Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1496–1516
Website:http://www.jaac-online.com
DOI:10.11948/20210316

DIMENSION ESTIMATES FOR REPELLERS AND EXPANDING MEASURES OF C^1 DYNAMICAL SYSTEMS*

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Abstract In this paper, we first conclude sharp upper and sharp lower bounds of dimensions of a repeller with dominated splitting for C^1 expanding maps, using the techniques in sub-additive and super-additive thermodynamic formalism. Furthermore, we prove a sharp upper bound for the Hausdorff dimension of an expanding measure is given by the unique solution of sub-additive measure-theoretic pressure equation for C^1 local diffeomorphisms.

Keywords Dimension, repeller, expanding measure, topological pressure.

MSC(2010) 37C45, 37D35, 37H15.

1. Introduction

The present paper is motivated by Cao etc [10] and Jordan etc [24]. Let $f : M \to M$ be a smooth map of an m_0-dimensional compact smooth Riemannian manifold M. For each $x \in M$, the following quantities

$$\|D_x f\| = \sup_{0\neq v\in T_xM} \frac{\|D_x f(v)\|}{\|v\|}, \quad m(D_x f) = \inf_{0\neq v\in T_xM} \frac{\|D_x f(v)\|}{\|v\|}$$

are respectively called *the maximal norm* and *minimum norm* of the differentiable operator $D_x f : T_xM \to T_{fx}M$, where $\|\cdot\|$ is the norm induced by the Riemannian metric on M. Let Λ be a compact f-invariant subset of M. We call Λ a *repeller* for f and f *expanding* if

(i) there exists an open neighborhood U of Λ such that $\Lambda = \{x \in U : f^n(x) \in U \text{ for all } n \geq 0\}$;

(ii) there is $\kappa > 1$ such that

$$\|D_x f(v)\| \geq \kappa\|v\| \text{ for all } x \in \Lambda \text{ and } v \in T_xM.$$

Assume that a repeller Λ admits a $\{\lambda_j\}_{1\leq j\leq k}$-dominated splitting $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ with $E_1 \succeq E_2 \succeq \cdots \succeq E_k$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ (See Section 2.3 for

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*The first author is partially supported by National Natural Science Foundation of China (11871361, 11801395) and the undergraduate training Program for innovation of Shanghai University of Engineering Science (cx2121009).

more details) for a C^1-expanding map f. Utilizing the techniques in sub-additive and super-additive thermodynamic formalism, sharp upper and sharp lower bounds of dimensions of Λ are given in this paper. Let μ be an ergodic Borel probability measure on M preserving a C^1-local diffeomorphism f. μ is said to be *expanding* if the Lyapunov exponents of μ with respect to f satisfies

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_{m_0}(\mu) > 0.$$

We also prove that the unique root of sub-additive measure-theoretic pressure equation can give an upper bound of the Hausdorff dimension of μ.

Let Λ be a conformal repeller for a $C^{1+\alpha}$ expanding map f. Assume that f is topologically mixing on Λ. Bowen [2] and Ruelle [36] found that

$$\dim_H \Lambda = t_*,$$

where t_* is the unique solution of the equation

$$P(f|_\Lambda, -t \log \|D_x f\|) = 0.$$

Gatzouras etc in [19] relaxed the smoothness to C^1. Bowen, Ruelle, and Gatzouras and Peres's approaches are to construct a measure of full dimension, which is equivalent to Hausdorff measure.

For a non-conformal repeller Λ of a C^1 map f, Barreira [4, Theorem 3.9] proved that

$$s \leq \dim_H \Lambda \leq \underline{\dim}_B \Lambda \leq \overline{\dim}_B \Lambda \leq t,$$

where s and t are the unique root of the following Bowen's equation

$$P(f|_\Lambda, -s \log \|D_x f\|) = 0, \quad P(f|_\Lambda, -t \log m(D_x f)) = 0.$$

Falconer in [12] defined topological pressure of sub-additive potential for a C^2 map f satisfying the bounded distortion condition

$$\|(D_x f)^{-1}\|^2 \cdot \|D_x f\| < 1,$$

and proved the zero of the topological pressure gives an upper bound of the upper box dimension of Λ. Zhang [41] introduced a new version of Bowen's equation, which involves the limit of a sequence of topological pressures for singular valued potentials, and obtained a sharp upper bound of Hausdorff dimension of Λ. Falconer's result automatically implied that for the Hausdorff dimension of Λ, and the bounded distortion condition is necessary. But Zhang's approach is to calculate the Hausdorff measure at each iteration, and is valid for all C^1 expanding maps. In [10], Cao etc considered an ergodic invariant measure μ with positive entropy for $C^{1+\alpha}$ non-conformal repellers, and constructed a compact expanding invariant set with dominated splitting corresponding to Oseledec splitting of μ, for which entropy and Lyapunov exponents approximate to entropy and Lyapunov exponents for μ. Moreover, they used this construction to give a sharp estimate for the lower bound estimate of Hausdorff dimension of non-conformal repellers. They also present a sharp upper bound of the upper box dimension of Λ. We also refer the reader to [16,17,34] for a detailed description of the recent progress in dimension estimates for repellers of C^1-dynamical systems. In this paper, our first main result extends the results of Cao etc [10] to C^1 expanding maps.

Let Λ be a repeller for a C^1 expanding map. Assume that the map $f|_\Lambda$ possesses a $\{\lambda_j\}_{1\le j\le k}$-dominated splitting $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ with $E_1 \succeq E_2 \succeq \cdots \succeq E_k$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ (See the definition in Section 2.3). Let $\dim E_j = m_j$, $r_j = m_1 + \cdots + m_j$ for $j \in \{1, 2, \ldots, k\}$ and $r_0 = 0$. For each $s \in [0, m_0]$, $n \ge 1$ and $x \in \Lambda$, define

$$\tilde{\psi}^s(x, f^n) = \sum_{j=1}^{d} m_j \log \|D_x f^n|_{E_j}\| + (s - r_d) \log \|D_x f^n|_{E_{d+1}}\| \tag{1.1}$$

if $r_d < s \le r_{d+1}$ for some $d \in \{0, 1, \cdots, k-1\}$ (We assume $\tilde{\psi}^0(x, f^n) = 0$.). It is clear $\widetilde{\Psi}_f(s) = \{-\tilde{\psi}^s(x, f^n)\}_{n\ge1}$ is super-additive. Let

$$\widetilde{P}_{\sup}(s) := P(f|_\Lambda, \widetilde{\Psi}_f(s)). \tag{1.2}$$

(See the definition of the super-additive topological presure $P(f|_\Lambda, \widetilde{\Psi}_f(s))$ in (2.1).) One can easily see that $\widetilde{P}_{\sup}(s)$ is continuous and strictly decreasing in s. Let $\ell_d = m_k + \cdots + m_{k-d+1}$ for $d = 1, 2, \ldots, k$ and $\ell_0 = 0$. For $t \in [0, m_0]$ and $n \ge 1$, define

$$\tilde{\varphi}^t(x, f^n) = \sum_{j=k-d+1}^{k} m_j \log m\left(D_x f^n|_{E_j}\right) + (t - \ell_d) \log m\left(D_x f^n|_{E_{k-d}}\right) \tag{1.3}$$

if $\ell_d < t \le \ell_{d+1}$ for some $d \in \{0, 1, \ldots, k-1\}$ (We assume $\tilde{\varphi}^0(x, f^n) = 0$.). It is easy to see that $\widetilde{\Phi}_f(t) = \{-\tilde{\varphi}^t(\cdot, f^n)\}_{n\ge1}$ is sub-additive and that the sub-additive topological presure function (see the definition in Definition 2.1)

$$\widetilde{P}_{\mathrm{sub}}(t) := P(f|_\Lambda, \widetilde{\Phi}_f(t)) \tag{1.4}$$

is continuous and strictly decreasing in t. We state the first main result of the present paper:

Theorem 1.1. *Let Λ be a repeller for a C^1 expanding map admitting a $\{\lambda_j\}_{1\le j\le k}$-dominated splitting $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ with $E_1 \succeq E_2 \succeq \cdots \succeq E_k$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. Then*

$$s^* \le \dim_H \Lambda \le \underline{\dim}_B \Lambda \le \overline{\dim}_B \Lambda \le t^*$$

where s^*, t^* are the unique roots of the equations $\widetilde{P}_{sup}(s) = 0$, $\widetilde{P}_{sub}(t) = 0$ respectively.

