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HIGH ORDER PARAMETER-UNIFORM CONVERGENT HDG METHOD FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEM

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Abstract In this paper, a high order hybridizable discontinuous Galerkin method (HDG) on two layer-adapted meshes have been developed for the singularly perturbed convection-diffusion problems in one and two-dimensional. The existence and uniqueness of the HDG solutions are verified. Thanks to the implementation of two-type different anisotropic meshes, i.e., the Shishkin and an improved grade meshes, the uniform $2k+1$-order super-convergence is obtained for both one-dimensional and two-dimensional cases.

Keywords Hybridizable discontinuous Galerkin method, uniform convergence, convection-diffusion, singularly perturbed.

MSC(2010) 65N15, 76D07, 65N30.

1. Introduction

In this paper, we consider a singularly perturbed convection-diffusion problem of the form

$$-\epsilon\Delta u + \beta\cdot\nabla u + cu = f \qquad \text{in} \quad \Omega, \tag{1.1}$$

$$u = g \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\Omega \in R^d(d=1,2,3)$, $0<\epsilon\ll 1$ denotes the diffusion parameter, $\beta(x,y) > (\beta_1,\beta_2) > (0,0)$ denotes the convection coefficient, $c(x,y)\geq 0$ denotes the reaction coefficient and the function $f(x,y)$ denotes a given source term. Further, we assume that β, c and f are sufficiently smooth on $\bar{\Omega}$ and $c-\frac{1}{2}\mathrm{div}\beta > c_0 > 0$ for some constant c_0. In deed, these hypotheses guarantee that our model problem has a unique solution in $H^2(\Omega)\cap H_0^1(\Omega)$ for all $f\in L^2(\Omega)$ [20]. On the other hand, it is known when $\epsilon\to 0$, the solution of the model problem usually exhibits boundary layers at the outflow boundary of Ω [13, 21]. Therefore, the standard method fails to produce an accurate numerical solution unless the mesh size is smaller than the singular perturbation parameter ϵ.

The HDG methods were introduced in [2] in the framework of steady-state diffusion as part of the effort of devising efficient implicit discontinuous Galerkin(DG)

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methods for solving elliptic PDE systems. In comparison with the DG method, the HDG methods guarantee that only degrees of freedom of scalar variable on inter-element boundaries are globally coupled and that the approximate gradient attains optimal order of convergence for elliptic problems [3]. Since the HDG methods inherit many attractive features of DG methods like high parallelize ability, easy to achieve hp-adaptivity, they have been widely used to solve many kinds of problems [4–6, 8, 15].

However, the parameter-uniform convergent HDG method had been bypassed in all developed numerical methods when the DG-related methods are applied to solve the singularly perturbed problem. For instance, in [9–11, 14, 16–21], the authors proposed some discontinuous Galerkin (DG) or local discontinuous Galerkin (LDG) method for singularly perturbed problem. Among them, the parameter-uniform numerical methods employing layer-adapted meshes, such as a Shishkin or a Bakhvalov mesh, ensure a uniform convergence for the singularly perturbed convection-dominated problems. We note that there is some related papers concerning with using the HDG method for convection-diffusion equation [1, 7, 12]. But, none of them reaches the parameter-uniform convergent which is important for the numerical method for the singularly perturbed problem. Hence, it remains unknown how the HDG method be applied to solve the singularly perturbed problem to achieve uniform convergence. Therefore, in this paper, we develop a high order parameter uniform HDG method on two-type layer-adapted meshes for one and two dimensional singularly perturbed problem. The numerical results exhibit that the HDG method does not produce any oscillation even under uniform meshes for arbitrary for both 1-D and 2-D cases. On the other hand, the $2k+1$ order uniform superconvergence of numerical fluxes are observed numerically for the HDG method under both meshes. Here the so-called "uniform convergence" means that the convergence rate is uniformly valid with respect to ϵ. It is worthwhile to point out that theoretical analysis of the uniform convergence is extremely difficult and remains an open problem for the HDG method.

The rest of this paper is organized as follows. In section 2, we present the detail of our numerical scheme and the two-type layer-adapted meshes. Section 3 gives several numerical experiments about uniform and two-type layer-adapted meshes to verify HDG's numerical accuracy, i.e., uniform convergence and super-convergence. Finally, we conclude in Section 4.

