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HOPF BIFURCATION PROBLEM FOR A CLASS OF KOLMOGOROV MODEL WITH A POSITIVE NILPOTENT CRITICAL POINT*

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Abstract In this paper, We discuss the Hopf bifurcation problem of a three-order positive nilpotent critical point $(1,1)$ of a class of Kolmogorov model. By using the method offered by [12], we obtain the expressions of quasi-Lyapunov constants with the help of computer algebra system-MATHEMATICA. By analyzing the structure of these quasi-lyapunov constants, we divide them into two kinds of cases and study their bifurcation behavior separately. For case 1, the nilpotent critical point $(1,1)$ can bifurcate 5 small amplitude limit cycles. For case 2, 6 small amplitude limit cycles can bifurcate from the three-order nilpotent critical point $(1,1)$. In addition, We also give the integrability conditions (i.e., center condition) for each case. In terms of limit cycle bifurcation for Kolmogorov model with nilpotent positive critical points, our result is new.

Keywords Hopf bifurcation, nilpotent critical point, kolmogorov model, quasi-Lyapunov constant.

MSC(2010) 34C07, 34C23.

1. Introduction

Our work focuses on investigating the limit cycle bifurcation and integrability condition of a three-order nilpotent critical point $(1,1)$ of the following Kolmogorov model

$$\begin{cases} \dfrac{dx}{dt} =x[\delta(x-1)^2-(x-2)(y-1)+a_{11}(x-1)(y-1)+a_{12}(y-1)^2] \\ \qquad \equiv xP(x,y), \\ \dfrac{dy}{dt} =-y[2(x-1)^3+2\delta(x-1)(y-1)(y-2)-b_{02}(y-1)^2+(b_{02}-b_{03}) \\ \qquad \times (y-1)^3-(x-1)(y-1)(b_{21}x+b_{12}y-b_{12}-b_{21})] \equiv yQ(x,y), \end{cases} \tag{1.1}$$

in which a_{11}, a_{12}, b_{02}, b_{03}, b_{12}, b_{21}, $\in R$, δ is a small real parameter and $\delta \to 0$. Here, x and y of system (1.1) denote prey and predator densities, and $P(x,y)$, $Q(x,y)$ are the intrinsic growth rates or biotic potential of the prey and predators,

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respectively. Obviously, the above system belongs to a class of famous ecologic model namely Kolmogorov model, Ref. [4, 14] described their realistic meanings.

Some natural predator-prey behavior can be summarized as an ecological model which further attracts people's attention and can be discussed and investigated by using mathematical method. In mathematical ecology, Many mathematic workers pay more and more attentions to the three most fundamental systems namely the predator-prey, the competition and the cooperation systems(see [1–4, 6–10, 13, 14, 16]). Theoretically, these systems can be reduced to some kinds of ecological models. From published references, it can be seen that Kolmogorov model is a class of investigated thermal ecological model. Kolmogorov model's equation is described as

$$\begin{cases} \dfrac{dx}{dt} =xf(x,y), \\ \dfrac{dy}{dt} =yg(x,y), \end{cases} \tag{1.2}$$

in which $f(x, y)$ and $g(y, x)$ are polynomials on x, y. Kolmogorov models are widely used in ecology to describe the interaction between two populations. Of course, in the context of meaningful research, attention of researchers is restricted to the behavior of orbits in the "realistic quadrant" $\{(x, y) : x > 0, y > 0\}$. Of particular significance in applications is the existence of limit cycles and the number of limit cycles that can bifurcate from positive equilibrium points, because a limit cycle corresponds to an equilibrium state of the system and the existence and stability of limit cycles is related to the positive equilibrium points. At the same time, the problem on the number of limit cycles gets in close touch with famous Hilbert 16^{th} problem; Hence, many articles studying Kolmogorov models pay more attention to the limit cycles bifurcation problem. For example, [2] showed a class of Kolmogorov system could bifurcate five limit cycles including 4 stable cycles; [3] gave an example about a class of Kolmogorov system which could bifurcate ten limit cycles; [6] studied the number of limit cycles of polynomial Lienard systems; [7] investigated hopf bifurcation problem about small amplitude limit cycles and the local bifurcation of critical periods for a quartic Kolmogrov system at the single positive equilibrium point $(1, 1)$ and proved that the maximum number of small amplitude limit cycles bifurcating from the equilibrium point $(1, 1)$ is 7; [9] considered the Kolmogorov system of degree 3 in R^2 and R^3 having an equilibrium point in the positive quadrant and studied their limit cycle bifurcation problem; [13] studied a class of cubic Kolmogorov system with three limit cycles; [10] showed a class of cubic Kolmogorov system could bifurcate six limit cycles; [8, 16] studied a general Kolmogorov model and obtained the conditions for the existence and uniqueness of limit cycles, at the same time it classify a series of models. As far as limit cycles of Kolmogorov models are concerned, many good results have been obtained by analyzing sole positive equilibrium point's state.

In terms of limit cycle problem on nilpotent critical points, less literatures has been published. [1] investigated the problem on limit cycle bifurcation for a class of Z3-equivariant Lyapunov system of five degrees with three third-order nilpotent critical points which lie in a Z3-equivariant vector field, and gave the result of existing 12 small amplitude limit cycles created from the three third-order nilpotent critical points. [15] characterized local behavior of an isolated nilpotent critical point for a class of septic polynomial differential systems including center conditions and bifurcation of limit cycles, and proved that there exist 16 small amplitude limit cycles created from the third-order nilpotent critical point. But it is hardly seen results on limit cycles bifurcation from nilpotent critical points for Kolmogorov model, here we will complement our work in this area, namely limit cycle bifurcation of system (1.1).

Clearly, model (1.1) has a nilpotent critical point namely the positive equilibrium point $(1, 1)$. We will focus on the limit cycles bifurcations of the positive equilibrium point $(1, 1)$. At first, we introduce a kind of research method about the limit cycles bifurcations from nilpotent critical point.

In the qualitative theory of ordinary differential equations, the limit cycles bifurcations problem for a critical point P of a planar analytic vector field X is a hot topics. Let $DX(P)$ denote the differential matrix of X at the critical point P. When the eigenvalues of the matrix $DX(P)$ are imaginary, we know that the origin is monodromic. When the matrix $DX(P)$ has its two eigenvalues equal to zero but the matrix is not identically null, it is said that P is a nilpotent critical point. In a suitable coordinate system, the Lyapunov system with the origin as a nilpotent critical point can be written as

$$\begin{cases} \dfrac{dx}{dt} = y + \displaystyle\sum_{i+j=2}^{\infty} a_{ij}x^i y^j = X(x, y), \\ \dfrac{dy}{dt} = \displaystyle\sum_{i+j=2}^{\infty} b_{ij}x^i y^j = Y(x, y), \end{cases} \tag{1.3}$$

in which the function $y = y(x)$ satisfies $X(x, y) = 0, y(0) = 0$.

In this paper, employing the integral factor method introduced in [11, 12], we will discuss several cases, and being based on these cases, we investigate the center-focus problem and prove the singular point $(1, 1)$ of model (1.1) can bifurcate 6 small limit cycles. To the best of our knowledge, our results on the lower bounds of cyclicity of a three-order nilpotent critical point for planar Kolmogorov model are new.