Let $f : M \to M$ be a C^2 map of an m_0-dimensional Riemannian manifold M, and Λ be a compact f-invariant set. Assume that f is expanding and conformal on Λ and μ is an ergodic probability measure on Λ. Ruelle [35] proved

$$\dim_H \mu = \frac{h_\mu(f)}{\lambda(\mu)},$$

where $\dim_H \mu$ is the Hausdorff dimension of μ, $h_\mu(f)$ is the measure-theoretic entropy, and $\lambda(\mu)$ is the Lyapunov exponent of μ. Hu [20] extended the result to non-conformal case. He obtained

$$\frac{h_\mu(f)}{\lambda_1(\mu)} \le D(\mu) \le \frac{h_\mu(f)}{\lambda_s(\mu)}, \tag{1.5}$$

where $\lambda_1(\mu)$ and $\lambda_s(\mu)$ are the largest and smallest Lyapunov exponents respectively, $D(\mu)$ is $\dim_H \mu$, $\underline{\dim}_B\mu$ or $\overline{\dim}_B\mu$. Wang etc [39] generalized Hu's result [20] in the C^2 setting. They stated that for a C^1 map f, the zero of the sub-additive measure-theoretic pressure $P_\mu(f, \{-t\log m(D_xf^n)\})$ gives the upper bound of dimensions of an ergodic measure μ, and the zero of the super-additive measure-theoretic pressure $P_\mu(f, \{-t\log \|D_xf^n\|\})$ gives the lower bound of dimensions of an ergodic measure μ. Using Theorem A and Theorem C in [9], we have that $\frac{h_\mu(f)}{\lambda_1(\mu)}$ and $\frac{h_\mu(f)}{\lambda_s(\mu)}$ are the unique roots of the equations

$$P_\mu(f, \{-t\log \|D_xf^n\|\}) = 0 \text{ and } P_\mu(f, \{-t\log m(D_xf^n)\}) = 0$$

respectively. Let f be a C^1 self-map on a smooth Riemannian manifold M, and μ be an f-invariant ergodic expanding Borel probability measure with a compact support Λ. Suppose f is non-degenerate on Λ, Huang etc [22] proved (1.5). In their paper, the non-degeneracy condition is used to give some estimates of the distortion of the differential D_xf. They removed the non-degeneracy condition of f if f is $C^{1+\alpha}$ self-map. Jordan etc [24] considered a measure μ supported on the limit set of an iterated function system in $\mathbb{R}^d$ which contracts on average, and presented a sharp upper bound for the Hausdorff dimension of μ. In [30], Mihailescu also obtained some interesting results for dimension estimates of invariant measures in iterated function systems with overlaps. Here we also refer the reader to [27–29, 31] for a detailed description about applications of thermodynamic formalism to dimension estimates for hyperbolic invariant sets and measures. In this paper we exploit Jordan and Pollicott's ideas [24] in an essential way to get a sharp upper bound for the Hausdorff dimension of an expanding measure μ for a C^1 local diffeomorphism.

Let $f : M \to M$ be a C^1 local diffeomorphism on the m_0 dimensional compact smooth Riemannian manifold M. Fixed any $x \in M$, for every $n \geq 1$, consider the differentiable operator $D_xf^n : T_xM \to T_{f^n(x)}M$ and denote the singular values of D_xf^n in the decreasing order by

$$\alpha_1(x, f^n) \geq \alpha_2(x, f^n) \geq \cdots \geq \alpha_{m_0}(x, f^n),$$

which are the positive square roots of the eigenvalues of $(D_xf^n)^*D_xf^n$, here $(D_xf^n)^*$ is the adjoint of D_xf^n. Let μ be an ergodic f-invariant expanding probability measure on M with the corresponding Lyapunov exponents

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_{m_0}(\mu) > 0.$$

Oseledec's Multiplicative Ergodic Theorem [25] tells us that for $i = 1, 2, \cdots, m_0$, $\mu.a.e.x$,

$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_i(x, f^n) = \lambda_i(\mu).$$

For $t \in [0, m_0]$, set

$$\varphi^t(x, f^n) = \sum_{i=m_0-[t]+1}^{m_0} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{m_0-[t]}(x, f^n) \tag{1.6}$$

for $t \in [0, m_0]$. Since f is smooth, the functions $x \mapsto \alpha_i(x, f^n)$ and $x \mapsto \varphi^t(x, f^n)$ are continuous. It is easy to see that for all $n, l \in \mathbb{N}$

$$\varphi^t(x, f^{n+\ell}) \geq \varphi^t(x, f^n) + \varphi^t(f^n(x), f^\ell).$$

It follows that the sequence of functions

$$\Phi_f(t) = \left\{-\varphi^t\left(\cdot, f^n\right)\right\}_{n\geq 1}$$

is sub-additive. We call them *sub-additive singular valued potentials.* We show that the unique solution of the sub-additive measure-theoretic pressure equation

$$P_\mu(f, \{-\varphi^t(\cdot, f^n)\}) = 0$$

is an upper bound for the Hausdorff dimension of an ergodic f-invariant expanding probability measure μ as follows.

Theorem 1.2. *Let $f : M \to M$ be a C^1 local diffeomorphism on the m_0 dimensional compact smooth Riemannian manifold M. Let μ be an ergodic f-invariant expanding probability measure with the corresponding Lyapunov exponents*

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_{m_0}(\mu) > 0.$$

Then we have

$$\dim_H \mu \leq t^*,$$

where t^* is the unique root of the equation $P_\mu\left(f, \{-\varphi^t\left(x, f^n\right)\}\right) = 0$.

For $s \in [0, m_0]$, $x \in M$ and $n \in \mathbb{N}$, denote

$$\psi^s\left(x, f^n\right) = \sum_{i=1}^{[s]} \log \alpha_i\left(x, f^n\right) + (s - [s]) \log \alpha_{[s]+1}\left(x, f^n\right). \tag{1.7}$$

It is easy to see that

$$\psi^s(x, f^{n+l}) \leq \psi^s(x, f^n) + \psi^s(f^n(x), f^l)$$

for every $x \in M$ and $n, l \in \mathbb{N}$. It is natural to ask whether $\dim_H \mu \geq s_*$ where s_* satisfies $P_\mu(f, \{-\psi^s(\cdot, f^n)\}) = 0$?

The paper is organized as follows. In Section 2 we recall definitions and preliminaries, such as dimensions, pressures, dominated splitting, Markov partition and weak Gibbs measures. In Section 3 we give the detailed proofs of the main results.

2. Preliminaries

2.1. Dimensions of sets and measures

We recall some notions and basic facts from dimension theory, see the books [13,32] for detailed introduction.

Let X be a compact metric space equipped with a metric d. Given a subset Z of X, for $s \geq 0$ and $\delta > 0$, define

$$\mathcal{H}^s_\delta(Z) = \inf\left\{\sum_i |U_i|^s : Z \subset \bigcup_i U_i,\ \ |U_i| \leq \delta\right\},$$

where $|\cdot|$ denotes the diameter of a set. The quantity $\mathcal{H}^s(Z) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(Z)$ is called the *s-dimensional Hausdorff measure* of Z. Define the *Hausdorff dimension* of Z, denoted by $\dim_H Z$, as follows:

$$\dim_H Z = \inf\left\{s : \mathcal{H}^s(Z) = 0\right\} = \sup\left\{s : \mathcal{H}^s(Z) = \infty\right\}.$$

Further define the *lower and upper box dimensions* of Z respectively by

$$\underline{\dim}_B Z = \liminf_{\delta \to 0} \frac{\log N(Z,\delta)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\delta \to 0} \frac{\log N(Z,\delta)}{-\log \delta},$$

where $N(Z,\delta)$ denotes the smallest number of balls of radius δ needed to cover the set Z. Clearly, $\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z$ for each subset $Z \subset X$.