2. Numerical Scheme

2.1. layer-adapted meshes

We simplify introduce the Shishkin mesh and the graded mesh which was used to solve our model problem when ϵ is small. Define the transition parameter

$$\tau_x = \min\{\frac{1}{2}, \frac{\kappa}{\beta_1}\epsilon \ln N\}. \tag{2.1}$$

Then the domain $[0, 1]$ is divided into two parts: $\Omega_0 = (0, 1-\tau_x)$, $\quad \Omega_x = (1-\tau_x, 1)$. Each sub-domain is equally decomposed into N interval. Therefor, the mesh is composed of $2N$ elements. While there are $2N+1$ nodes $x_i, i = 0, \cdots, 2N$.

For the graded meshes, we also need to use the parameter τ_x, the domain Ω_0, Ω_1 above. The difference of the graded mesh and Shishkin mesh lies that the partition

$x_j (j = N/2+2, \cdots, N+1)$ are given by

$$x_j = 1 - \tau_x((N+1-j)h)^{\lambda}, j = N/2+2, \cdots, N+1, \tag{2.2}$$

where λ is a mesh parameter which is greater than or equal to 1 in practical computation. It is apparent that the Shishkin mesh is the special case of improved grade mesh with $\lambda = 1$. With increasing λ, more and more mesh points will concentrate in the neighborhood of 1. Consequently the solution is approximated well on the boundary layer.

For the 2D/3D case, we use the tensor product Shishkin meshes and improved graded meshes.

2.2. HDG scheme

To give a clear presentation of the HDG method, here we introduce some computational concepts and notations. Denote by $\mathcal{T}_h$ the finite element partition of Ω. For $K \in \mathcal{T}_h$, denote by $h_K = \mathrm{diam}(K)$ its diameter, and $h = \max_{K\in\mathcal{T}_h} h_K$ the mesh size of $\mathcal{T}_h$. Denote by $\mathcal{E}_h = \cup_{K\in\mathcal{T}_h}\partial K$ the skeleton of the mesh, and set $\mathcal{E}_h^B = \mathcal{E}_h \cap \partial\Omega, \mathcal{E}_h^I = \mathcal{E}_h \setminus \mathcal{E}_h^B$.

In order to define the HDG scheme, we first rewrite (1.1) into a first order system

$$\begin{aligned} &\mathbf{q} + \epsilon\nabla u = 0 \quad \text{in } \Omega, \\ &\nabla\cdot\mathbf{q} + \beta\cdot\nabla u + cu = f \text{ in } \Omega. \end{aligned} \tag{2.3}$$

On each element $K \in \mathcal{T}_h$, we denote by $\mathbf{n}$ the outward normal direction on the boundary ∂K, and define the following local finite element spaces

$$\begin{aligned} &\mathbf{V}_h = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{v}|_K \in [P^k(K)]^d, \forall K \in \mathcal{T}_h\}, \\ &W_h = \{w \in L^2(\Omega) : w|_K \in [P^k(K)], \forall K \in \mathcal{T}_h\}, \\ &M_h = \{\mu \in L^2(\mathcal{E}_h) : \mu|_K \in P^k(e), \forall e \in \mathcal{E}_h\}, \\ &M_h(g) = \{\mu \in M_h : \langle \mu, \xi\rangle_{\partial\Omega} = \langle g, \xi\rangle_{\partial\Omega}, \forall \xi \in M_h\}, \end{aligned}$$

where $P^k(K)$ denotes the space of polynomials of total degree not larger than $k \geq 0$ defined on K, and $\langle u, v\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}\langle u, v\rangle_{\partial K}$.