Our work will be expanded as follows. In Section 2, we state some preliminary knowledge given in [11] which is useful throughout the paper. In Section 3, using the linear recursive formulae in [11] to do direct computation, we obtain with relative ease the first 6 quasi-Lyapunov constants of critical point $(1, 1)$. Moreover, we investigate the center-focus problem and give the sufficient and necessary condition that critical point $(1, 1)$ of model (1.1) can become a center, at the same time, we show the fact that critical point $(1, 1)$ can also become a 6-order weak focus and model (1.1) can bifurcate 6 small limit cycles from $(1, 1)$.

2. The method to compute the Lyapunov constants of nilpotent critical points

Ref. [12] offered a kind of method to compute the Lyapunov constants of nilpotent critical points. We will use this method to carry out our research work. For convenience, let's introduce this method.

The origin of system (1.3) is a three-order monodromic critical point if and only if $b_{20} = 0, (2a_{20} - b_{11})^2 + 8b_{30} < 0$. Without loss of generality, we can assume that

$$a_{20} = \mu, \quad b_{20} = 0, \quad b_{11} = 2\mu, \quad b_{30} = -2. \tag{2.1}$$

Otherwise, by letting $(2a_{20}-b_{11})^2+8b_{30}=-16\lambda^2, 2a_{20}+b_{11}=4\lambda\mu$, and making the transformation $\xi=\lambda x, \eta=\lambda y+\frac{1}{4}(2a_{20}-b_{11})\lambda x^2$, we can also arrive at this aim. From (2.1), system (1.3) becomes the following real autonomous planar system

$$\begin{cases} \dfrac{dx}{dt}=y+\mu x^2+\displaystyle\sum_{i+2j=3}^{\infty} a_{ij}x^iy^j=X(x,y), \\ \dfrac{dy}{dt}=-2x^3+2\mu xy+\displaystyle\sum_{i+2j=4}^{\infty} b_{ij}x^iy^j=Y(x,y). \end{cases} \tag{2.2}$$

Write that

$$X(x,y)=y+\sum_{k=2}^{\infty}X_k(x,y),\ Y(x,y)=\sum_{k=2}^{\infty}Y_k(x,y), \tag{2.3}$$

where for $k=1,2,\cdots$,

$$X_k(x,y)=\sum_{i+j=k}a_{ij}x^iy^j,\ Y_k(x,y)=\sum_{i+j=k}b_{ij}x^iy^j. \tag{2.4}$$

For the computation about quasi-Lyapunov constants of the origin of system (2.2), Ref. [12] offered the following method.

Lemma 2.1 ([12]). *For system* (2.2) *and any positive integer s and a given number sequence*

$$\{c_{0\beta}\},\quad \beta\geq 3, \tag{2.5}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ satisfying $\alpha\neq 0$ of the formal series

$$M(x,y)=y^2+\sum_{\alpha+\beta=3}^{\infty}c_{\alpha\beta}x^{\alpha}y^{\beta}=\sum_{k=2}^{\infty}M_k(x,y), \tag{2.6}$$

such that

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right)+\frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right)=\frac{1}{M^{s+2}}\sum_{m=5}^{\infty}\omega_m(s,\mu)x^m, \tag{2.7}$$

where for all k, $M_k(x,y)$ is a k-homogeneous polynomial of x,y and $s\mu=0$.

Now, (2.7) can be written by

$$\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)M-(s+1)\left(\frac{\partial M}{\partial x}X+\frac{\partial M}{\partial y}Y\right)=\sum_{m=3}^{\infty}\omega_m(s,\mu)x^m. \tag{2.8}$$

It is easy to see that (2.8) is linear with respect to the function M, so that we can easily find the following recursive formulae for the calculation of $c_{\alpha\beta}$ and $\omega_m(s,\mu)$.

Lemma 2.2 ([12]). *For $\alpha\geq 1, \alpha+\beta\geq 3$ in* (2.6) *and* (2.7), *$c_{\alpha\beta}$ can be uniquely determined by the recursive formula*

$$c_{\alpha\beta}=\frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1}+B_{\alpha-1,\beta+1}). \tag{2.9}$$

For $m\geq 1$, $\omega_m(s,\mu)$ can be uniquely determined by the recursive formula

$$\omega_m(s,\mu)=A_{m,0}+B_{m,0}, \tag{2.10}$$

where

$$A_{\alpha\beta}=\sum_{k+j=2}^{\alpha+\beta-1}[k-(s+1)(\alpha-k+1)]a_{kj}c_{\alpha-k+1,\beta-j},$$
$$B_{\alpha\beta}=\sum_{k+j=2}^{\alpha+\beta-1}[j-(s+1)(\beta-j+1)]b_{kj}c_{\alpha-k,\beta-j+1}. \tag{2.11}$$

Notice that in (2.11)*, we set*

$$\begin{aligned}&c_{00}=c_{10}=c_{01}=0,\\&c_{20}=c_{11}=0,\ c_{02}=1,\\&c_{\alpha\beta}=0,\ \text{if}\ \alpha<0\ \text{or}\ \beta<0.\end{aligned} \tag{2.12}$$

Lemma 2.3 ([12]). *The expressions of $m-th$ quasi-Lyapunov constants at the origin of system* (2.2) *are as follows:*

$$\lambda_m=\frac{\omega_{2m+4}(s,\mu)}{2m-4s-1}. \tag{2.13}$$

Lemma 2.4 ([12]). *If system* (2.2) *has a three-order nilpotent center at the origin, then there always exists a formal integral factor of the form* (2.6)*.*

Clearly, the recursive formulae given by Lemma 2.2 is linear with respect to all $c_{\alpha\beta}$. Therefore, it is convenient to realize the computation of quasi-Lyapunov constants by using computer algebraic system like MATHEMATICA.

3. Bifurcation of limit cycles and center condition of the critical point $(1,1)$ of model (1.1)

In order to obtain the quasi-Lyapunov constants of the critical point $(1,1)$ of model (1.1) and study the limit cycles bifurcations. May as well make the following transformations:

$$x=x_1+1,\ y=y_1+1, \tag{3.1}$$

system (1.1) is changed into

$$\begin{cases}\dfrac{dx_1}{dt}=(x_1+1)[\delta x_1^2-(x_1-1)y_1+a_{11}x_1y_1+a_{12}y_1^2],\\ \dfrac{dy_1}{dt}=-(y_1+1)[2x_1^3+2\delta x_1y_1(y_1-1)-b_{02}y_1^2+(b_{02}-b_{03})y_1^3\\ \qquad -x_1y_1(b_{21}x_1+b_{12}y_1)].\end{cases} \tag{3.2}$$

Comparing system (2.2) with system (3.2), it is clear that the origin of system (3.2) is a nilpotent critical point. According to the translation's invariable property, model (1.1) has a three-order nilpotent critical point $(1,1)$. Hence the study on the origin of system (3.2) will can derive the similar property for the three-order nilpotent critical point $(1,1)$ of model (1.1). Next we will investigate the bifurcation behavior of the origin of system (3.2).