If μ is a probability measure on X, then the *Hausdorff dimension of* μ is defined by

$$\dim_H \mu = \inf \{\dim_H Y : Y \subseteq X,\ \mu(Y) = 1\}.$$

Finally, we define the *lower and upper pointwise dimensions of the measure* μ at the point $x \in X$ by

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \bar{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r) = \{y \in X : d(x,y) < r\}$. In particular, if there exists a number s such that

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = s$$

for μ-almost every $x \in X$, then $\dim_H \mu = s$, see [40].

The following lemma gives a method for calculating an upper bound to the Hausdorff dimension on a measure.

Lemma 2.1 (Lemma 6, [24]). *Let μ be a probability measure on the m_0 -dimensional compact Riemannian manifold M. If we can find a sequence of subsets $A_n \subseteq M$ such that*

(i) $\lim_{n\to\infty} \mu(A_n) = 1$;

(ii) $\lim_{n\to\infty} \mathcal{H}^t_{\beta_n}(A_n) = 0$ *for a sequence* $\{\beta_n\}_{n\in\mathbb{N}}$ *satisfying* $\lim_{n\to\infty} \beta_n = 0$.

Then it follows that $\dim_H \mu \leq t$.

2.2. Pressures

Let (X,f) be a topological dynamical systems (TDS), that is, X is a compact metric space X with a metric d, and $f: X \to X$ is a continuous transformation. Denote by $\mathcal{M}(X,f)$ and $\mathcal{M}^e(X,f)$ the set of all f-invariant and respectively, ergodic Borel probability measures on X. A sequence of continuous functions $\Phi = \{\varphi_n\}_{n\geq 1}$ is called *sub-additive*, if

$$\varphi_{m+n} \leq \varphi_n + \varphi_m \circ f^n, \quad \text{for all} \quad m,n \geq 1.$$

Similarly, we call a sequence of continuous functions $\Psi = \{\psi_n\}_{n\geq 1}$ *super-additive* if $-\Psi = \{-\psi_n\}_{n\geq 1}$ is sub-additive.

For $x,y \in X$ and $n \geq 0$ define the d_n-metric on X by

$$d_n(x,y) = \max\left\{d\left(f^i(x), f^i(y)\right) : 0 \leq i < n\right\}.$$

Given $\varepsilon > 0$ and $n \geq 0$, denote by $B_n(x,\varepsilon) = \{y \in X : d_n(x,y) < \varepsilon\}$ Bowen's ball centered at x of radius ε and length n and we call a subset $E \subset X$ (n,ε)-separated if $d_n(x,y) > \varepsilon$ for any two distinct points $x, y \in E$. A set $F \subset X$ is said to be an (n,ε)-spanning subset of X with respect to f if for any $x \in X$, there exists $y \in F$ with $d_n(x,y) \le \varepsilon$. For each $\mu \in \mathcal{M}(X,f)$, $0 < \delta < 1, n \ge 1$ and $\varepsilon > 0$, a subset $F \subset X$ is an (n,ε,δ)-spanning set if the union $\cup_{x\in F} B_n(x,\varepsilon)$ has μ-measure more than or equal to $1-\delta$.

Definition 2.1. Given a sub-additive sequence of continuous potentials Φ=$\{\varphi_n\}_{n\ge1}$, let

$$P_n(\Phi,\varepsilon) := \sup\left\{\sum_{x\in E} e^{\varphi_n(x)} : E \text{ is an } (n,\varepsilon)-\text{separated subset of } X\right\}.$$

The quantity

$$P(f,\Phi) := \lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(\Phi,\varepsilon)$$

is called the sub-additive topological pressure of Φ.

The authors in [21] proved that it satisfies the following variational principle:

$$P(f,\Phi) = \sup\left\{h_\mu(f) + \mathcal{F}_*(\Phi,\mu) : \mu \in \mathcal{M}(X,f), \mathcal{F}_*(\Phi,\mu) \ne -\infty\right\},$$

where $h_\mu(f)$ is the metric entropy of f with respect to μ and

$$\mathcal{F}_*(\Phi,\mu) = \lim_{n\to\infty}\frac{1}{n}\int\varphi_n\, d\mu = \inf_{n\ge1}\frac{1}{n}\int\varphi_n\, d\mu.$$

Existence of the above limit can be shown by the standard sub-additive argument.

Remark 2.1. If $\mathit{\Phi} = \{\varphi_n\}_{n\ge1}$ is *additive* in the sense that $\varphi_n(x) = \varphi(x)+\varphi(fx)+\cdots+\varphi(f^{n-1}x) \triangleq S_n\varphi(x)$ for some continuous function $\varphi : X \to \mathbb{R}$, we simply denote the topological pressure $P(f,\mathit{\Phi})$ as $P(f,\varphi)$.

Given a super-additive sequence of continuous potentials $\Psi = \{\psi_n\}_{n\ge1}$, we define the *super-additive topological pressure* of $\Psi = \{\psi_n\}_{n\ge1}$ by

$$P(f,\Psi) := \sup\left\{h_\mu(f) + \mathcal{F}_*(\Psi,\mu) : \mu \in \mathcal{M}(X,f)\right\}. \tag{2.1}$$

Note that for any super-additive sequence of continuous potentials and any f invariant measure μ we have

$$\mathcal{F}_*(\Psi,\mu) = \lim_{n\to\infty}\frac{1}{n}\int\psi_n\, d\mu = \sup_{n\ge1}\frac{1}{n}\int\psi_n\, d\mu.$$

It was proved in [10] that the following relation between the super-additive topological pressure and the topological pressure for additive potentials holds.

Proposition 2.1. *Let $\Psi = \{\psi_n\}_{n\ge1}$ be a super-additive sequence of continuous potentials on X. Then*

$$P(f,\Psi) = \lim_{n\to\infty}P(f,\frac{\psi_n}{n}) = \lim_{n\to\infty}\frac{1}{n}P\left(f^n,\psi_n\right).$$

Definition 2.2. For a sub-additive potential $\Phi = \{\varphi_n\}_{n\ge1}$, for $\mu \in \mathcal{M}^e(M,f), 0 < \delta < 1, n \ge 1$, and $\varepsilon > 0$, put

$$P_\mu(f,\Phi,n,\varepsilon,\delta) := \inf\left\{\sum_{x\in F}\exp\left(\sup_{y\in B_n(x,\varepsilon)}\varphi_n(y)\right) : F \text{ is an } (n,\varepsilon,\delta)-\text{spanning set}\right\},$$

$$P_\mu(f,\Phi,\varepsilon,\delta) := \limsup_{n\to\infty} \frac{1}{n} \log P_\mu(f,\Phi,n,\varepsilon,\delta),$$
$$P_\mu(f,\Phi,\delta) := \liminf_{\varepsilon\to 0} P_\mu(f,\Phi,\varepsilon,\delta),$$
$$P_\mu(f,\Phi) := \lim_{\delta\to 0} P_\mu(f,\Phi,\delta),$$

we call $P_\mu(f,\Phi)$ the sub-additive measure-theoretic pressure of f with respect to Φ.

Remark 2.2. (i) It is easy to see that $P_\mu(f,\Phi,\delta)$ increases with δ. So the limit in the last formula exists. In fact, it is proved in [11] that $P_\mu(f,\Phi,\delta)$ is independent of δ. Hence, the limit of $\delta \to 0$ is redundant in the definition.

(ii) If $\Phi = \{\varphi_n\}$ is additive generated by a continuous function, that is, $\varphi_n(x) = \sum_{i=0}^{n-1} \varphi_1(f^i x)$, then we simply write $P_\mu(f,\Phi)$ as $P_\mu(f,\varphi_1)$.

Theorem 2.1 (Theorem A, [9])**.** *Let (X,f) be a TDS and $\Phi = \{\varphi_n\}_{n\geq 1}$ a sub-additive potential on X. For every $\mu \in \mathcal{M}^e(X,f)$ with $\Phi_*(\mu) \neq -\infty$, we have*

$$P_\mu(f,\Phi) = h_\mu(f) + \mathcal{F}_*(\Phi,\mu).$$

Remark 2.3. (i) The results still apply for $\mathcal{F}_*(\Phi,\mu) = -\infty$ if $h_\mu(f) < \infty$.