Multiplying (2.3) by the test function v, w and integrating by parts leads to

$$\begin{aligned} &(\mathbf{q}, \mathbf{v})_K + \epsilon(u, \nabla\cdot\mathbf{v})_K - \epsilon\langle u, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} = 0, \\ &-(\mathbf{q}, \nabla w)_K - (u, \beta\cdot\nabla w)_K + \langle \mathbf{q}\cdot\mathbf{n}, w\rangle_{\partial K} + \langle u\beta\cdot\mathbf{n}, w\rangle_{\partial K} + c(u, w)_K = (f, w)_K. \end{aligned} \tag{2.4}$$

Based on the above weak formulation, we can define our the HDG formulation as: find $(\mathbf{q}_h, u_h, \hat{u}_h) \in \mathbf{V}_h^k \times W_h^k \times M_h(g)$ such that

$$\begin{aligned} &(\mathbf{q}_h, \mathbf{v})_{\mathcal{T}_h} + \epsilon(u_h, \nabla\cdot\mathbf{v})_{\mathcal{T}_h} - \epsilon\langle \hat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0, \\ &-(\mathbf{q}_h, \nabla w)_{\mathcal{T}_h} - (u_h, \beta\cdot\nabla w)_{\mathcal{T}_h} + \langle \hat{\mathbf{q}}_h\cdot\mathbf{n}, w\rangle_{\partial\mathcal{T}_h} + \langle \hat{u}_h\beta\cdot\mathbf{n}, w\rangle_{\partial\mathcal{T}_h} + c(u_h, w)_{\mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \\ &\langle \hat{u}_h, \mu\rangle_{\partial\Omega} = \langle g, \mu\rangle_{\partial\Omega}, \\ &\langle \hat{\mathbf{q}}_h\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \end{aligned} \tag{2.5}$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where the numerical flux $\hat{\mathbf{q}}_h$ is choosing as

$$\hat{\mathbf{q}}_h\cdot\mathbf{n} = \mathbf{q}_h\cdot\mathbf{n} + \tau(u_h - \hat{u}_h), \tag{2.6}$$

with τ denotes some positive function defined on $\partial\mathcal{T}_h$ satisfies $\tau - \frac{1}{2}\beta\cdot\mathbf{n} \geq 0$.

Theorem 2.1. *The system* (2.5) *has a unique solution* $\mathbf{q}_h, u_h, \hat{u}_h$.

Proof. Taking $\mathbf{v} = \mathbf{q}_h, w = u_h, \mu = \hat{u}_h$ in (2.5), and then summing them together, we have

$$\epsilon^{-1}\|\mathbf{q}_h\|_0^2 + \langle(\tau - \frac{1}{2}\beta \cdot \mathbf{n})u_h - \hat{u}_h, u_h - \hat{u}_h\rangle_{\partial\mathcal{T}_h} + ((c - \frac{1}{2}\nabla \cdot \beta)u_h, u_h)_{\mathcal{T}_h} = (f, u_h)_{\mathcal{T}_h}.$$

where we have used the fact that

$$(\beta \cdot \nabla u_h, u_h)_K = -\frac{1}{2}((\mathrm{div}\beta)u_h, u_h)_K + \frac{1}{2} < (\beta \cdot \mathbf{n})u_h, u_h >_{\partial K},$$

Hence, if $f = 0$, we can derive $\mathbf{q}_h = 0, \hat{u}_h = 0$ and $u_h = 0$. □

3. Numerical result

In this section, we present some numerical examples using simple model problems in both 1D and 2D to verify display the performance of the HDG methods when the exact solution with or without layers.

3.1. 1D numerical result

Example 3.1 (A smooth solution test in 1D)**.** In this example, we choose $\epsilon = 1, b = 1, c = 0$ and the right hand side f, such that the exact solution is $u = \sin(\pi x)$. This numerical experiment was performed on uniform meshes. The corresponding error and convergent order are list in Table 1. From it, we conclude that the HDG method approximate are convergent at a rate of $O(h^{k+1})$ for both u and $\mathbf{q}$ in L^2 norms. Meanwhile, it has a super-convergent order of $O(h^{2k+1})$ for the numerical flux $\hat{u}_h$.