According to Lemma 2.2 and applying the recursive formulae presented in Lemma 2.1–Lemma 2.3 to carry out calculations by using MATHEMATICA, we can obtain

$$
\begin{aligned}
&\omega_3=\omega_4=\omega_5=0,\quad \omega_6=-\frac{1}{3}b_{21}(-1+4s),\\
&\omega_7\sim -1-2a_{12}+2s-2a_{12}s+3c_{03}+3sc_{03},\\
&\textit{Let } c_{03}=\frac{1+2a_{12}-2s+2a_{12}s}{3(1+s)},\ \textit{then}\\
&\omega_8\sim \frac{2}{5}(-a_{12}+a_{11}a_{12}+b_{02}+2a_{12}b_{02}-3b_{03})(4s-3),\\
&\textit{Let } b_{03}=\frac{1}{3}(-a_{12}+a_{11}a_{12}+b_{02}+2a_{12}b_{02}),\ \textit{then}\\
&\omega_9\sim -\frac{2}{3}(2a_{12}-3a_{11}a_{12}+a_{11}^2a_{12}-2a_{12}b_{02}+2a_{11}a_{12}b_{02}+2b_{12}-2a_{12}b_{12})(s-1).
\end{aligned}
\tag{3.3}
$$

From (2.13) and (3.3), we obtain the first two quasi-Lyapunov constants at the origin of system (3.3) as follows:

$$
\begin{aligned}
&\lambda_1\sim\frac{1}{3}b_{21},\\
&\lambda_2\sim-\frac{2}{5}(-a_{12}+a_{11}a_{12}+b_{02}+2a_{12}b_{02}-3b_{03}).
\end{aligned}
\tag{3.4}
$$

From (3.4), we know that $\omega_9\sim 0$ will deduce $s=1$ or $2a_{12}-3a_{11}a_{12}+a_{11}^2a_{12}-2a_{12}b_{02}+2a_{11}a_{12}b_{02}+2b_{12}-2a_{12}b_{12}=0$. Then we investigate the quasi-Lyapunov constants according to the following cases.

3.1. Case 1: $s=1$

If $s=1$, then the origin of system (3.2) is a three-order nilpotent critical point of 1-class. At this kind of case, we can compute the quasi-Lyapunov constants at the origin of system (3.2), namely the following Theorem.

Theorem 3.1. *If the origin of system* (3.2) *is a three-order nilpotent critical point of* 1*-class, then the first 5 quasi-Lyapunov constants at the origin of system* (3.2) *are as follows:*

$$
\begin{aligned}
&\lambda_1\sim\frac{1}{3}b_{21};\\
&\lambda_2\sim-\frac{2}{5}(-a_{12}+a_{11}a_{12}+b_{02}+2a_{12}b_{02}-3b_{03});\\
&\lambda_3\sim-\frac{2}{105}(-10a_{12}+29a_{11}a_{12}-26a_{11}^2a_{12}+7a_{11}^3a_{12}-12a_{12}b_{02}\\
&\qquad+9a_{11}a_{12}b_{02}+3a_{11}^2a_{12}b_{02}+22a_{12}b_{02}^2-22a_{11}a_{12}b_{02}^2+4a_{11}b_{12}+10a_{12}b_{12}\\
&\qquad-14a_{11}a_{12}b_{12}-22b_{02}b_{12}+22a_{12}b_{02}b_{12});\\
&\lambda_4\sim\frac{4(a_{11}-1)n_1}{4725(7a_{11}-11b_{02}-5)^2(2-3a_{11}+a_{11}^2-2b_{02}+2a_{11}b_{02}-2b_{12})^2};
\end{aligned}
\tag{3.5}
$$

in which n_1 is the function about a_{11}, b_{02}, b_{12}, whose expression is shown in Appendix.

And
(1)If $a_{11} = 1$, *then* $\lambda_5 \sim 0$.
(2) If $a_{11} \neq 1, n_1 = 0$, *then*

$$\lambda_5 \sim \frac{a_{11}-1}{6767145000(7a_{11}-11b_{02}-5)^3(2-3a_{11}+a_{11}^2-2b_{02}+2a_{11}b_{02}-2b_{12})^2 b_{12}} n_2;$$

in which n_2 *is the function about* a_{11}, b_{02}, b_{12}, *whose expression is shown in Appendix.*

By analyzing the construction of quasi-Lyapunov constants of Theorem 3.1, it is easy to obtain the following result.

Theorem 3.2. *If the origin of system* (3.2) *is a three-order nilpotent critical point of* 1*-class, then the origin for system* (3.2) *can become a 5-order weak focus if and only if the following condition holds:*

$$\begin{aligned} &b_{21} = 0,\, b_{03} = \frac{1}{3}(-a_{12} + a_{11}a_{12} + b_{02} + 2a_{12}b_{02}),\\ &a_{12} = \frac{2b_{12}(11b_{02} - 2a_{11})}{(7a_{11}-11b_{02}-5)(2-3a_{11}+a_{11}^2-2b_{02}+2a_{11}b_{02}-2b_{12})},\\ &n_1 = 0,\, a_{11} \neq 1. \end{aligned} \tag{3.6}$$

Proof. From the expression of $\lambda_k, k = 1, 2, 3, 4, 5$, clearly the necessity holds. Next we prove sufficiency . In order to prove the origin of system (3.2) is a 5-order weak focus, we only need to prove that there exists a group of solutions about $a_{11}, b_{21}, b_{03}, a_{12}, b_{12}$ such that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \lambda_5 \neq 0$.

Clearly, $b_{21} = 0$ if $\lambda_1 = 0$. According to $\lambda_2 = 0$, we can obtain $b_{03} = \frac{1}{3}(-a_{12} + a_{11}a_{12} + b_{02} + 2a_{12}b_{02})$ and the expression of λ_3, moreover, $\lambda_3 = 0$ deduces

$$a_{12} = \frac{2b_{12}(11b_{02} - 2a_{11})}{(7a_{11}-11b_{02}-5)(2-3a_{11}+a_{11}^2-2b_{02}+2a_{11}b_{02}-2b_{12})}. \tag{3.7}$$

Next let $\lambda_4 = 0$, then $n_1 = 0$ or $a_{11} = 1$, but $a_{11} = 1$ will deduce $\lambda_5 = 0$, while $n_1 = 0$ and $a_{11} \neq 1$ will deduce

$$\lambda_5 \sim \frac{(a_{11}-1)n_2}{6767145000(7a_{11}-11b_{02}-5)^3(2-3a_{11}+a_{11}^2-2b_{02}+2a_{11}b_{02}-2b_{12})^2 b_{12}} \neq 0, \tag{3.8}$$

otherwise $n_1 = 0, n_2 = 0$ will deduce $a_{11} = 1$. Proof end. ☐

From the translation's invariable property, it is clear that the following Theorem holds.

Theorem 3.3. *The critical points* $(1, 1)$ *of model* (1.1) *become a 5-order weak focuses if the condition of Theorem 3.2 holds.*

After discussing the weak focus problem of system (1.1), we will investigate the limit cycles bifurcations of system (1.1). According to Theorem 3.1, we can obtain the following theorem.

Theorem 3.4. *If the critical points* $(1, 1)$ *of model* (1.1) *become a 5-order weak focuses, then model* (1.1) *can bifurcate 5 small limit cycles from* $(1, 1)$.