(ii) If $\Phi = \{\varphi_n\}_{n\geq 1}$ is an additive sequence, then we have

$$P_\mu(f,\varphi_1) = h_\mu(f) + \int \varphi_1 d\mu.$$

The above equality is also in [32].

2.3. Dominated splitting

We recall the definition of a dominated splitting. Consider a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M of dimension m_0 and let $\Lambda \subset M$ be a compact invariant set. We say that Λ admits a *dominated splitting* if there is continuous invariant splitting $T_\Lambda M = E \oplus F$ and constants $C > 0$, $\lambda \in (0,1)$ such that for each $x \in \Lambda$, $n \in \mathbb{N}$, $0 \neq u \in E(x)$, and $0 \neq v \in F(x)$

$$\frac{\|D_x f^n(u)\|}{\|u\|} \leq C\lambda^n \frac{\|D_x f^n(v)\|}{\|v\|}.$$

We write $E \preceq F$ if F dominates E. Furthermore, we say that Df-invariant splitting on Λ

$$T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k, \quad (k \geq 2)$$

is a $\{\lambda_j\}_{1\leq j\leq k}$-*dominated splitting*, if there are numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, constants $C > 0$ and $0 < \varepsilon < \min_{1\leq i\leq k} \left\{\frac{\lambda_i - \lambda_{i+1}}{100}\right\}$ such that for every $x \in \Lambda$, $n \in \mathbb{N}$ and $1 \leq j \leq k$ and each unit vector $u \in E_j(x)$, it holds that

$$C^{-1} e^{n(\lambda_j - \varepsilon)} \leq \|D_x f^n(u)\| \leq C e^{n(\lambda_j + \varepsilon)}.$$

We write $E_1 \succeq E_2 \succeq \cdots \succeq E_k$.

2.4. Markov partition and weak Gibbs measures

Let Λ be a repeller of a C^1 expanding map f. Assume that $f|_\Lambda$ is topologically mixing. A finite closed cover $P_1, P_2, \cdots, P_k$ of Λ is called *a Markov partition of* Λ (with respect to f) if:

(i) $P_i \neq \emptyset$ and $\overline{\text{int}\,(P_i)} = P_i$;

(ii) int $(P_i) \cap$ int $(P_j) = \emptyset$ if $i \neq j$;

(iii) for any i the set $f(P_i)$ is the union of some of the sets P_j from the partition.

Here int$(\cdot)$ denotes the interior of a set relative to Λ. It is well known that any repeller Λ of a continuously differentiable expanding map f has Markov partition of arbitrary small diameter [33] and (Λ, f) is semi-conjugated to (Σ_A, σ), a subshift space of finite type $\Sigma^k = \{1, 2, \cdots, k\}^{\mathbb{N}}$, where

$$\Sigma_A = \left\{(i_0 i_1 \cdots i_n \cdots) \in \Sigma^k : a_{i_j i_{j+1}} = 1 \text{ for every } n \in \mathbb{N}\right\},$$

and $A = a_{ij}$ is the transfer matrix of the Markov partition, namely, $a_{ij} = 1$ if $\text{int}(P_i) \cap f^{-1}(\text{int}(P_j)) \neq \emptyset$ and $a_{ij} = 0$ otherwise. For any $n \geq 1$, $\Sigma_{A,n}$ denotes the set of finite sequence $\mathbf{i} = (i_0 i_1 \cdots i_{n-1})$ such that $a_{i_j i_{j+1}} = 1$ for all $0 \leq j \leq n-2$. These sequences $\mathbf{i}$ are called *admissible words*. The length of the word is denoted by $|\mathbf{i}|$. For $\mathbf{i} = (i_0 i_1 \cdots i_{n-1}) \in \Sigma_{A,n}$, we define

$$P_{i_0 i_1 \ldots i_{n-1}} = \bigcap_{j=0}^{n-1} f^{-j}\left(P_{i_j}\right). \tag{2.2}$$

Definition 2.3. Let $\varphi : \Lambda \to \mathbb{R}$ be a continuous function. We call a (not necessarily invariant) Borel probability measure μ on Λ a weak Gibbs measure for φ if for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that for all $n \geq N$, every admissible sequence $(i_0 i_1 \ldots i_{n-1}) \in \Sigma_{A,n}$, and $x \in P_{i_0 i_1 \ldots i_{n-1}}$, we have

$$e^{-n\varepsilon} \leq \frac{\mu\left(P_{i_0 i_1 \ldots i_{n-1}}\right)}{\exp\left[-nP + S_n\varphi(x)\right]} \leq e^{n\varepsilon},$$

where P is a constant and $S_n\varphi(x) = \sum_{j=0}^{n-1} \varphi\left(f^j(x)\right)$.

Remark 2.4. (i) The authors in [23, 26] proved the existence of such a weak Gibbs measure μ for a continuous function $\varphi : \Lambda \to \mathbb{R}$.

(ii) If there exists a constant $K > 0$ such that for every $n \in \mathbb{N}$,

$$K^{-1} \leq \frac{\mu\left(P_{i_0 i_1 \ldots i_{n-1}}\right)}{\exp\left[-nP + S_n\varphi(x)\right]} \leq K.$$

Definition 2.3 is recovered the classical notion of Gibbs measure (See [1] for more details.).

3. Proofs of Main Results

3.1. Proof of Theorem 1.1

This section is divided into two parts which provide the proof of Theorem 1.1 stated in Section 1.

3.1.1. Lower bound for the Hausdorff dimension of repellers.

In this section we prove $\dim_H \Lambda \geq s^*$, where s^* is the unique root of the equation $\widetilde{P}_{\sup}(s) = 0$, which is defined in (1.2). We first obtain a coarse lower bound of Hausdorff dimension of a non-conformal repeller as follows.

Lemma 3.1. *Assume that $f : M \to M$ is a C^1 map on the m_0-dimensional compact smooth Riemannian manifold M. Let Λ be a repeller for the map f, if $f|_\Lambda$ possesses a $\{\lambda_j\}_{1\leq j\leq k}$-dominated splitting $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ with $E_1 \succeq E_2 \succeq \cdots \succeq E_k$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. Then*

$$\dim_H \Lambda \geq s_1,$$

where s_1 is the unique root of Bowen's equation $P(f|_\Lambda, -\tilde{\psi}^s(\cdot, f)) = 0$.

Proof. Note that $x \mapsto E_i(x)$ is continuous on Λ since the splitting $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is dominated, and the continuity of the map $x \mapsto E_i(x)$ can be extended to U (here U is an open neighborhood of Λ in the definition of the repeller), so the map $x \mapsto \left\| D_x f|_{E_i} \right\|$ is continuous for every $i = 1, 2, \ldots, k$. Therefore, for any sufficiently small $\varepsilon > 0$, there exists $\delta > 0$, for any $x, y \in U$ with $d(x, y) < \delta$, so that

$$e^{-\varepsilon} \leq \frac{\|D_x f|_{E_i}\|^{-1}}{\|D_y f|_{E_i}\|^{-1}} \leq e^{\varepsilon}.$$

Let $\{P_1, P_2 \ldots, P_l\}$ be a Markov partition of Λ, with $\max_i \operatorname{diam}(P_i) < \frac{\delta}{2}$. It follows that for each $i = 1, 2, \ldots, l$ the closed $\frac{\delta}{4}$ neighborhood $\widetilde{P}_i$ of P_i is such that $\widetilde{P}_i \subseteq U$ and $\widetilde{P}_i \cap \widetilde{P}_j = \emptyset$ whenever $P_i \cap P_j = \emptyset$. Given an admissible sequence $\mathbf{i} = (i_0 i_1 \ldots i_{n-1})$ and the cylinder $P_{i_0 i_1 \ldots i_{n-1}}$, we denote by $\widetilde{P}_{i_0 i_1 \ldots i_{n-1}}$ the corresponding cylinder. Since for any $x, y \in \widetilde{P}_{i_0 i_1 \ldots i_{n-1}}$, we have $f^j(x) \in \widetilde{P}_{i_j}, f^j(y) \in \widetilde{P}_{i_j}$ for every $j = 0, 1, \ldots, n-1$. Then $d(f^j(x), f^j(y)) < \delta$ and for any $n \in \mathbb{N}$, we have

$$e^{-n\varepsilon} \leq \frac{\prod_{j=0}^{n-1} \|D_{f^j(x)} f|_{E_i}\|^{-1}}{\prod_{j=0}^{n-1} \|D_{f^j(y)} f|_{E_i}\|^{-1}} \leq e^{n\varepsilon}. \tag{3.1}$$

