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EXISTENCE OF AT LEAST TWO SOLUTIONS FOR DOUBLE PHASE PROBLEM

Bin Ge[1,†] and Wen-Shuo Yuan[1]

Abstract This paper concerns with a class of double phase Dirichlet problem depending of one real parameter. Under some appropriate assumptions, we obtain the existence of at least two solutions for this problem using a direct consequence of the celebrated Pucci and Serrin theorem. Our results generalize some existing results.

Keywords Double phase problem, variational method, existence results.

MSC(2010) 35D30, 35J20, 35J60, 35J70, 35J92.

1. Introduction and main results

It is well-known that a great attention in the last years has been focused by many authors on the study of double phase problems on bounded domains. In this direction, we can refer to [3, 5, 6, 9–14, 20, 23, 24, 26–28] and references therein. For more physical background, for instance, Lavrentiev's phenomenonwe [25], quantum physics [2], transonic flows [4] and so on. For other existence results on elliptic equations with double phase operators we refer to the papers of Papageorgiou-Radulescu-Repovs [18,19], Mingione-Radulescu [16]. We refer the interested reader to [1, 7, 17, 22] and references therein for recent regularity results.

Motivated by this large interest in the current literature, we are interested in the existence of solutions of the following double phase problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, λ is a positive real parameter, $\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)$ denotes the double phase operator, $1 < p < q < N$ and

$$\frac{q}{p} < 1 + \frac{1}{N},\ \mu : \overline{\Omega} \to [0, +\infty) \text{ is Lipschitz continuous.} \tag{1.1}$$

Moreover, $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that

(h_1) there exist $1 < r < p^*$ and some positive constant C such that

$$|f(x,t)| \leq C(1 + |t|^{r-1}),$$

where $p^* = \frac{Np}{N-p}$ is the critical exponent;

[†]The corresponding author. Email: gebin791025@hrbeu.edu.cn(B. Ge)
[1]College of Mathematical Sciences, Harbin Engineering University, Harbin, 150001, China

(h_2) there exist $\theta > q$, $M > 0$ such that for any $x \in \Omega$, $|t| \geq M$

$$0 < \theta F(x,t) \leq tf(x,t),$$

where $F(x,t) = \int_0^t f(x,s)ds$.

The aim of this paper is to present the existence of at least two weak solutions for our problem when nonlinear term f satisfies (h_1)-(h_2) but does not satisfy the usual additional assumption at zero, that is,

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-1}} = 0 \text{ uniformly in } x,$$

which comes from [12]. This is a more natural and general case, but needs different tricks and estimate. To overcome this difficulty, we shall use the Ricceri's variational principle due to Ricceri [21, Theorem 6] .

We are now in the position to state our main results.

Theorem 1.1. *Suppose that (h_1) and (h_2) hold. Then, for every $\sigma > 0$ and each*

$$\lambda \in \Big(0, \frac{\sigma}{Cs_1 q_\sigma + Cs_r^r q_\sigma^r}\Big), \tag{1.2}$$

where $q_\sigma = \max\{(q\sigma)^{\frac{1}{p}}, (q\sigma)^{\frac{1}{q}}\}$ and $s_1, s_r > 0$ are the best constants for the embeddings of $W_0^{1,H}(\Omega) \hookrightarrow L^1(\Omega)$ and $W_0^{1,H}(\Omega) \hookrightarrow L^r(\Omega)$, respectively, the problem (P) has at least two weak solutions one of which lies in

$$B_\sigma := \{u \in W_0^{1,H}(\Omega) : \|u\| < \min\{(q\sigma)^{\frac{1}{p}}, (q\sigma)^{\frac{1}{q}}\}.$$

The rest of this paper is organized as follows. In Sect. 2, we state some preliminary knowledge on space $W_0^{1,H}(\Omega)$ and the main lemmas. In Sect. 3, we prove the main results.

2. Abstract framework

In this section, firstly we summarize some relevant results on the Musielak-Orlicz-Sobolev space $W_0^{1,H}(\Omega)$. For more details, we can see Refs. [5, 8, 15].