Proof. According to Theorem 2.1 in Ref. [12], we only need to prove

$$\frac{D(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{D(b_{21},b_{03},a_{12},b_{02})}\neq 0, \tag{3.9}$$

In fact,

$$\begin{aligned}&\frac{D(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{D(b_{21},b_{03},a_{12},b_{02})}\\ =&\frac{16(a_{11}-1)n_1^2}{2480625(7a_{11}-11b_{02}-5)^2(2-3a_{11}+a_{11}^2-2b_{02}+2a_{11}b_{02}-2b_{12})^2}.\end{aligned} \tag{3.10}$$

If the critical points $(1,1)$ of model (1.1) become a 5-order weak focus, then $a_{11}\neq 1$ and $n_1\neq 0$, hence Eq.(3.9) holds. According to Theorem 2.1 in Ref. [12], the conclusions of Theorem 3.4 holds. Proof end. □

Next we will consider the center problem of system (3.2) or model (1.1) under case 1, by analyzing the construction of λ_k, $k=1,2,3,4,5$ of Theorem 3.1, it is easy to obtain the following results.

Theorem 3.5. *If the origin of system* (3.2) *is a three-order nilpotent critical point of* 1*-class , then the first 5 quasi-Lyapunov constants of the origin for system* (3.2) *vanish if and only if the following condition is satisfied:*

$$b_{21}=0,\ a_{11}=1,\ a_{12}=1,\ b_{03}=b_{02}. \tag{3.11}$$

Moreover, we have the following Theorem.

Theorem 3.6. *If the origin of system* (3.2) *is a three-order nilpotent critical point of* 1*-class, then the origin of system* $(3.2)|_{\delta=0}$ *is a center (or the critical point* $(1,1)$ *of model* $(1.1)|_{\delta=0}$ *is a center) if and only if the condition of Theorem 3.5 holds, namely* (3.11) *holds.*

Proof. Obviously, necessity holds. Next, we prove sufficiency. If condition (3.11) holds, then system $(3.2)|_{\delta=0}$ is changed into the following form:

$$\begin{cases}\dfrac{dx_1}{dt}=&y_1(x_1+1)(1+y_1),\\ \dfrac{dy_1}{dt}=&-(y_1+1)(2x_1^3-b_{02}y_1^2-b_{12}x_1y_1^2),\end{cases} \tag{3.12}$$

make time transformations $dT=(1+y_1)dt$, system (3.12) becomes

$$\begin{cases}\dfrac{dx_1}{dT}=&y_1(x_1+1),\\ \dfrac{dy_1}{dT}=&-2x_1^3+b_{02}y_1^2+b_{12}x_1y_1^2,\end{cases} \tag{3.13}$$

whose vector field is symmetric with respect to $y-$axis. Therefore, under condition (3.11), the origin of system $(3.2)|_{\delta=0}$ is a center, According to the transformations' invariant property, the critical point $(1,1)$ of model $(1.1)|_{\delta=0}$ is a center. Proof end. □

Next we will consider the second case, namely the following case.

3.2. Case 2: $s \neq 1$

If $s \neq 1$, then $\omega_9 = 0$ will deduce $2a_{12} - 3a_{11}a_{12} + a_{11}^2 a_{12} - 2a_{12}b_{02} + 2a_{11}a_{12}b_{02} + 2b_{12} - 2a_{12}b_{12} = 0$, namely

$$b_{02} = \frac{2a_{12} - 3a_{11}a_{12} + a_{11}^2 a_{12} + 2b_{12} - 2a_{12}b_{12}}{2a_{12}(a_{11} - 1)}. \tag{3.14}$$

At this time,

$$\lambda_2 \sim 0, \ \lambda_3 \sim \frac{4}{21} b_{12}(a_{11} - 1), \tag{3.15}$$

from the expressions of b_{02}, we know $a_{11} \neq 1$. Next let $b_{12} = 0$, we obtain

$$\lambda_3 \sim 0, \ \lambda_4 \sim -\frac{4}{45}(5a_{12} - 2)(a_{12} + a_{11} - 2), \tag{3.16}$$

at this time, $\lambda_4 \sim 0$ will deduce $a_{12} = \frac{2}{5}$ or $a_{12} = 2 - a_{11}$, moreover, we will consider the following two cases.

3.2.1. Case 2.1: $a_{12} = \frac{2}{5}$

If $a_{12} = \frac{2}{5}$, then

$$\omega_{13} \sim -\frac{4}{75}(a_{11} - 1)(5a_{11} - 8)(s - 2), \tag{3.17}$$

let $\omega_{13} \sim 0$, then $a_{11} = \frac{8}{5}$ or $s = 2$. Next let's study it in two different cases.

Case 2.1.1: $a_{12} = \frac{2}{5}$, $a_{11} = \frac{8}{5}$

If $a_{11} = \frac{8}{5}$, then $\lambda_5 \sim 0$.

Case 2.1.2: $a_{12} = \frac{2}{5}$, $a_{11} \neq \frac{8}{5}$, $s = 2$

If $s = 2$, then we obtain

$$\lambda_5 \sim -\frac{4}{5775}(a_{11} - 1)(5a_{11} - 8)(35a_{11} - 52), \tag{3.18}$$

let $a_{11} = \frac{52}{35}$, then

$$\lambda_6 \sim -\frac{6893504}{4389328125} \neq 0. \tag{3.19}$$

From the above analysis, we have the following Theorem.

Theorem 3.7. *If the origin of system* (3.2) *is a three-order nilpotent critical point of* 2*-class and* $a_{12} = \frac{2}{5}$*, then the first 6 quasi-Lyapunov constants of the origin for system* (3.2) *are as follows:*

$$\begin{aligned}
\lambda_1 &\sim \frac{1}{3} b_{21}; \\
\lambda_2 &\sim -\frac{2}{5}(-a_{12} + a_{11}a_{12} + b_{02} + 2a_{12}b_{02} - 3b_{03}); \\
\lambda_3 &\sim \frac{4}{21} b_{12}(a_{11} - 1); \\
\lambda_4 &\sim -\frac{4}{45}(5a_{12} - 2)(a_{12} + a_{11} - 2); \\
\lambda_5 &\sim -\frac{4}{5775}(a_{11} - 1)(5a_{11} - 8)(35a_{11} - 52); \\
\lambda_6 &\sim -\frac{6893504}{4389328125}.
\end{aligned} \tag{3.20}$$

In the above expressions of λ_k, we have already let $\lambda_1 = \lambda_2 = \cdots \lambda_{k-1} = 0$.

By analyzing the construction of quasi-Lyapunov constants of Theorem 3.7, it is easy to obtain the following result.

Theorem 3.8. *If the origin of system* (3.2) *is a three-order nilpotent critical point of* 2*-class, then the origin for system* (3.2) *can become a* 6*-order weak focus if and only if the following condition holds:*

$$\begin{aligned} &b_{21} = 0,\, b_{03} = \frac{1}{3}(-a_{12} + a_{11}a_{12} + b_{02} + 2a_{12}b_{02}), \\ &b_{02} = \frac{2a_{12} - 3a_{11}a_{12} + a_{11}^2 a_{12} + 2b_{12} - 2a_{12}b_{12}}{2a_{12}(a_{11} - 1)}, \\ &b_{12} = 0,\, a_{12} = \frac{2}{5},\, a_{11} = \frac{52}{35}. \end{aligned} \tag{3.21}$$

Proof. From the expression of $\lambda_k, k = 1, 2, 3, 4, 5$, clearly the necessity holds. Next we prove sufficiency. Submit (3.21) into λ_k of Theorem 3.7, we have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \lambda_6 \neq 0$. Hence, conclusion of Theorem 3.8 holds. Proof end. □

Of course, condition (3.21) can also be expressed as follows:

$$b_{21} = 0,\, b_{03} = \frac{23}{105},\, b_{02} = \frac{9}{35},\, b_{12} = 0,\, a_{12} = \frac{2}{5},\, a_{11} = \frac{52}{35}. \tag{3.22}$$

From the translation's invariable property, it is clear that the following Theorem holds.

