.

Therefore, all of the assumptions of Theorem 1.1 hold and so the self mapping $g^{-1}\Upsilon$ has a fixed point in A_0 called $x^\star$, that is, $g^{-1}\Upsilon x^\star = x^\star$ which deduces that $\Upsilon x^\star = gx^\star$ and so,

$$d(gx^\star, Tx^\star) = d(\Upsilon x^\star, Tx^\star) = \text{dist}(A, B).$$

Moreover, if we define $x_n = g^{-1}\Upsilon x_{n-1}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to a fixed point of the self mapping $g^{-1}\Upsilon$. In this way, we have

$$d(gx_n, Tx_{n-1}) = d(\Upsilon x_{n-1}, Tx_{n-1}) = \text{dist}(A, B), \quad \forall n \in \mathbb{N},$$

and this completes the proof. □

Theorem 3.2. *Theorem 2.2 is a straightforward consequence of Theorem 1.2.*

Proof. At first, it is worth noticing that A_0 is closed. To prove this let $\{u_n\}$ be a sequence in the set A_0 such that $u_n \to p \in A$. Notice that for any natural number n there exists $v_n \in B_0$ such that $d(u_n, v_n) = \text{dist}(A, B)$. In view of the fact that (A, B) has the P-property, we must have

$$d(v_n, v_m) = d(u_n, u_m),$$

for any natural numbers m, n. Thus $\{v_n\}$ is a Cauchy sequence in B and so by the closedness of the set B, there exists an element $q \in B$ such that $v_n \to q$. Therefore,

$$d(p, q) = \lim_{n\to\infty} d(u_n, v_n) = \operatorname{dist}(A, B),$$

which ensures that $p \in A_0$. Similarly, the set B_0 is closed.

Now assume that $x \in A_0$. then there exists a point $w \in B_0$ for which $d(x, w) = \operatorname{dist}(A, B)$. We note that if there is another element $w' \in B_0$ such that $d(x, w') = \operatorname{dist}(A, B)$, then from the fact that (A, B) has the P-property, we must have $w = w'$. Thereby, we can define a mapping $\Delta : A_0 \to B_0$ such that

$$d(x, \Delta x) = \operatorname{dist}(A, B), \quad \forall x \in A_0.$$

It is interesting to note that for any $u_1, u_2 \in A_0$, we have $d(u_1, \Delta u_1) = \operatorname{dist}(A, B) = d(u_2, \Delta u_2)$. Again by this reality that (A, B) has the P-property, we obtain,

$$d(u_1, u_2) = d(\Delta u_1, \Delta u_2), \quad \forall u_1, u_2 \in A_0,$$

that is, Δ is an isometry. Hence, Δ is a bijective isometry mapping. Now consider the continuous self-mapping $\Delta^{-1}T : A_0 \to A_0$. Let $x, y \in A_0$ be such that $x \preceq y$. Because of the fact that Δ^{-1} is an isometry, we conclude that

$$\begin{aligned}\int_0^{d(\Delta^{-1}Tx,\Delta^{-1}Ty)} \varphi(t)dt &= \int_0^{d(Tx,Ty)} \varphi(t)dt \\ &\leq \alpha \int_0^{d(x,y)} \varphi(t)dt,\end{aligned}$$

where $\alpha \in (0, 1)$. Besides, from the conditions of Theorem 2.2 there exist $x_0, x_1 \in A_0$ with $x_0 \preceq x_1$ such that $d(x_1, Tx_0) = \operatorname{dist}(A, B)$. Since $d(x_1, \Delta x_1) = \operatorname{dist}(A, B)$ we obtain $Tx_0 = \Delta x_1$, that is, $x_1 = \Delta^{-1}Tx_0$ which implies that $x_0 \preceq \Delta^{-1}Tx_0$. We claim that $\Delta^{-1}T$ is monotone nondecreasing. Let $y_1, y_2 \in A_0$ be such that $y_1 \preceq y_2$. Then there exist $x_1, x_2 \in A_0$ such that $d(x_1, Ty_1) = \operatorname{dist}(A, B) = d(x_2, Ty_2)$. Since T is proximally increasing, we deduce that $x_1 \preceq x_2$. On the other hand, by the definition of Δ we have $x_1 = \Delta^{-1}T(y_1)$ and $x_2 = \Delta^{-1}T(y_2)$ and so, $\Delta^{-1}T(y_1) \preceq \Delta^{-1}T(y_2)$, that is, the self-mapping $\Delta^{-1}T$ is monotone nondecreasing. It now follows from Theorem 1.2 that the self-mapping $\Delta^{-1}T : A_0 \to A_0$ has fixed point, called $x^\star \in A_0$. Thus $\Delta^{-1}Tx^\star = x^\star$ and so $Tx^\star = \Delta x^\star$ We now have

$$d(x^\star, Tx^\star) = d(x^\star, \Delta x^\star) = \operatorname{dist}(A, B),$$

which ensures that $x^\star \in A$ is a best proximity point of T and the result follows. □