Since $P(f|_\Lambda, -\tilde{\psi}^{s_1}(\cdot, f)) = 0$ and $\tilde{\psi}^{s_1}(\cdot, f)$ is a continuous function on Λ, there is a weak Gibbs measure μ, for the above $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for any $n > N$ and each $x \in P_{i_0 i_1 \ldots i_{n-1}}$, we have

$$e^{-n\varepsilon} \leq \frac{\mu\left(P_{i_0 i_1 \ldots i_{n-1}}\right)}{\exp\left(-\sum_{i=0}^{n-1} \tilde{\psi}^{s_1}\left(f^i(x), f\right)\right)} \leq e^{n\varepsilon}.$$

Since $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is a dominated splitting, the angles between different subspaces E_i are uniformly bounded away from zero. Therefore there exists $a > 0$ and $\xi_i \in \widetilde{P}_{i_0 i_1 \ldots i_{n-1}}$ for each $i = 1, 2, \ldots, k$ with a rectangle of sides

$$\overbrace{a\|D_{\xi_1} f^n|_{E_1}\|^{-1}, \ldots, a\|D_{\xi_1} f^n|_{E_1}\|^{-1}}^{m_1}, \ldots, \overbrace{a\|D_{\xi_k} f^n|_{E_k}\|^{-1}, \ldots, a\|D_{\xi_k} f^n|_{E_k}\|^{-1}}^{m_k}$$

contained in $\widetilde{P}_{i_0 i_1 \ldots i_{n-1}}$. Note that

$$\|D_{\xi_i} f^n|_{E_i}\|^{-1} \geq \prod_{j=0}^{n-1} \|D_{f^j(\xi_i)} f|_{E_i}\|^{-1}.$$

Thus by (3.1), there exists $x \in P_{i_0 i_1 \ldots i_{n-1}}$ such that $\widetilde{P}_{i_0 i_1 \ldots i_{n-1}}$ contains a rectangle of sides

$$\overbrace{ae^{-n\varepsilon}A_1(x,n),\ldots,ae^{-n\varepsilon}A_1(x,n)}^{m_1},\ldots,\overbrace{ae^{-n\varepsilon}A_k(x,n),\ldots,ae^{-n\varepsilon}A_k(x,n)}^{m_k},$$

where $A_i(x,n) = \prod_{j=0}^{n-1} \|D_{f^j(x)} f|_{E_i}\|^{-1}$ for any $i \in \{1,2,\ldots,k\}$. Note that there is $i \in \{0,1,2,\ldots,k-1\}$ such that $r_i < s_1 \le r_{i+1}$. Fix $r > 0$ small enough and set

$$\begin{aligned}\mathcal{Q} =\Big\{ &\mathbf{i} = (i_0 i_1 \cdots i_{n-1}) : \; ae^{-n\varepsilon} A_{i+1}(x,n) \le r \; \text{ for all } \; x \in P_{i_0 i_1 \cdots i_{n-1}}, \\ &\text{but } \; ae^{-n\varepsilon} A_{i+1}(y, n-1) > r \; \text{ for some } \; y \in P_{i_0 i_1 \cdots i_{n-1}} \Big\}.\end{aligned}$$

Therefore, for every $\mathbf{i} = (i_0 i_1 \cdots i_{n-1}) \in \mathcal{Q}$, we have

$$bre^{-n\varepsilon} < ae^{-n\varepsilon} A_{i+1}(x,n) \le r \text{ for all } x \in P_{i_0 i_1 \ldots i_{n-1}}, \tag{3.2}$$

where $b = \min_{x\in\Lambda} \|D_x f\|^{-1} < 1$. Let $b_1 = \max_{x\in\Lambda} \|D_x f\|^{-1} < 1$, therefore $a{b_1}^n e^{-n\varepsilon_1} > bre^{-n\varepsilon_1}$. Combining with (3.2) one has

$$n < \frac{\log r + \log b - \log a}{\log b_1}. \tag{3.3}$$

Let B be a ball of radius r and $\tilde{B}$ a ball of radius $2r$ with the same center as that of B. Put

$$\mathcal{Q}_1 = \Big\{ \mathbf{i} \in \mathcal{Q} \mid P_{\mathbf{i}} \cap B \ne \emptyset \Big\}.$$

Recall that $A_1(x,n) \le A_2(x,n) \le \cdots \le A_k(x,n)$. Hence, for each $\mathbf{i} \in \mathcal{Q}_1$, we have $\widetilde{P}_{\mathbf{i}} \cap \tilde{B}$ contains a rectangle of sides

$$\begin{aligned}&\overbrace{ae^{-n\varepsilon}A_1(x,n),\ldots,ae^{-n\varepsilon}A_1(x,n)}^{m_1},\ldots,\overbrace{ae^{-n\varepsilon}A_i(x,n),\ldots,ae^{-n\varepsilon}A_i(x,n)}^{m_i},\\ &\overbrace{ae^{-n\varepsilon}A_{i+1}(x,n),\ldots,ae^{-n\varepsilon}A_{i+1}(x,n)}^{m_0-r_i}.\end{aligned}$$

(Recall $r_i = m_1 + \cdots + m_i$.) It follows that

$$\begin{aligned}\mathrm{vol}_{m_0}\left(\widetilde{P}_{\mathbf{i}} \cap \tilde{B}\right) &\ge a^{m_0} \cdot e^{-nm_0\varepsilon} \cdot A_{i+1}(x,n)^{m_0-r_i} A_i(x,n)^{m_i} \; \cdots \; A_1(x,n)^{m_1} \\ &= a^{m_0} \cdot e^{-nm_0\varepsilon} \cdot A_{i+1}(x,n)^{s_1-r_i} A_{i+1}(x,n)^{m_0-s_1} \cdot \\ &\quad A_i(x,n)^{m_i} \; \cdots \; A_1(x,n)^{m_1} \\ &\ge a^{m_0} \cdot e^{-nm_0\varepsilon} \cdot A_{i+1}(x,n)^{s_1-r_i} \left(\frac{b}{a}\right)^{m_0-s_1} \cdot r^{m_0-s_1} \cdot \\ &\quad A_i(x,n)^{m_i} \; \cdots \; A_1(x,n)^{m_1} \\ &\ge a_2 \cdot e^{-nm_0\varepsilon} \cdot r^{m_0-s_1} \cdot A_{i+1}(x,n)^{s_1-r_i} A_i(x,n)^{m_i} \cdots A_1(x,n)^{m_1}\end{aligned}$$

where $a_2 = a^{m_0}\left(\dfrac{b}{a}\right)^{m_0-s_1}$, and the second inequality is by (3.2). Thus there is a positive constant a_3 such that

$$\begin{aligned}
2^{m_0} a_3 r^{m_0} &\geq \mathrm{vol}_{m_0}\left(\tilde{B}\right) \\
&= \sum_{\mathbf{i}\in\mathcal{Q}_1} \mathrm{vol}_{m_0}\left(\widetilde{P}_{\mathbf{i}} \cap \tilde{B}\right) \\
&\geq \sum_{\mathbf{i}\in\mathcal{Q}_1} a_2 \cdot e^{-nm_0\varepsilon} \cdot r^{m_0-s_1} \cdot A_{i+1}(x,n)^{s_1-r_i} A_i(x,n)^{m_i} \cdots A_1(x,n)^{m_1}.
\end{aligned}$$