Theorem 3.9. *The critical points* $(1, 1)$ *of model* (1.1) *become a* 6*-order weak focuses if condition* (3.21) *or* (3.22) *holds.*

After discussing the weak focus problem of system (1.1), we will investigate the limit cycles bifurcations of system (1.1). According to the above analysis, we can obtain the following theorem.

Theorem 3.10. *If the critical points* $(1, 1)$ *of model* (1.1) *become a* 6*-order weak focuses (namely under the condition* (3.22)*), then disturbed model* (1.1) *can bifurcate* 6 *small limit cycles from* $(1, 1)$*.*

Proof. According to Theorem 2.1 in Ref. [12], we only need to prove

$$\frac{D(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)}{D(b_{21}, b_{02}, b_{12}, a_{12}, a_{11})} \neq 0.$$

In fact,

$$\begin{aligned} M &= \frac{D(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)}{D(b_{21}, b_{02}, b_{12}, a_{12}, a_{11})} \\ &= \frac{128(a_{11} - 1)(956 - 1430a_{11} + 525a_{11}^2)(1 + 2a_{12})(-12 + 5a_{11} + 10a_{12})}{81860625}, \end{aligned}$$

and $M|_{(3.10)} = -\frac{591872}{77994984375} \neq 0$. Hence, the conclusion of Theorem 3.10 holds. Proof end. □

3.2.2. Case 2.2: $a_{12} = 2 - a_{11}$

Under condition $a_{12} = 2 - a_{11}$, we can obtain $\lambda_4 \sim 0, \lambda_5 \sim 0, \lambda_6 \sim 0$.

From the above investigation about quasi-Lyapunov constants at the origin of system (3.2) under case 2 (including case 2.1 and case 2.2), we will also consider the center problem of the origin of system $(3.2)|_{\delta=0}$ (or the critical point $(1,1)$ of model $(1.1)|_{\delta=0}$) under case 2. Analyze the expressions of quasi-Lyapunov constants under case 2, we can easily obtain the following results.

Theorem 3.11. *If the origin of system* (3.2) *is a three-order nilpotent critical point of non-*1*-class , then the first 6 quasi-Lyapunov constants of the origin for system* (3.2) *vanish if and only if one of the following two conditions is satisfied:*

$$C_1: \ b_{21} = 0,\, b_{12} = 0,\, a_{12} = \frac{2}{5},\, a_{11} = \frac{8}{5},\, b_{03} = \frac{1}{5},\, b_{02} = \frac{1}{5}, x \tag{3.23}$$

$$C_2: \ b_{21} = 0,\, b_{12} = 0,\, a_{12} = 2 - a_{11},\, b_{03} = \frac{1}{2}(2 - a_{11}),\, b_{02} = \frac{1}{2}(2 - a_{11}). \tag{3.24}$$

Moreover, we have the following Theorem.

Theorem 3.12. *If the origin of system* (3.2) *is a three-order nilpotent critical point of non-*1*-class, then the origin of system* $(3.2)|_{\delta=0}$ *is a center (or the critical point* $(1,1)$ *of model* $(1.1)|_{\delta=0}$ *is a center) if and only if one of the two conditions of Theorem 3.11 holds, namely* (3.23) *or* (3.24) *holds.*

Proof. Obviously, necessity holds. Next, we prove sufficiency. If condition (3.23) holds, then system $(3.2)|_{\delta=0}$ is changed into the following form:

$$\begin{cases} \dfrac{dx_1}{dt} =& y_1(x_1+1)(1+\dfrac{3}{5}x_1+\dfrac{2}{5}y_1), \\ \dfrac{dy_1}{dt} =& -(y_1+1)(2x_1^3-\dfrac{1}{5}y_1^2), \end{cases} \tag{3.25}$$

system (3.25) has an integrating factor

$$\frac{1}{(1+x_1)^2(1+y_1)}, \tag{3.26}$$

and a first integral

$$f_1 = \frac{5(-15-45x_1-15x_1^2+5x_1^3+3y_1+3x_1y_1+y_1^2)}{1+x_1} + 15\ln\frac{(1+x_1)^{10}}{1+y_1}. \tag{3.27}$$

If condition (3.24) holds, then system $(3.2)|_{\delta=0}$ are changed into the following form:

$$\begin{cases} \dfrac{dx_1}{dt} =& = y_1(x_1+1)[1+(a_{11}-1)x_1+(2-a_{11})y_1)], \\ \dfrac{dy_1}{dt} =& = -(y_1+1)[2x_1^3-\dfrac{1}{2}(2-a_{11})y_1^2], \end{cases} \tag{3.28}$$

system (3.28) has an integrating factor

$$\frac{1}{(1+x_1)^2(1+y_1)}, \tag{3.29}$$

and a first integral and a first integral

$$f_2 = \frac{-6 - 18x - 6x^2 + 2x^3 - 2y + 2a_{11}y - 2xy + 2a_{11}xy + 2y^2 - a_{11}y^2}{2(1+x_1)} + \ln(1+x_1)^6(1+y_1)^{1-a_{11}}. \tag{3.30}$$

Therefore, under condition (3.23) or (3.24), the origin of system $(3.2)|_{\delta=0}$ is a center. According to the transformations' invariant property, the critical point $(1,1)$ of model $(1.1)|_{\delta=0}$ is a center under condition (3.23) or (3.24). Proof end. □

4. Conclusion

The work of this paper focuses on investigating the limit cycle bifurcation problem of a three-order positive nilpotent critical point $(1,1)$ of a class of Kolmogorov model. By computing and analyzing the expressions of quasi-Lyapunov constants carefully, we divide them into two kinds of cases and study their bifurcation behavior separately. For case 1, the nilpotent critical point $(1,1)$ can bifurcate 5 small amplitude limit cycles. For case 2, 6 small amplitude limit cycles can bifurcate from the three-order nilpotent critical point $(1,1)$. In addition, We also give the integrability conditions (i.e., center condition) for each case. In terms of limit cycle bifurcation for Kolmogorov model with nilpotent positive critical points, our result is new.