The Musielak-Orlicz space $L^H(\Omega)$ associated with the function

$$H : \Omega \times [0, +\infty) \to [0, +\infty),\ (x,t) \mapsto t^p + \mu(x)t^q.$$

Thus, the Musielak-Orlicz space $L^H(\Omega)$ is defined

$$L^H(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega H(x,|u|)dx < +\infty\Big\}.$$

On $L^H(\Omega)$ we consider the Luxemburg norm

$$|u|_H = \inf\Big\{\lambda > 0 : \int_\Omega H(x, |\frac{u}{\lambda}|)dx \leq 1\Big\}.$$

Then the space $L^H(\Omega)$ is a separable, uniformly convex Banach space. We define the generalized Musielak-Orlicz-Sobolev space $W^{1,H}(\Omega)$ is defined by putting

$$W^{1,H}(\Omega) = \{u \in L^H(\Omega) : |\nabla u| \in L^H(\Omega)\}$$

and it is endowed with the following norm

$$\|u\| = |u|_H + |\nabla u|_H.$$

By $W_0^{1,H}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,H}(\Omega)$. We recall that, thanks to (1.1), the Poincaré inequality also is true. Furthermore, from Colasuonno-Squassina [5, Proposition 2.18], it is known that $\|u\|$ and $|\nabla u|_H$ are equivalent norms on $W_0^{1,H}(\Omega)$. In what follows, we equip the space $W_0^{1,H}(\Omega)$ with the equivalent norm $|\nabla u|_H$. Moreover, it is known that the embedding

$$W_0^{1,H}(\Omega) \hookrightarrow L^\gamma(\Omega) \tag{2.1}$$

is compact whenever $\gamma \in [1, p^*)$, see Colasuonno-Squassina [5, Proposition 2.15]. Furthermore, if we define the H-modular function

$$\rho_H(u) = \int_\Omega (|u|^p + \mu(x)|u|^q)dx, \ \forall u \in L^H(\Omega),$$

then from Liu-Dai [12, Proposition 2.1] we have the following facts:

$$\min\{|u|_H^p, |u|_H^q\} \le \rho_H(u) \le \max\{|u|_H^p, |u|_H^q\}. \tag{2.2}$$

Finally, we present the following abstract theorem which will play a crucial role in the proof of Theorem 1.1. First, let us recall that the definition of the Palais-Smale condition is as follows:

Definition 2.1. Let X be a real Banach space and X^* its topological dual. We say that $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition ((PS)-condition in short), if any sequence $u_n \subset X$ satisfying

$$I(u_n) \text{ is bounded}, \ \|I'(u_n)\|_{X^*} \to 0,$$

contains a convergent subsequence.

Lemma 2.1 ([21, Theorem 6]). *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive. Further, assume that Ψ is sequentially weakly continuous. In addition, assume that, for each $\alpha > 0$, the functional $I_\alpha := \alpha\Phi - \Psi$ satisfies (PS)-condition. Then, for each $\sigma > \inf_{u\in X} \Phi(u)$ and each*

$$\alpha > \inf_{u\in\Phi^{-1}(-\infty,\sigma)} \frac{\sup\limits_{v\in\Phi^{-1}(-\infty,\sigma)} \Psi(v) - \Psi(u)}{\sigma - \Phi(u)}$$

the following alternative holds: either the functional I_α has a strict global minimum which lies in $\Phi^{-1}(-\infty, \sigma)$, or I_α has at least two critical points one of which lies in $\Phi^{-1}(-\infty, \sigma)$.

3. Variational Setting and Proof of Theorem 1.1

Firstly, fix $\lambda > 0$, we define the functional $I_\lambda : W_0^{1,H}(\Omega) \to \mathbb{R}$ as

$$\begin{aligned} I_\lambda(u) =& \frac{1}{\lambda}\int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q\right)dx - \int_\Omega F(x,u)dx \\ =& \frac{1}{\lambda}\Phi(u) - \Psi(u), \end{aligned} \tag{3.1}$$

where $\Phi(u)=\int_\Omega(\frac{1}{p}|\nabla u|^p+\frac{\mu(x)}{q}|\nabla u|^q)dx$ and $\Psi(u)=\int_\Omega F(x,u)dx$. Under the assumption (f_1), we can easily check that Ψ is well-defined and of class C^1 on $W_0^{1,H}(\Omega)$, and its Fréchet derivative is

$$\langle\Psi'(u),v\rangle=\int_\Omega f(x,u)vdx,\ \forall u,v\in W_0^{1,H}(\Omega).$$

Furthermore, it follows that the functional $I_\lambda\in C^1(W_0^{1,H}(\Omega),\mathbb{R})$ and its Fréchet derivative

$$\langle I'_\lambda(u),v\rangle=\frac{1}{\lambda}\int_\Omega(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u)\cdot\nabla vdx+\int_\Omega f(x,u)vdx$$

for any $u,v\in W_0^{1,H}(\Omega)$.