Table 1. The convergence rate of L^2 error on uniform mesh

k	h	$\|\|u - u_h\|\|_0$	order	$\|\|q - q_h\|\|_0$	order	$\|\|\hat{u} - u\|\|_\infty$	order
k=1	1/8	5.0707e-03		4.7136e-03		1.1214e-04	
	1/16	1.2985e-03	1.9653	1.2047e-03	1.9681	1.4597e-05	2.9416
	1/32	3.2830e-04	1.9838	3.0429e-04	1.9852	1.8588e-06	2.9731
	1/64	8.2520e-05	1.9922	7.6446e-05	1.9929	2.3461e-07	2.9860
	1/128	2.0685e-05	1.9962	1.9158e-05	1.9965	2.9541e-08	2.9895
k=2	1/8	1.2327e-04		1.2081e-04		8.4854e-08	
	1/16	1.5661e-05	2.9765	1.5338e-05	2.9776	2.7143e-09	4.9663
	1/32	1.9720e-06	2.9895	1.9305e-06	2.9900	8.5843e-11	4.9827
	1/64	2.4734e-07	2.9951	2.4209e-07	2.9953	4.0762e-12	4.3964

Example 3.2 (A boundary layer test in 1D)**.** We consider the one dimension case in this example. We take $b = 1, c = 0, f = e^x$ and $u_0 = u_1 = 0$. Therefore, the exact solution is

$$u = \frac{e^x(1 - e^{-\frac{1}{\epsilon}}) + e^{1-\frac{1}{\epsilon}} - 1 + (1 - e)e^{\frac{x-1}{\epsilon}}}{(1 - \epsilon)(1 - e^{-\frac{1}{\epsilon}})}. \tag{3.1}$$

Plotted in Fig.1 are the numerical traces $\hat{u}_h$ and $\hat{q}_h$ under uniform mesh with $\epsilon = 10^{-6}$and $N = 32$, respectively. We see that the HDG solutions do not have any oscillatory behavior even for small ϵ under uniform meshes. In other words, the HDG method is more local than the finite element and finite difference methods. Meanwhile, we can obtain that the condition $c - \frac{1}{2}\mathrm{div}\beta \geq c_0$ is not an necessary condition in this HDG scheme. On the other hand, numerical results presented in Table 2–3 show that the convergence rate of $u_h, \mathbf{q}_h, \hat{u}_h$ for Shishkin mesh and improved graded meshes, respectively. We conclude that, under both meshes, the $2k+1$-order uniform super-convergence of numerical flux $\hat{u}$ is observed for 1D case. This uniform convergence rate of $\mathbf{q}_h$ and super-convergence result of $\hat{u}_h$ are a remarkable observation which is reported for the first time in the literature to our knowledge.

Table 2. The convergence rate of L^2 error for $k = 2$ on Shishkin mesh

ϵ	h	$\|\|e_u\|\|_0$	order	$\epsilon^{-\frac{1}{2}}\|\|e_q\|\|_0$	order	$\|\|\hat{u} - u\|\|_\infty$	order
1.0e-10	1/8	2.0156e-04		2.5474e-01		5.8555e-02	
	1/16	2.4361e-05	3.0486	7.3149e-02	1.8001	1.2236e-02	2.6522
	1/32	2.9967e-06	3.0231	1.3631e-02	2.4240	9.6365e-04	3.9780
	1/64	3.7167e-07	3.0113	1.9924e-03	2.7743	3.7373e-05	4.8807
	1/128	4.6278e-08	3.0056	2.6426e-04	2.9144	1.2390e-06	4.8242
1.0e-12	1/8	2.0155e-04		3.2537e-01		5.8555e-02	
	1/16	2.4350e-05	3.0491	2.2537e-01	1.6109	1.2236e-02	3.3789
	1/32	2.9937e-06	3.0239	2.1902e-02	2.2820	9.6365e-04	4.4901
	1/64	3.7115e-07	3.0118	3.3432e-03	2.7118	3.7373e-05	4.8633
	1/128	4.6205e-08	3.0059	4.3923e-04	2.9282	1.2390e-06	4.7267