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Appendix: Expressions of n_1, n_2

$$\begin{aligned}
n_1 =& 42000b_{02} - 201600a_{11}b_{02} + 370020a_{11}^2b_{02} - 322140a_{11}^3b_{02} + 132300a_{11}^4b_{02} \\
&- 20580a_{11}^5b_{02} + 100800b_{02}^2 - 267120a_{11}b_{02}^2 + 189000a_{11}^2b_{02}^2 - 5040a_{11}^3b_{02}^2 \\
&- 17640a_{11}^4b_{02}^2 - 124320b_{02}^3 + 505680a_{11}b_{02}^3 - 506940a_{11}^2b_{02}^3 + 125580a_{11}^3b_{02}^3 \\
&- 221760b_{02}^4 + 166320a_{11}b_{02}^4 + 55440a_{11}^2b_{02}^4 + 203280b_{02}^5 - 203280a_{11}b_{02}^5 \\
&+ 6000a_{11}b_{12} - 7200a_{11}^2b_{12} - 28800a_{11}^3b_{12} + 74178a_{11}^4b_{12} - 73284a_{11}^5b_{12} \\
&+ 38856a_{11}^6b_{12} - 11052a_{11}^7b_{12} + 1302a_{11}^8b_{12} - 118000b_{02}b_{12} + 260400a_{11}b_{02}b_{12} \\
&+ 33470a_{11}^2b_{02}b_{12} - 482221a_{11}^3b_{02}b_{12} + 515942a_{11}^4b_{02}b_{12} - 269752a_{11}^5b_{02}b_{12} \\
&+ 73092a_{11}^6b_{02}b_{12} - 7891a_{11}^7b_{02}b_{12} + 54100b_{02}^2b_{12} - 563620a_{11}b_{02}^2b_{12} \\
&+ 863447a_{11}^2b_{02}^2b_{12} - 655834a_{11}^3b_{02}^2b_{12} + 275732a_{11}^4b_{02}^2b_{12} - 36490a_{11}^5b_{02}^2b_{12} \\
&- 2855a_{11}^6b_{02}^2b_{12} + 482480b_{02}^3b_{12} - 123664a_{11}b_{02}^3b_{12} - 439402a_{11}^2b_{02}^3b_{12} \\
&+ 625151a_{11}^3b_{02}^3b_{12} - 343720a_{11}^4b_{02}^3b_{12} + 62495a_{11}^5b_{02}^3b_{12} - 502232b_{02}^4b_{12} \\
&+ 698364a_{11}b_{02}^4b_{12} - 845687a_{11}^2b_{02}^4b_{12} + 385530a_{11}^3b_{02}^4b_{12} - 40895a_{11}^4b_{02}^4b_{12} \\
&- 42496b_{02}^5b_{12} - 38624a_{11}b_{02}^5b_{12} + 204736a_{11}^2b_{02}^5b_{12} - 123616a_{11}^3b_{02}^5b_{12}
\end{aligned}$$

$$
\begin{aligned}
&+ 126148b_{02}^6b_{12} - 252296a_{11}b_{02}^6b_{12} + 126148a_{11}^2b_{02}^6b_{12} - 200a_{11}b_{12}^2 - 38720a_{11}^2b_{12}^2\\
&+ 83530a_{11}^3b_{12}^2 - 79826a_{11}^4b_{12}^2 + 36574a_{11}^5b_{12}^2 - 6398a_{11}^6b_{12}^2 + 53600b_{02}b_{12}^2\\
&+ 210320a_{11}b_{02}b_{12}^2 - 496490a_{11}^2b_{02}b_{12}^2 + 517312a_{11}^3b_{02}b_{12}^2 - 246762a_{11}^4b_{02}b_{12}^2\\
&+ 42660a_{11}^5b_{02}b_{12}^2 - 460980b_{02}^2b_{12}^2 + 658580a_{11}b_{02}^2b_{12}^2 - 857174a_{11}^2b_{02}^2b_{12}^2\\
&+ 429664a_{11}^3b_{02}^2b_{12}^2 - 61150a_{11}^4b_{02}^2b_{12}^2 - 192640b_{02}^3b_{12}^2 + 307728a_{11}b_{02}^3b_{12}^2\\
&- 9508a_{11}^2b_{02}^3b_{12}^2 - 105580a_{11}^3b_{02}^3b_{12}^2 + 144864b_{02}^4b_{12}^2 - 495804a_{11}b_{02}^4b_{12}^2\\
&+ 350940a_{11}^2b_{02}^4b_{12}^2 + 252296b_{02}^5b_{12}^2 - 252296a_{11}b_{02}^5b_{12}^2 - 9200a_{11}b_{12}^3 + 29680a_{11}^2b_{12}^3\\
&- 30640a_{11}^3b_{12}^3 + 9968a_{11}^4b_{12}^3 + 32600b_{02}b_{12}^3 - 137580a_{11}b_{02}b_{12}^3 + 172320a_{11}^2b_{02}b_{12}^3\\
&- 64636a_{11}^3b_{02}b_{12}^3 + 120620b_{02}^2b_{12}^3 - 282840a_{11}b_{02}^2b_{12}^3 + 156252a_{11}^2b_{02}^2b_{12}^3\\
&+ 164920b_{02}^3b_{12}^3 - 204884a_{11}b_{02}^3b_{12}^3 + 126148b_{02}^4b_{12}^3 + 3400a_{11}b_{12}^4 - 4760a_{11}^2b_{12}^4\\
&- 10200b_{02}b_{12}^4 + 21760a_{11}b_{02}b_{12}^4 - 22440b_{02}^2b_{12}^4.
\end{aligned}
$$