It yields that

$$\sum_{\mathbf{i}\in\mathcal{Q}_1} e^{-nm_0\varepsilon} A_{i+1}(x,n)^{s_1-r_i} A_i(x,n)^{m_i} \cdots A_1(x,n)^{m_1} \leq C r^{s_1},$$

where $C = \dfrac{2^{m_0} a_3}{a_2}$. Therefore

$$\begin{aligned}
\mu(B) &\leq \sum_{(i_0 i_1 \ldots i_{n-1})\in\mathcal{Q}_1} \mu\left(P_{i_0 i_1 \ldots i_{n-1}}\right) \\
&\leq \sum_{\mathbf{i}\in\mathcal{Q}_1} \exp\left(n\varepsilon - \sum_{i=0}^{n-1} \tilde{\psi}^{s_1}\left(f^i(x), f\right)\right) \\
&\leq \sum_{\mathbf{i}\in\mathcal{Q}_1} e^{n\varepsilon} \cdot A_{i+1}(x,n)^{s_1-r_i} A_i(x,n)^{m_i} \cdots A_1(x,n)^{m_1} \\
&= \sum_{\mathbf{i}\in\mathcal{Q}_1} e^{-nm_0\varepsilon} \cdot A_{i+1}(x,n)^{s_1-r_i} A_i(x,n)^{m_i} \cdots A_1(x,n)^{m_1} \cdot e^{n(m_0+1)\varepsilon} \\
&\leq C C_1 r^{s_1 + \frac{m_0+1}{\log b_1}\varepsilon},
\end{aligned}$$

where $C_1 = (\frac{b}{a})^{\frac{m_0+1}{\log b_1}\varepsilon}$, and the last inequality is by (3.3). Thus

$$\liminf_{r\to 0} \frac{\log \mu(B)}{\log r} \geq \frac{m_0+1}{\log b_1}\varepsilon + s_1.$$

Since ε is arbitrary, we have $\underline{d}_\mu(x) \geq s_1$, which implies that $\dim_H \mu \geq s_1$. Hence $\dim_H \Lambda \geq s_1$. □

For any $k \in \mathbb{N}$, the set Λ is also a repeller for f^{2^k}. By Lemma 3.1, for every $k \in \mathbb{N}$, we have $\dim_H \Lambda \geq s_k$ where s_k is the unique root of the equation

$$P\left(f^{2^k}\Big|_\Lambda, -\tilde{\psi}^s\left(\cdot, f^{2^k}\right)\right) = 0.$$

Note that any f^{2^k}-invariant measure μ must be $f^{2^{k+1}}$-invariant. This together with the super-additivity of $\left\{-\tilde{\psi}^s(\cdot, f^n)\right\}_{n\geq 1}$ (See definition in (1.1).) yields that for any f^{2^k}-invariant measure μ,

$$\begin{aligned}
\frac{1}{2^{k+1}} P\left(f^{2^{k+1}}, -\tilde{\psi}^s(\cdot, f^{2^{k+1}})\right) &\geq \frac{1}{2^{k+1}}\left(h_\mu(f^{2^{k+1}}) - \int \tilde{\psi}^s(x, f^{2^{k+1}})\, d\mu\right) \\
&\geq \frac{1}{2^{k+1}}\left(h_\mu(f^{2^{k+1}}) - 2\int \tilde{\psi}^s(x, f^{2^k})\, d\mu\right) \\
&= \frac{1}{2^k}\left(h_\mu(f^{2^k}) - \int \tilde{\psi}^s(x, f^{2^k})\, d\mu\right).
\end{aligned}$$

Hence,

$$\frac{1}{2^{k+1}}P\left(f^{2^{k+1}},-\tilde{\psi}^s(\cdot,f^{2^{k+1}})\right)\geq\frac{1}{2^k}P\left(f^{2^k},-\tilde{\psi}^s(\cdot,f^{2^k})\right). \tag{3.4}$$

By Proposition 2.1, we have

$$\widetilde{P}_{\sup}(s)=\lim_{k\to\infty}\frac{1}{2^k}P\left(f^{2^k},-\tilde{\psi}^s(\cdot,f^{2^k})\right)=P_{\mathrm{var}}\left(f|_\Lambda,\left\{-\tilde{\psi}^s\left(\cdot,f^n\right)\right\}\right). \tag{3.5}$$

By (3.4), we have that $s_k\leq s_{k+1}$ and hence, there is a limit $s^*=\lim_{k\to\infty}s_k$. We have that $\dim_H\Lambda\geq s^*$. Note that $s^*\geq s_k$ for every $k\geq 1$. It now follows from (3.5) and $\widetilde{P}_{\sup}(\cdot)$ is continuous, strictly decreasing in s that $\widetilde{P}_{\sup}\ (s^*)\leq 0$. On the other hand for any $\varepsilon>0$, there is a positive integer K such that $s_k\geq s^*-\varepsilon$ for any $k\geq K$. Thus $\widetilde{P}_{\sup}(s^*-\varepsilon)\geq 0$ for every $\varepsilon>0$. Therefore $\widetilde{P}_{\sup}(s^*)\geq 0$. So $\widetilde{P}_{\sup}(s^*)=0$.

3.1.2. Upper bound for the box dimension of repellers

Secondly we prove $\overline{\dim}_B\Lambda\leq t^*$, where t^* is the unique root of the equation $\widetilde{P}_{\mathrm{sub}}(t)=0$, which is defined in (1.4).

For any sufficiently small $\varepsilon>0$, let $\{P_1,P_2,\cdots,P_l\}$ be a Markov partition of Λ and $\delta>0$ as that in the proof of Lemma 3.1. We still denote by $\widetilde{P}_i$ the $\frac{\delta}{4}$-neighborhood of P_i as that in the proof of Lemma 3.1. Note that the map $x\mapsto m(D_xf|_{E_i})$ is continuous on U for each $i\in\{1,2,\ldots,k\}$, and the $\{\lambda_j\}_{1\leq j\leq k}$-dominated splitting can be extended to U, where U is an open neighborhood of Λ in the definition of the repeller.

Choose a number s such that $t^*<s\leq m_0$ and assume that $\ell_d<s\leq\ell_{d+1}$ for some $d\in\{0,1,\ldots,k-1\}$ (Recall $\dim E_j=m_j$ and $\ell_d=m_k+m_{k-1}+\cdots+m_{k-d+1}$.). Since $\widetilde{P}_{\mathrm{sub}}\ (s)<0$, we may find a positive integer q for which

$$\sum_{\mathbf{i}\in S_q}e^{-\tilde{\varphi}^s(y_{\mathbf{i}},f^q)}<1$$

for all $y_{\mathbf{i}}\in P_{\mathbf{i}}$, where $\mathbf{i}=(i_0i_1\cdots i_{q-1})$ is an admissible sequence, S_q is the collection of all admissible sequences of length q, $P_{\mathbf{i}}$ is a cylinder (see (2.2)), and $\tilde{\varphi}^s\left(\cdot,f^q\right)$ is as in (1.3). For any $n\geq 1$, let

$$B_i(x,nq)=\prod_{j=0}^{n-1}m(D_{f^{jq}(x)}f^q|_{E_i})^{-1}$$

for $i\in\{1,2,\ldots,k\}$, it follows that for all $y_{\mathbf{i}}\in P_{\mathbf{i}}$,

$$\sum_{\mathbf{i}\in S_q}B_{k-d}(y_{\mathbf{i}},q)^{s-\ell_d}\cdot B_{k-d+1}(y_{\mathbf{i}},q)^{m_{k-d+1}}\cdots B_{k-1}(y_{\mathbf{i}},q)^{m_{k-1}}B_k(y_{\mathbf{i}},q)^{m_k}<1. \tag{3.6}$$

Given $0<r\leq 1$, set

$$\begin{aligned}\mathcal{Q}=\Big\{\mathbf{i}=\left(i_0i_1\ldots i_{nq-1}\right):B_{k-d}(x,nq)\leq r\text{ for all }x\in P_{i_0i_1\ldots i_{nq-1}}\\ \text{but }r<B_{k-d}(y,(n-1)q)\ \text{ for some }\ y\in P_{i_0i_1\ldots i_{(n-1)q-1}}\Big\}.\end{aligned}$$