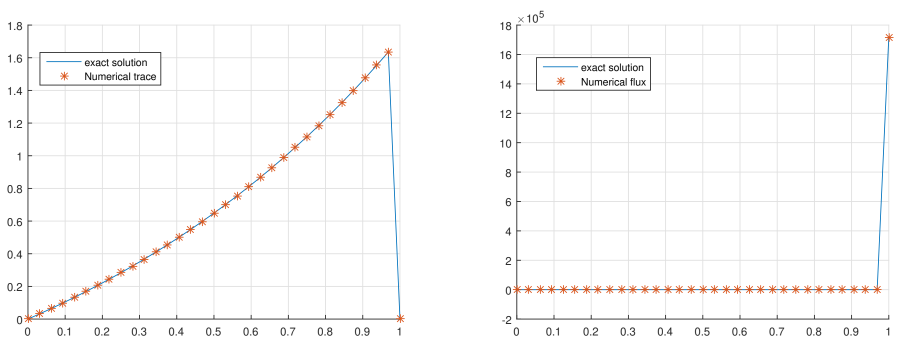

Figure 1. u and $\hat{u}_h$ (left), q and $\hat{q}_h$ (right) under uniform mesh, $N = 32, k = 1, \epsilon = 10^{-6}$.

3.2. Numerical result in 2D

Example 3.3. Firstly, we consider the problem with $\beta = (1, 1), c = 1, \epsilon = 1$ on a unit square $\Omega = [0, 1]^2$. The right hand side is chosen such that the exact solution is

$$u = \sin(x)\sin(y).$$

Table 3. The convergence rate of L^2 error for $k=2$ on grade mesh

ϵ	h	$\|\|e_u\|\|_0$	order	$\epsilon^{-\frac{1}{2}}\|\|e_q\|\|_0$	order	$\|\|\hat{u}-u\|\|_\infty$	order
1.0e-10	1/8	2.0162e-04		4.1909e-01		4.3007e-02	
	1/16	2.4361e-05	3.0486	7.1986e-02	2.5145	6.8413e-03	2.6522
	1/32	2.9967e-06	3.0231	9.9784e-03	2.8509	4.3415e-04	3.9780
	1/64	3.7167e-07	3.0113	1.3115e-03	2.9276	1.4736e-05	4.8807
	1/128	4.6278e-08	3.0056	1.6804e-04	2.9643	5.2019e-07	4.8242
	1/256	5.7735e-09	3.0028	2.2112e-05	2.9259	1.8657e-08	4.8013
1.0e-12	1/8	2.0155e-04		3.2537e-01		5.8555e-02	
	1/16	2.4350e-05	3.0491	1.0652e-01	1.6109	1.2236e-02	2.2587
	1/32	2.9937e-06	3.0239	2.1902e-02	2.2820	9.6365e-04	3.6665
	1/64	3.7115e-07	3.0118	3.3432e-03	2.7118	3.7373e-05	4.6885
	1/128	4.6205e-08	3.0059	4.3923e-04	2.9282	1.2390e-06	4.9147
	1/256	5.7641e-09	3.0029	7.9211e-05	2.4712	4.2261e-08	4.8738

This numerical experiment was performed on uniform rectangle partitions of the domain. The corresponding error and convergent order are list in Table 4. From it, we conclude that the HWG method approximate are convergent at a rate of $O(h^{k+1})$ for both u and $\mathbf{q}$ in L^2 norms.

Table 4. The convergence rate for example 3.1

k	N	Error_u	order	Error_q	order
1	4	1.0843e-3		3.7391e-3	
	8	2.8654e-4	1.9200	1.0774e-3	1.7951
	16	7.4635e-5	1.9408	3.0697e-4	1.8114
	32	1.9196e-5	1.9590	8.7475e-5	1.8112
	64	4.8920e-6	1.9723	2.5057e-5	1.8037
2	4	3.2293e-5		9.4870e-5	
	8	4.0544e-6	2.9937	1.2465e-5	2.9281
	16	5.0895e-7	2.9939	1.6384e-6	2.9275
	32	6.3831e-8	2.9952	2.1672e-7	2.9184
	64	7.9986e-9	2.9964	2.8974e-8	2.9030
3	4	3.6486e-7		1.3215e-6	
	8	2.3658e-8	3.9469	9.3925e-8	3.8145
	16	1.5144e-9	3.9655	6.6827e-9	3.8130
	32	9.6159e-11	3.9772	4.7804e-10	3.8052
	64	6.0795e-12	3.9834	3.4647e-11	3.7863