Now, we say that $u\in W_0^{1,H}(\Omega)$ is a weak solution of problem (P) if it satisfies

$$\langle\Phi'(u),v\rangle=\lambda\langle\Psi'(u),v\rangle$$

for any $v\in W_0^{1,H}(\Omega)$. Hence, if if $u\in W_0^{1,H}(\Omega)$ is a critical point of the functional I_λ, then u is a weak solution of (P).

We are now turning to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $X=W_0^{1,H}(\Omega)$, $\sigma>0$ and set $\alpha:=\frac{1}{\lambda}$. By the definition of Φ and by virtue of (2.2), Φ is coercive. Moreover, since Φ is convex and continuous, we can see that Φ is sequentially weakly lower semicontinuous. It follows from Rellich-Kondrachov compactness theorem that Ψ is sequentially weakly continuous. In view of Lemma 2.1, it suffices to show that if,

(A_1) the functional I_λ satisfies the (PS) condition;

(A_2) there exists $u_0\in W_0^{1,H}(\Omega)$ such that $I_\lambda(tu_0)\to-\infty$ as $t\to+\infty$;

(A_3) we prove that

$$\frac{1}{\lambda}>\tau(\sigma):=\inf_{u\in\Phi^{-1}(-\infty,\sigma)}\frac{\sup\limits_{v\in\Phi^{-1}(-\infty,\sigma)}\Psi(v)-\Psi(u)}{\sigma-\Phi(u)}.$$

We shall firstly show that the relation (A_1) is valid. For fixed $\lambda>0$, suppose that $\{u_n\}\subset X$, $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n)\to0$ as $n\to+\infty$. Then by virtue of conditions (h_1) and (h_2), there exist constants $C_1,c_1>0$ such that

$$\begin{aligned}C_1\geq&I_\lambda(u_n)=\int_\Omega(\frac{1}{\lambda p}|\nabla u_n|^p+\frac{\mu(x)}{q}|\nabla u_n|^q)dx-\lambda\int_\Omega F(x,u_n)dx\\\geq&\frac{1}{q}\int_\Omega(|\nabla u_n|^p+\mu(x)|\nabla u_n|^q)dx-\frac{1}{\theta}\int_\Omega u_nf(x,u_n)dx-c\\=&\frac{1}{\lambda}\Big[\frac{1}{q}-\frac{1}{\theta}\Big]\int_\Omega(|\nabla u_n|^p+\mu(x)|\nabla u_n|^q)dx+\frac{1}{\theta}\langle I'_\lambda(u_n),u_n\rangle-c_1\\\geq&\frac{1}{\lambda}\Big[\frac{1}{q}-\frac{1}{\theta}\Big]\min\{\|u_n\|^p,\|u_n\|^q\}-\frac{1}{\theta}\|I'_\lambda(u_n)\|\|u_n\|-c.\end{aligned}\tag{3.2}$$

Since $1<p<q<\theta$, (3.2) implies that the sequence $\{u_n\}$ is bounded. As X is reflexive, $\{u_n\}$ has a weakly convergent subsequence, without loss of generality, we may assume that there exists $u \in X$ such that $u_n \rightharpoonup u$ in X. Hence, owing to (2.1), we conclude that

$$u_n \to u \text{ in } L^{\gamma}(\Omega) \text{ for } \gamma \in [1, p^*). \tag{3.3}$$

Furthermore, it follows from (h_1), (3.3) and Hölder inequality that

$$\begin{aligned} \Big| \int_\Omega f(x,u_n)(u_n-u)dx \Big| \leq & C\int_\Omega (1+|u_n|^{r-1}|u_n-u|)dx \\ \leq & C|u_n-u|_1 + C||u_n|^{r-1}|_{\frac{r}{r-1}}|u_n-u|_r \\ = & C|u_n-u|_1 + C|u_n|_r^{r-1}|u_n-u|_r \to 0. \end{aligned} \tag{3.4}$$

Since $u_n \rightharpoonup u$ it follows that

$$\lim_{n\to+\infty} \langle I'_\lambda(u_n), u_n-u\rangle = 0. \tag{3.5}$$

Taking into account (3.4) and (3.5) one has

$$\begin{aligned} \lim_{n\to+\infty} \langle \Phi'(u_n), u_n-u\rangle &= \lambda \lim_{n\to+\infty} \langle \Psi'(u_n), u_n-u\rangle \\ &= \lambda \lim_{n\to+\infty} \int_\Omega f(x,u_n)(u_n-u)dx = 0. \end{aligned} \tag{3.6}$$

Since Φ' is of type (S_+) (see [12, Proposition 3.1]), $u_n \rightharpoonup u$ in X, we conclude that $u_n \to u$ in X. Therefore, the relation (A_1) follows.