Since $x \mapsto B_i(x,q)$ is continuous and f^q is expanding, then for each $n \geq 1$, all $i \in \{1,2,\cdots,k\}$ and all $x,y \in \widetilde{P}_{i_0 i_1 \dots i_{nq-1}}$,

$$e^{-n\varepsilon} \leq \frac{B_i(x,nq)}{B_i(y,nq)} \leq e^{n\varepsilon}.$$

Therefore for any $\mathbf{i} = (i_1 i_2 \cdots i_{nq-1}) \in \mathcal{Q}$,

$$bre^{-n\varepsilon} < B_{k-d}(x,nq) \leq r, \tag{3.7}$$

for all $x \in P_{i_0 i_1 \dots i_{nq-1}}$, where $b = \min_{x\in\Lambda} B_{k-d}(x,q) < 1$. Let

$$b_1 = \max_{x\in\Lambda} m(D_x f^q)^{-1} < 1$$

and hence $bre^{-n\varepsilon} \leq b_1^n$, we have

$$n \leq \frac{\log b + \log r}{\log b_1 + \varepsilon}. \tag{3.8}$$

For every admissible sequence $(i_0 i_1 \dots)$, there is a unique integer $n \in \mathbb{Z}$ such that $(i_0 i_1 \dots i_{nq-1}) \in \mathcal{Q}$. In particular, $\Lambda \subset \bigcup_{\mathbf{i}\in\mathcal{Q}} P_{\mathbf{i}}$. Note that

$$m\left(D_x f^{nq}|_{E_1}\right)^{-1} \leq m\left(D_x f^{nq}|_{E_2}\right)^{-1} \leq \cdots \leq m\left(D_x f^{nq}|_{E_k}\right)^{-1}$$

and for all $i \in \{1,\dots,k\}$ and any $x \in \Lambda$,

$$m\left(D_x f^{nq}|_{E_i}\right)^{-1} \leq B_i(x,nq).$$

Since $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is dominated, we conclude that there is a constant $C_1 > 0$ such that

$$\begin{aligned}
&N(\Lambda,r)\\
&\leq C_1 \cdot \sum_{\mathbf{i}\in\mathcal{Q}} \frac{m(D_{y_{\mathbf{i}}} f^{nq}|_{E_k})^{-m_k}}{r^{m_k}} \cdots \frac{m(D_{y_{\mathbf{i}}} f^{nq}|_{E_{k-d+1}})^{-m_{k-d+1}}}{r^{m_{k-d+1}}}\\
&\leq C_1 \cdot \sum_{\mathbf{i}\in\mathcal{Q}} \frac{B_k\left(y_{\mathbf{i}},nq\right)^{m_k}}{B_{k-d}\left(y_{\mathbf{i}},nq\right)^{m_k}} \cdots \frac{B_{k-d+1}\left(y_{\mathbf{i}},nq\right)^{m_{k-d+1}}}{B_{k-d}\left(y_{\mathbf{i}},nq\right)^{m_{k-d+1}}}\\
&= C_1 \cdot \sum_{\mathbf{i}\in\mathcal{Q}} B_k\left(y_{\mathbf{i}},nq\right)^{m_k} \cdots B_{k-d+1}\left(y_{\mathbf{i}},nq\right)^{m_{k-d+1}} B_{k-d}\left(y_{\mathbf{i}},nq\right)^{-\ell_d}\\
&= C_1 \cdot \sum_{\mathbf{i}\in\mathcal{Q}} B_k\left(y_{\mathbf{i}},nq\right)^{m_k} \cdots B_{k-d+1}\left(y_{\mathbf{i}},nq\right)^{m_{k-d+1}} B_{k-d}\left(y_{\mathbf{i}},nq\right)^{s-\ell_d} B_{k-d}\left(y_{\mathbf{i}},nq\right)^{-s}\\
&\leq C_1 \cdot 1 \cdot b^{-s} r^{-s} e^{n\varepsilon s}\\
&\leq C_1 \cdot b^{-s} r^{-s} \cdot (br)^{\frac{\varepsilon s}{\log b_1 + \varepsilon}},
\end{aligned}$$

where $N(\Lambda,r)$ denotes the smallest number of balls of radius r required to cover Λ. Here the penultimate inequality is by (3.6) and (3.7), and the last inequality is by (3.8). Therefore

$$\limsup_{r\to 0} \frac{\log N(\Lambda,r)}{-\log r} \leq s + \frac{-\varepsilon s}{\log b_1 + \varepsilon}.$$

Let ε tend to 0, we have

$$\limsup_{r\to 0} \frac{\log N(\Lambda,r)}{-\log r} \leq s.$$

So that $\overline{\dim}_B \Lambda \leq s$. By the arbitrariness of s one has $\overline{\dim}_B \Lambda \leq t^*$.

3.2. Proof of Theorem 1.2

To prove Theorem 1.2, we need a coarse upper bound for the Hausdorff dimension of an ergodic f-invariant expanding probability measure μ first. We now provide the following useful lemma, which estimates the Hausdorff measure of the image of a small set under a C^1 local diffeomorphism. It is similar to Lemma 3 and Corollary 1 in [41].

Lemma 3.2. *Let $f : M \to M$ be a C^1 local diffeomorphism on the m_0-dimensional compact smooth Riemannian manifold M. Fix $t \in [0, m_0]$, then for any $b_0 > 2\sqrt{m_0}$ and $C_0 > 2^t m_0^{\frac{t}{2}}$, there is $\rho_0 > 0$ such that for all $x \in M$ and all $A \subseteq B(x, \rho_0)$ we have*

$$\mathcal{H}^t_{b\rho}(A) \leq C\mathcal{H}^t_\rho(f(A))$$

for all $0 < \rho < \rho_0$, where

$$b = b_0 \exp\left\{-\log \alpha_{m_0-[t]}(x, f)\right\} \text{ and } C = C_0 \exp\left\{-\varphi^t(x, f)\right\}.$$

Proof. For simplicity, we just prove the lemma on the assumption that M is the Euclid space $\mathbb{R}^{m_0}$. For the general case, we can use local charts to prove it.

Since $f : M \to M$ is a C^1 local diffeomorphism on the m_0-dimensional compact smooth Riemannian manifold M, then there is a constant $\rho_1 > 0$, for every $x \in M$ such that $f|_{B(x_1,\rho_1)} : B(x, \rho_1) \to f(B(x, \rho_1))$ is a C^1 diffeomorphism. Let ε be a small positive number with $(1+\varepsilon)e^\varepsilon < 2$. For such ε, there exists $0 < \rho_0 < \rho_1$ such that for $y, z \in M$ with $d(y, z) \leq \rho_0$ the following properties hold:

(a) $\left\|y - z - D_y f^{-1}(f(y) - f(z))\right\| \leq \varepsilon\|y - z\|$;

(b) $|\log \alpha_i(y, f) - \log \alpha_i(z, f)| \leq \varepsilon$ for $i = 1, 2, \cdots, m_0$.

Assume $A \subseteq B(x, \rho_0)$. Fix any $0 < \rho < \rho_0$. Let $a = \mathcal{H}^t_\rho(f(A))$ (a is finite). Then for any $\eta > 0$, there are $\{z_j\} \subseteq f(B(x, \rho_0))$ such that $f(A) \subseteq \cup_j B(z_j, r_j)$ and

$$\sum_j r_j^t < a + \eta \quad \text{where} \quad r_j \leq \rho.$$

Let $B'_j = \{y \in B(x, \rho_0) : f(y) \in B(z_j, r_j)\}$. Note that $\cup_j B'_j \supseteq A$. By (a) we conclude B'_j is contained in an ellipse with principal axes

$$(1+\varepsilon)r_j \cdot \alpha_1(y_j, f)^{-1}, (1+\varepsilon)r_j \cdot \alpha_2(y_j, f)^{-1}, \cdots, (1+\varepsilon)r_j \cdot \alpha_{m_0}(y_j, f)^{-1},$$

where $y_j \in B(x, \rho_0)$, $f(y_j) = z_j$. Then by (b) we obtain that B'_j is contained in an ellipse with principal axes

$$(1+\varepsilon)r_j \cdot e^\varepsilon \alpha_1(x, f)^{-1}, (1+\varepsilon)r_j \cdot e^\varepsilon \alpha_2(x, f)^{-1}, \cdots, (1+\varepsilon)r_j \cdot e^\varepsilon \alpha_{m_0}(x, f)^{-1}.$$