Theorem 3.3. *Theorem 2.3 is a straightforward consequence of Theorem 1.3.*

Proof. It follows from the proof of Theorem 3.2 that A_0 is closed and there exists a bijective isometry $\Delta : A_0 \to B_0$ such that $d(x, \Delta x) = \operatorname{dist}(A, B)$ for all $x \in A_0$. Now if we consider the self-mapping $\Delta^{-1}T : A_0 \to A_0$, then for all $x, y \in A_0$ with $x \preceq y$ we have

$$\int_0^{d(\Delta^{-1}Tx,\Delta^{-1}Ty)} \varphi(t)dt$$

$$
\begin{aligned}
&= \int_0^{d(Tx,Ty)} \varphi(t)dt \\
&\leq \alpha \int_0^{d^*(x,Tx)} \varphi(t)dt + \beta \int_0^{d^*(y,Ty)} \varphi(t)dt \\
&\leq \alpha \int_0^{[\underbrace{d^*(x,\Delta x)}_{0} + d(\Delta x, Tx)]} \varphi(t)dt + \beta \int_0^{[\underbrace{d^*(y,\Delta y)}_{0} + d(\Delta y, Ty)]} \varphi(t)dt \\
&= \alpha \int_0^{d(x,\Delta^{-1}Tx)} \varphi(t)dt + \beta \int_0^{d(y,\Delta^{-1}Ty)} \varphi(t)dt, \quad (\Delta^{-1} \text{is an isometry})
\end{aligned}
$$

where $\varphi \in \Phi$ and $\alpha, \beta \in (0, \frac{1}{2})$. Moreover, by a similar argument of the proof of Theorem 3.2 for the elements $x_0, x_1 \in A_0$ with $x_0 \preceq x_1$ and $d(x_1, Tx_0) = \text{dist}(A, B)$, we obtain $x_0 \preceq \Delta^{-1}Tx_0$. In view of the fact that T is proximally increasing, by a similar manner of the proof of Theorem 3.2, the self-mapping $\Delta^{-1}T$ is monotone nondecreasing. Therefore, by Theorem 1.3 the continuous self-mapping $\Delta^{-1}T : A_0 \to A_0$ has a fixed point and this point is a best proximity point of the non-self mapping T and we are finished. □

Acknowledgements

The authors would like to thank the reviewers for their useful comments.

References

[1] G. V. R. Babu, P. D. Sailaja and K. T. Kidane, *A fixed point theorem in orbitally complete partially ordered metric spaces*, J. Operators, 2013, 2013, 1–8.

[2] S. Chandok, B. S. Choudhury and N. Metiya, *Fixed point results in ordered metric spaces forrational type expressions with auxiliary functions*, J. Egyptian Math. Soc., 2015, 23, 95–101.

[3] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., 1974, 45, 267–273.

[4] K. Chairaa, A. Eladraouib and M. Kabilc, *Extensions of some fixed point theorems for weak-contraction mappings in partially ordered modular metric spaces*, Iranian Journal of Mathematical Sciences and Informatics, 2020, 15, 111–124.

[5] H. Ding, Z. Kadelburg and H. K. Nashine, *Common fixed point theorems for weakly increasing mappings on ordered orbitally complete metric spaces*, Fixed Point Theory Apll., 2012, 2012, 1–14.

[6] K. S. Eke and G. J. Oghonyon, *Some fixed point theorems in ordered partial metric spaces with applications*, Cogent Mathematics & Statistics, 2018, 5, 1–11.

[7] K. Fallahi, H. Ghahramani and G. Soleimani Rad, *Integral type contractions in partially ordered metric spaces and best proximity point*, Iranian J. Sci. Tech., Transactions A: Sci., 2020, 44, 177–183.

[8] M. Gabeleh and J. Markin, *A note on the paper "Best proximity point results for p -proximal contractions"*, Acta Math. Hungar., 2021, 164, 326–329.

[9] M. Gabeleh and J. Markin, *Some notes on the paper "On best proximity points of interpolative proximal contractions"*, Quaestiones Mathematicae, 2021. DOI: 10.2989/16073606.2021.1951872.

[10] V. Gupta and R. Deep, *Some coupled fixed point theorems in partially ordered S-metric spaces*, Miskolc Mathematical Notes, 2015, 16, 181–194.

[11] V. Gupta, G. Jungck and N. Mani, *Some novel fixed point theorems in partially ordered metric spaces*, AIMS Math., 2020,5, 4444–4452.

[12] R. Kannan, *Some results on fixed points-II*, Am. Math. Mon., 1969, 76, 405–408.

[13] N. Mani and J. Jindal, *Some new fixed point theorems for generalized weak contraction in partially ordered metric spaces*, Computational and Mathematical Methods, 2020, 2(5), 1–9.

[14] J. J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, 2005, 22, 223–239.

[15] M. A. Olona, T. A. Alakoya, O. E. Owolabi and O. T. Mewomo, *Inertial shrinking projection algorithm with self-adaptive step size for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings*, Demonstratio Mathematica, 2021, 54, 47–67.

[16] V. Sankar Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal., 2011, 74, 4804–4808.

[17] S. Sadiq Basha, *Discrete optimization in partially ordered sets*, J. Global Optim., 2012, 54, 511–517.

[18] N. Seshagiri Rao, K. Kalyani and K. Khatri, *Contractive mapping theorems in Partially ordered metric spaces*, CUBO, 2020, 22, 203–214.

[19] H. Zegeye and G. Bekele Wega, *Approximation of a common f-fixed point of f-pseudocontractive mappings in Banach spaces*, Rendiconti del Circolo Matematico di Palermo Series 2 volume, 2021, 70, 1139–1162.