Let us second examine the relation (A_2). Indeed, we observe that by assumption (h_2), there exists a constant $C_2 > 0$ such that

$$F(x,t) \geq C_2|t|^\theta, \ \forall (x,t) \in \Omega \times (-\infty, M] \cup [M, +\infty). \tag{3.7}$$

Let $u_0 \in X \setminus \{0\}$ and $t > 1$, then we have

$$\begin{aligned} I_\lambda(tu_0) = & \frac{1}{\lambda}\int_\Omega (\frac{1}{p}|\nabla tu_0|^p + \frac{\mu(x)}{q}|\nabla tu_0|^q)dx - \int_\Omega F(x,tu_0)dx \\ \leq & \frac{t^q}{\lambda p}\int_\Omega (|\nabla u_0|^p + \frac{\mu(x)}{q}|\nabla u_0|^q)dx - C_2 t^\theta \int_\Omega |u_0|^\theta dx - C_3, \end{aligned} \tag{3.8}$$

for some constant $C_3 > 0$. In view of $1 < q < \theta$, it follows from (3.8) that $I_\lambda(tu_0) \to -\infty$ as $t \to +\infty$. As a consequence, we deduce that (A_2) holds.

It remains to prove (A_3). Let us fix $\sigma > 0$. Obviously, $0 \in \Phi^{-1}(-\infty, \sigma)$. Then, we conclude that

$$\tau(\sigma) \leq \frac{\sup\limits_{v\in\Phi^{-1}(-\infty,\sigma)} \Psi(v)}{\sigma}.$$

Using (h_1) and (2.1), we deduce

$$\begin{aligned} \Psi(u) = & \int_\Omega F(x,u)dx \leq C\int_\Omega (|u| + |u|^r)dx = C(|u|_1 + |u|_r^r) \\ \leq & Cs_1\|u\| + Cs_r^r\|u\|^r, \end{aligned} \tag{3.9}$$

where s_1, s_r are the best constants for the continuous embeddings $X \hookrightarrow L^1(\Omega)$ and $X \hookrightarrow L^r(\Omega)$, respectively. Now, for each $u \in \Phi^{-1}(-\infty, \sigma)$, it follows from (2.2) that

$$\begin{aligned} q\sigma \geq q\Phi(u) =& q\int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q)dx \\ \geq& \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)dx \\ \geq& \begin{cases} \|u\|^p, & \text{if } \|u\| \geq 1, \\ \|u\|^q, & \text{if } \|u\| \leq 1, \end{cases} \end{aligned}$$

and consequently

$$\|u\| \leq q_\sigma, \tag{3.10}$$

where q_σ is defined in Theorem 1.1.

Therefore, combining (3.9), (3.10) together with (1.2), we have

$$\frac{1}{\sigma} \sup_{u \in \Phi^{-1}(-\infty,\sigma)} \Psi(u) \leq \frac{Cs_1 q_\sigma + Cs_r q_\sigma^r}{\sigma} < \frac{1}{\lambda} \tag{3.11}$$

and the relation (A_3) follows.

Therefore, all the assumptions of Lemma 2.1 are satisfied, so that, the problem (P) has at least two weak solutions one of which lies in $\Phi^{-1}(-\infty, \sigma)$. The proof of Theorem 1.1 is now complete. □

Remark 3.1. If we assume that $k(\sigma) := \frac{\sigma}{Cs_1 q_\sigma + Cs_r q_\sigma^r}$ with $\sigma > 0$, then one has $\max_{\sigma>0} k(\sigma) < +\infty$ since $r > q > p$. Therefore, in this case, for each $\lambda \in \big(0, \max_{\sigma>0} k(\sigma)\big)$ Theorem 1.1 ensures the existence of at least two weak solutions. Among other things, compared with the results obtained by [9, 12], our results are new and very different because of the following facts:

(C_1) assumption (h_1) is somewhat weaker than (f_2) in [12, Theorem 1.3] or (h_2) in [9, Theorem 1.3];

(C_2) in our results, there is no need to assume

$$(h_3) \qquad \lim_{t\to 0} \frac{f(x,t)}{|t|^{p-1}} = 0 \text{ uniformly in } x;$$

(C_3) Theorem 1.1 (and its consequences) represents a more precise version of Theorem 1.3 in [12].

Remark 3.2. It is easy to check that the following the nonlinearities f satisfy assumptions (h_1) and (h_2):

$$f(x,t) = \begin{cases} 1 + rt^{r-1}, & \text{if } t \geq 0, \\ 1 - r(-t)^{r-1}, & \text{if } t < 0, \end{cases}$$

where $r \in (1, p^*)$. But it does not satisfy the assumption (h_3).

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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