Example 3.4. A boundary layer test in 2D. We consider our model Problem with $\Omega=[0,1]^2$,$\beta=[1,1]'$, $c=1$. The right side term f is properly chosen such that the exact solution is

$$u = xy(1-e^{\frac{-(1-x)}{\epsilon}})(1-e^{\frac{-(1-y)}{\epsilon}}).$$

The convergence behaviors of the HDG solution of Example 3.3 are similar to Example 3.2. Hence, we only list the convergence behaviors on Shishkin mesh in Table 5. Our numerical results show that, under Shishkin meshes, the ϵ uniform convergence rate of $\mathbf{q}_h$ is observed for 2-D case. Moreover, it is clear that higher order methods lead to better approximation results and are computationally cheaper than lower order methods for similar numerical results.

Table 5. The convergence rate for example 3.2

k	N	ϵ	$\|\|e_u\|\|_0$	order	$\epsilon^{-\frac{1}{2}}\|\|e_q\|\|_0$	order	ϵ	$\|\|e_u\|\|_0$	order	$\epsilon^{-\frac{1}{2}}\|\|e_q\|\|_0$	order
1	4	1.0e-4	1.07e-3		1.10e-3		1.0e-6	2.94e-4		2.10e-3	
	8		6.05e-4	1.99	6.40e-4	1.89		1.36e-4	2.69	6.40e-4	4.13
	16		2.91e-4	1.80	3.12e-4	1.76		3.33e-5	3.47	3.12e-4	1.77
	32		1.25e-4	1.80	1.31e-4	1.84		1.26e-5	2.06	1.31e-5	1.83
	64		4.81e-5	1.88	4.99e-5	1.90		4.81e-6	1.88	4.99e-5	1.90
2	8		1.54e-4		9.07e-5			1.27e-4		9.07e-5	
	16		3.02e-5	4.01	2.82e-5	2.89		1.38e-5	5.47	2.82e-5	2.89
	32		7.11e-6	3.07	7.21e-6	2.90		1.50e-6	4.72	7.21e-6	2.90
	64		1.58e-6	2.95	1.61e-6	2.94		2.00e-7	3.94	1.61e-6	2.94

In Fig 2, we plot the exact solution and computational results for $\epsilon = 10^{-6}$ in a Shishkin mesh with 256 elements. It can be concluded that the HDG method plays well in solving singularly perturbed problem.

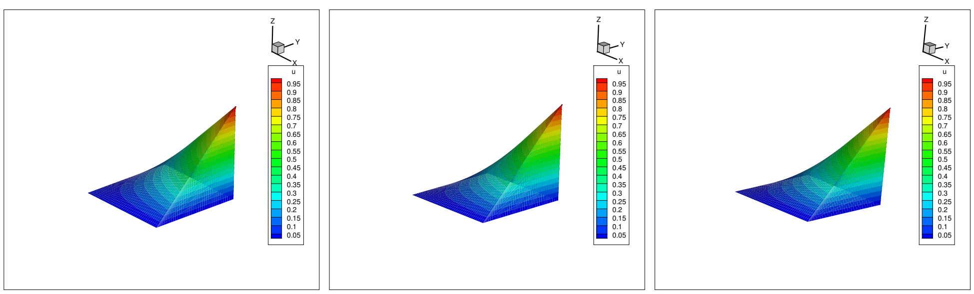

Figure 2. From left to right: exact solution, Numerical solution $k = 2$, and Numerical solution $k = 3$ for $\epsilon = 1.0e - 6$.

4. Conclusion

In this paper, the HDG method was implemented to solve the singularly perturbed convection-diffusion equations. The existence and uniqueness of the HDG solution is verified first. Then, under the uniform and two-type layer-adapted meshes in one and two dimensional settings, numerically we demonstrate that the combination of HDG methods and the layer-adapted meshes is a robust approach for solving singularly perturbed problems. Our numerical results show that the HDG method does not produce any oscillation even under the uniform mesh for 1-D. More significantly, under the layered adapted meshes, the optimal convergent order and $2k+1$-order uniform super-convergence of numerical fluxes are observed for both 1-D and 2-D cases. This uniform super-convergence result is a remarkable observation which is reported for the first time in the literature to our knowledge. The analysis of uniform convergence and super-convergence property will be considered in our further study.

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