$$
\begin{aligned}
n_2 =&30810976560000b_{02}^2 - 147892687488000a_{11}b_{02}^2 + 271444703493600a_{11}^2b_{02}^2 - 236320190215200a_{11}^3b_{02}^2\\
&+ 97054576164000a_{11}^4b_{02}^2 - 15097378514400a_{11}^5b_{02}^2 + 73946343744000b_{02}^3 - 195957810921600a_{11}b_{02}^3\\
&+ 138649394520000a_{11}^2b_{02}^3 - 3697317187200a_{11}^3b_{02}^3 - 12940610155200a_{11}^4b_{02}^3 - 91200490617600b_{02}^4\\
&+ 370964157782400a_{11}b_{02}^4 - 371888487079200a_{11}^2b_{02}^4 + 92124819914400a_{11}^3b_{02}^4 - 162681956236800b_{02}^5\\
&+ 122011467177600a_{11}b_{02}^5 + 40670489059200a_{11}^2b_{02}^5 + 149125126550400b_{02}^6 - 149125126550400a_{11}b_{02}^6\\
&+ 433080500832000b_{02}b_{12} - 2233381933641600a_{11}b_{02}b_{12} + 4605486204604320a_{11}^2b_{02}b_{12}\\
&- 4899248153845920a_{11}^3b_{02}b_{12} + 2927753812526760a_{11}^4b_{02}b_{12} - 1025180765461440a_{11}^5b_{02}b_{12}\\
&+ 217966012341840a_{11}^6b_{02}b_{12} - 28089030362400a_{11}^7b_{02}b_{12} + 1613353006440a_{11}^8b_{02}b_{12}\\
&+ 2048625220000800b_{02}^2b_{12} - 8435120032069920a_{11}b_{02}^2b_{12} + 13842516766909320a_{11}^2b_{02}^2b_{12}\\
&- 11864370285370680a_{11}^3b_{02}^2b_{12} + 5773529873326740a_{11}^4b_{02}^2b_{12} - 1565624147664780a_{11}^5b_{02}^2b_{12}\\
&+ 219720142417860a_{11}^6b_{02}^2b_{12} - 15580220362140a_{11}^7b_{02}^2b_{12} + 1435613156570880b_{02}^3b_{12}\\
&- 2474619113242080a_{11}b_{02}^3b_{12} + 236500649602560a_{11}^2b_{02}^3b_{12} + 1507434908066640a_{11}^3b_{02}^3b_{12}\\
&- 910480961688480a_{11}^4b_{02}^3b_{12} + 175457831965920a_{11}^5b_{02}^3b_{12} - 17971594709040a_{11}^6b_{02}^3b_{12}\\
&- 5256761511682800b_{02}^4b_{12} + 16987259209105920a_{11}b_{02}^4b_{12} - 18157750313754660a_{11}^2b_{02}^4b_{12}\\
&+ 8215537780193100a_{11}^3b_{02}^4b_{12} - 1795612130011560a_{11}^4b_{02}^4b_{12} + 200511789181200a_{11}^5b_{02}^4b_{12}\\
&- 4669268518152000b_{02}^5b_{12} + 5004834041536560a_{11}b_{02}^5b_{12} - 657639017705880a_{11}^2b_{02}^5b_{12}\\
&- 6343754971680a_{11}^3b_{02}^5b_{12} + 104729559467400a_{11}^4b_{02}^5b_{12} + 5598142850280960b_{02}^6b_{12}\\
&- 8643293133896400a_{11}b_{02}^6b_{12} + 3814699136704260a_{11}^2b_{02}^6b_{12} - 769548853088820a_{11}^3b_{02}^6b_{12}\\
&+ 1356904316940480b_{02}^7b_{12} - 1161874565082000a_{11}b_{02}^7b_{12} - 195029751858480a_{11}^2b_{02}^7b_{12}\\
&- 946336014790320b_{02}^8b_{12} + 946336014790320a_{11}b_{02}^8b_{12} + 61728361776000a_{11}b_{12}^2\\
&- 96375180535200a_{11}^2b_{12}^2 - 265197927528000a_{11}^3b_{12}^2 + 864193892927688a_{11}^4b_{12}^2\\
&- 1047245204783016a_{11}^5b_{12}^2 + 720476017627176a_{11}^6b_{12}^2 - 306492003578448a_{11}^7b_{12}^2\\
&+ 80225339705208a_{11}^8b_{12}^2 - 12179708508744a_{11}^9b_{12}^2 + 866412897336a_{11}^{1}0b_{12}^2\\
&- 1313183911890000b_{02}b_{12}^2 + 3765003863478000a_{11}b_{02}b_{12}^2 - 1902758630958900a_{11}^2b_{02}b_{12}^2\\
&- 4646449960820016a_{11}^3b_{02}b_{12}^2 + 8700636376201392a_{11}^4b_{02}b_{12}^2 - 7209914913510654a_{11}^5b_{02}b_{12}^2\\
&+ 3534652185864414a_{11}^6b_{02}b_{12}^2 - 1069426612778736a_{11}^7b_{02}b_{12}^2 + 191414835716760a_{11}^8b_{02}b_{12}^2\\
&- 17703207261138a_{11}^9b_{02}b_{12}^2 + 618608774238a_{11}^{1}0b_{02}b_{12}^2 - 2811572911827200b_{02}^2b_{12}^2\\
&+ 2712081258615600a_{11}b_{02}^2b_{12}^2 + 8658595516552052a_{11}^2b_{02}^2b_{12}^2 - 21817498985973734a_{11}^3b_{02}^2b_{12}^2\\
&+ 21626028962543714a_{11}^4b_{02}^2b_{12}^2 - 11962876028090647a_{11}^5b_{02}^2b_{12}^2 + 3973369699464226a_{11}^6b_{02}^2b_{12}^2\\
&- 761693239781386a_{11}^7b_{02}^2b_{12}^2 + 77164890414672a_{11}^8b_{02}^2b_{12}^2 - 3668789532737a_{11}^9b_{02}^2b_{12}^2\\
&+ 5964575133957800b_{02}^3b_{12}^2 - 17939694466334384a_{11}b_{02}^3b_{12}^2 + 22318026167430378a_{11}^2b_{02}^3b_{12}^2
\end{aligned}
$$

$$
\begin{aligned}
&- 14735218650821308a_{11}^3b_{02}^3b_{12}^2 + 5329620360327143a_{11}^4b_{02}^3b_{12}^2 - 372043315505272a_{11}^5b_{02}^3b_{12}^2\\
&- 324710534850138a_{11}^6b_{02}^3b_{12}^2 + 89073183143800a_{11}^7b_{02}^3b_{12}^2 - 8127601038499a_{11}^8b_{02}^3b_{12}^2\\
&+ 8686621642788888b_{02}^4b_{12}^2 - 62888501067656a_{11}b_{02}^4b_{12}^2 - 20327450894869064a_{11}^2b_{02}^4b_{12}^2\\
&+ 27486980368797409a_{11}^3b_{02}^4b_{12}^2 - 17208313003002872a_{11}^4b_{02}^4b_{12}^2 + 5481271297882039a_{11}^5b_{02}^4b_{12}^2\\
&- 885145354153178a_{11}^6b_{02}^4b_{12}^2 + 66645574401294a_{11}^7b_{02}^4b_{12}^2 - 13085124106321980b_{02}^5b_{12}^2\\
&+ 23070057464239700a_{11}b_{02}^5b_{12}^2 - 28433322106804959a_{11}^2b_{02}^5b_{12}^2 + 16745528154544130a_{11}^3b_{02}^5b_{12}^2\\
&- 4627859687526003a_{11}^4b_{02}^5b_{12}^2 + 538483491464792a_{11}^5b_{02}^5b_{12}^2 - 649885280040a_{11}^6b_{02}^5b_{12}^2\\
&- 2931481772915424b_{02}^6b_{12}^2 - 2226100854005816a_{11}b_{02}^6b_{12}^2 + 9004246880868680a_{11}^2b_{02}^6b_{12}^2\\
&- 7568683603910065a_{11}^3b_{02}^6b_{12}^2 + 2459223371694754a_{11}^4b_{02}^6b_{12}^2 - 355353113837829a_{11}^5b_{02}^6b_{12}^2\\
&+ 5586599907276144b_{02}^7b_{12}^2 - 10594463032771372a_{11}b_{02}^7b_{12}^2 + 8927957712983633a_{11}^2b_{02}^7b_{12}^2\\
&- 2740909868015806a_{11}^3b_{02}^7b_{12}^2 + 240319302712881a_{11}^4b_{02}^7b_{12}^2 + 344305034440904b_{02}^8b_{12}^2\\
&- 98964076947056a_{11}b_{02}^8b_{12}^2 - 834986949428600a_{11}^2b_{02}^8b_{12}^2 + 589645991934752a_{11}^3b_{02}^8b_{12}^2\\
&- 589556032293932b_{02}^9b_{12}^2 + 1179112064587864a_{11}b_{02}^9b_{12}^2 - 589556032293932a_{11}^2b_{02}^9b_{12}^2\\
&- 15791019040200a_{11}b_{12}^3 - 378988663137120a_{11}^2b_{12}^3 + 1065697829092410a_{11}^3b_{12}^3\\
&- 1336030289250684a_{11}^4b_{12}^3 + 922024329715608a_{11}^5b_{12}^3 - 370551538066176a_{11}^6b_{12}^3\\
&+ 88014941090538a_{11}^7b_{12}^3 - 11463106613700a_{11}^8b_{12}^3 + 501566206764a_{11}^9b_{12}^3 + 823270584383600b_{02}b_{12}^3\\
&+ 1304120543691520a_{11}b_{02}b_{12}^3 - 6824400572993030a_{11}^2b_{02}b_{12}^3\\
&+ 10880335110555623a_{11}^3b_{02}b_{12}^3 - 8886195476719627a_{11}^4b_{02}b_{12}^3 + 4115663682406832a_{11}^5b_{02}b_{12}^3\\
&- 1086500661017238a_{11}^6b_{02}b_{12}^3 + 151333879271113a_{11}^7b_{02}b_{12}^3 - 8862615635993a_{11}^8b_{02}b_{12}^3\\
&- 3179166599086980b_{02}^2b_{12}^3 + 14303757179371960a_{11}b_{02}^2b_{12}^3 - 27821375837704111a_{11}^2b_{02}^2b_{12}^3\\
&+ 27315702828360809a_{11}^3b_{02}^2b_{12}^3 - 14440744119124418a_{11}^4b_{02}^2b_{12}^3 + 4057465602673654a_{11}^5b_{02}^2b_{12}^3\\
&- 567707346024123a_{11}^6b_{02}^2b_{12}^3 + 36314951785049a_{11}^7b_{02}^2b_{12}^3 - 15094671599463680b_{02}^3b_{12}^3\\
&+ 25360709448656992a_{11}b_{02}^3b_{12}^3 - 28829454903762688a_{11}^2b_{02}^3b_{12}^3 + 15447407491720941a_{11}^3b_{02}^3b_{12}^3\\
&- 3454311259925265a_{11}^4b_{02}^3b_{12}^3 + 140995589951903a_{11}^5b_{02}^3b_{12}^3 + 33482109131217a_{11}^6b_{02}^3b_{12}^3\\
&- 4168401626452144b_{02}^4b_{12}^3 + 1561381746811636a_{11}b_{02}^4b_{12}^3 + 8083809233690693a_{11}^2b_{02}^4b_{12}^3\\
&- 8786014632746015a_{11}^3b_{02}^4b_{12}^3 + 2939347485612195a_{11}^4b_{02}^4b_{12}^3 - 410544966705765a_{11}^5b_{02}^4b_{12}^3\\
&+ 9652063969809484b_{02}^5b_{12}^3 - 21526479767881948a_{11}b_{02}^5b_{12}^3 + 18521519748817496a_{11}^2b_{02}^5b_{12}^3\\
&- 4984635553061720a_{11}^3b_{02}^5b_{12}^3 + 492651544311708a_{11}^4b_{02}^5b_{12}^3 + 7463471217925436b_{02}^6b_{12}^3\\
&- 9264562240459356a_{11}b_{02}^6b_{12}^3 + 1072171687798096a_{11}^2b_{02}^6b_{12}^3 + 728919334735824a_{11}^3b_{02}^6b_{12}^3\\
&- 671452602478552b_{02}^7b_{12}^3 + 2621251225414484a_{11}b_{02}^7b_{12}^3 - 1949798622935932a_{11}^2b_{02}^7b_{12}^3\\
&- 1179112064587864b_{02}^8b_{12}^3 + 1179112064587864a_{11}b_{02}^8b_{12}^3 - 94393267991000a_{11}b_{12}^4\\
&+ 429205635419600a_{11}^2b_{12}^4 - 638693619085630a_{11}^3b_{12}^4 + 451995235349324a_{11}^4b_{12}^4\\
&- 172689224812888a_{11}^5b_{12}^4 + 35713543384820a_{11}^6b_{12}^4 - 2518103047586a_{11}^7b_{12}^4 + 213085652420000b_{02}b_{12}^4\\
&- 2244024610098400a_{11}b_{02}b_{12}^4 + 4433965040427970a_{11}^2b_{02}b_{12}^4 - 3922219151642558a_{11}^3b_{02}b_{12}^4\\
&+ 1732937799673278a_{11}^4b_{02}b_{12}^4 - 382638593386938a_{11}^5b_{02}b_{12}^4 + 34562240504568a_{11}^6b_{02}b_{12}^4\\
&+ 3022363333749300b_{02}^2b_{12}^4 - 9267217588901760a_{11}b_{02}^2b_{12}^4 + 11372679837292086a_{11}^2b_{02}^2b_{12}^4\\
&- 5967901378964926a_{11}^3b_{02}^2b_{12}^4 + 1403794513757394a_{11}^4b_{02}^2b_{12}^4 - 148005598398126a_{11}^5b_{02}^2b_{12}^4\\
&+ 6369389198356560b_{02}^3b_{12}^4 - 13149313438274792a_{11}b_{02}^3b_{12}^4 + 8684596524474492a_{11}^2b_{02}^3b_{12}^4\\
&- 1781649667952968a_{11}^3b_{02}^3b_{12}^4 + 122330412001460a_{11}^4b_{02}^3b_{12}^4 + 5697253565779424b_{02}^4b_{12}^4\\
&- 5906103481893784a_{11}b_{02}^4b_{12}^4 - 321870295180704a_{11}^2b_{02}^4b_{12}^4 + 583554551072060a_{11}^3b_{02}^4b_{12}^4\\
&+ 2014063890247692b_{02}^5b_{12}^4 + 2198211861693888a_{11}b_{02}^5b_{12}^4 - 1573347824572872a_{11}^2b_{02}^5b_{12}^4\\
&- 1196708243691952b_{02}^6b_{12}^4 + 1541103237773676a_{11}b_{02}^6b_{12}^4 - 589556032293932b_{02}^7b_{12}^4\\
&+ 56570891332600a_{11}b_{12}^5 - 134861384906440a_{11}^2b_{12}^5 + 103657131846240a_{11}^3b_{12}^5 - 38917637205608a_{11}^4b_{12}^5\\
&+ 4053617709176a_{11}^5b_{12}^5 - 178936013927800b_{02}b_{12}^5 + 696055160643540a_{11}b_{02}b_{12}^5\\
&- 738972301255100a_{11}^2b_{02}b_{12}^5 + 319960374810356a_{11}^3b_{02}b_{12}^5 - 47762701617988a_{11}^4b_{02}b_{12}^5\\
&- 846064393377660b_{02}^2b_{12}^5 + 1710215641090420a_{11}b_{02}^2b_{12}^5 - 990677990744772a_{11}^2b_{02}^2b_{12}^5\\
&+ 211573610369436a_{11}^3b_{02}^2b_{12}^5 - 1250355346513840b_{02}^3b_{12}^5 + 1304907048249684a_{11}b_{02}^3b_{12}^5\\
&- 416207025366892a_{11}^2b_{02}^3b_{12}^5 - 643391050491468b_{02}^4b_{12}^5 + 407355819249004a_{11}b_{02}^4b_{12}^5\\
&- 180950606772496b_{02}^5b_{12}^5 - 8114966077400a_{11}b_{12}^6 + 12823347424160a_{11}^2b_{12}^6 - 2047352882120a_{11}^3b_{12}^6\\
&+ 22683188182200b_{02}b_{12}^6 - 60368215327560a_{11}b_{02}b_{12}^6 + 18279625917840a_{11}^2b_{02}b_{12}^6\\
&+ 72341359365240b_{02}^2b_{12}^6 - 55083101815000a_{11}b_{02}^2b_{12}^6 + 49364359801680b_{02}^3b_{12}^6.
\end{aligned}
$$

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