Therefore B'_j is covered by

$$\frac{\exp\left\{-\sum_{i=m_0-[t]+1}^{m_0} \log \alpha_i(x, f)\right\}}{\exp\left\{-[t] \log \alpha_{m_0-[t]}(x, f)\right\}}$$

balls with radius $(1+\varepsilon)e^\varepsilon\sqrt{m_0}r_j \exp\left\{-\log \alpha_{m_0-[t]}(x, f)\right\}$. In fact the radius

$$\begin{aligned} & (1+\varepsilon)e^\varepsilon\sqrt{m_0}r_j \exp\left\{-\log \alpha_{m_0-[t]}(x, f)\right\} \\ \leq\ & 2\sqrt{m_0} \exp\left\{-\log \alpha_{m_0-[t]}(x, f)\right\} \cdot \rho \\ \leq\ & b\rho. \end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{H}_{b\rho}^{t}\left(B_j'\right) \leq &\ \exp\Big\{\sum_{i=m_0-[t]+1}^{m_0}(-\log\alpha_i(x,f))+[t]\log\alpha_{m_0-[t]}(x,f)\Big\}\cdot\\ &\left[(1+\varepsilon)e^{\varepsilon}\sqrt{m_0}\right]^t\cdot r_j^t\exp\left\{-t\log\alpha_{m_0-[t]}(x,f)\right\}\\ \leq &\ (2\sqrt{m_0})^t\exp\left[-\varphi^t(x,f)\right]\cdot r_j^t\ .\end{aligned}$$

Summing up over all j we get

$$\begin{aligned}\mathcal{H}_{b\rho}^{t}(A) &\leq \sum_j \mathcal{H}_{b\rho}^{t}\left(B_j'\right)\\ &\leq 2^t(\sqrt{m_0})^t\exp\left[-\varphi^t(x,f)\right]\sum_j r_j^t\\ &\leq 2^t(\sqrt{m_0})^t\cdot\exp\left[-\varphi^t(x,f)\right]\cdot(a+\eta).\end{aligned}$$

Since $\eta>0$ is arbitrary, this proves the lemma. □

The following lemma gives a coarse upper bound for the Hausdorff dimension of an ergodic f-invariant expanding probability measure μ, which is the zero point of the additive measure-theoretic pressure. The method of proof involves applying Lemma 2.1. We initially define a suitable sequence of sets $\{A_n\}_{n\geq 1}$. We utilize Lemma 3.2 to estimate how one iteration of a C^1 local diffeomorphism f effects the Hausdorff measure of some dynamical balls, which cover A_n.

Lemma 3.3. *Let $f: M\to M$ be a C^1 local diffeomorphism on the m_0-dimensional compact smooth Riemannian manifold M. Let μ be an ergodic f-invariant expanding probability measure with the corresponding Lyapunov exponents*

$$\lambda_1(\mu)\geq\lambda_2(\mu)\geq\cdots\geq\lambda_{m_0}(\mu)>0.$$

Then we have

$$\dim_H\mu\leq t_1,$$

where t_1 is the unique root of the equation $P_\mu\left(f,-\varphi^t(\cdot,f)\right)=0$.

Proof. Fix any $\varepsilon>0$ such that $-\lambda_{m_0}(\mu)+2\varepsilon<0$. We choose $s>t_1$ such that

$$h_\mu(f)-\int\varphi^s(x,f)\ d\mu=-3\varepsilon.$$

Recall the definition of $\varphi^s(x,f)$ in (1.6). By Lemma 2.2 in [38] we conclude that for the above $\varepsilon>0$, there exists an integer $N_1=N_1(\varepsilon)$ such that, for μ-almost every points $x\in M$ and any $L\geq N_1$,

$$\lambda_{m_0}(\mu)-\varepsilon\leq\lim_{n\to\infty}\frac{1}{nL}\sum_{i=0}^{n-1}\log m\left(D_{f^{iL}x}f^L\right)\leq\lambda_{m_0}(\mu)+\varepsilon.$$

Let ρ_0 be as in Lemma 3.2. Fix $\delta\in(0,\rho_0)$ and consider a finite measurable partition ξ with $\operatorname{diam}\xi=\max\{\operatorname{diam}C_\xi: C_\xi\in\xi\}\leq\frac{\delta}{2}$. Let $C_{\xi_n}(x)$ be the element of the partition

$$\xi_n=\xi\vee f^{-1}(\xi)\vee\cdots\vee f^{-(n-1)}(\xi)$$

containing x. It is easy to see $C_{\xi_n}(x) \subseteq B_n\left(x, \frac{\delta}{2}\right)$. By the Shannon-McMillan-Breiman theorem in [37], we obtain that for μ almost every $x \in M$,

$$\lim_{n\to\infty} \frac{-\log \mu\left(C_{\xi_n}(x)\right)}{n} = h_\mu(f,\xi),$$

where $h_\mu(f,\xi)$ is the measure-theoretic entropy of f with respect to ξ. Therefore

$$\limsup_{n\to\infty} \frac{-\log \mu\left(B_n\left(x, \frac{\delta}{2}\right)\right)}{n} \leq \lim_{n\to\infty} \frac{-\log \mu\left(C_{\xi_n}(x)\right)}{n} = h_\mu(f,\xi) \leq h_\mu(f).$$

Let $b_0 > 2\sqrt{m_0}$, $C_0 > 2^s m_0^{\frac{s}{2}}$ and choose $N > N_1$ large enough such that

$$C_0 e^{-N\varepsilon} < 1 \text{ and } e^{\left[\lambda_{m_0}(\mu)-2\varepsilon\right]N} > b_0. \tag{3.9}$$

Since the measure μ is ergodic, the Birkhoff Ergodic Theory says that for μ almost every $x \in M$,

$$\lim_{n\to\infty} \frac{1}{nN} \sum_{i=0}^{nN-1} \varphi^s\left(f^i x, f\right) = \int \varphi^s(x,f)\, d\mu.$$

We would like to choose sets $A_n \subseteq M$ such that any $x \in A_n$ satisfies

(a) $\exp\left\{-nN\left[h_\mu(f)+\varepsilon\right]\right\} \leq \mu\left(B_{nN}(x, \frac{\delta}{2})\right)$,

(b) $nN\left[-\int \varphi^s(x,f)\, d\mu - \varepsilon\right] \leq -\sum_{i=0}^{nN-1} \varphi^s\left(f^i x, f\right) \leq nN\left[-\int \varphi^s(x,f)\, d\mu + \varepsilon\right]$;

(c) $nN\left[\lambda_{m_0}(\mu)-2\varepsilon\right] \leq \sum_{i=0}^{n-1} \log m\left(D_{f^{iN}x} f^N\right) \leq nN\left[\lambda_{m_0}(\mu)+2\varepsilon\right]$.

Let E be a maximal (nN,δ)-separated subset of A_n, then $A_n \subseteq \bigcup_{x_j\in E} B_{nN}\left(x_j,\delta\right)$. Furthermore the balls $B_{nN}\left(x_j, \frac{\delta}{2}\right)$ less than or equal to $\exp\left\{nN\left[h_\mu(f)+\varepsilon\right]\right\}$. For $x \in A_n$, let

$$b_k(x) = \left(b_0\right)^k \exp\left\{-\sum_{i=n-k}^{n-1} \log \alpha_{m_0-[s]}\left(f^{iN}x, f^N\right)\right\}$$

for $k = 1, 2, \cdots, n$ and $\beta_n = \left\{b_0 e^{\left[-\lambda_{m_0}(\mu)+2\varepsilon\right]N}\right\}^n \rho$ where $0 < \rho < \rho_0$. For $x \in A_n$, by (c) we have

$$\begin{aligned} b_n(x)\rho &\leq \left(b_0\right)^n \cdot \exp\left\{-\sum_{i=0}^{n-1} \log m\left(D_{f^{iN}x} f^N\right)\right\} \cdot \rho \\ &\leq \left(b_0\right)^n \cdot \exp\left\{-nN\left[\lambda_{m_0}(\mu)-2\varepsilon\right]\right\} \cdot \rho \\ &= \beta_n. \end{aligned}$$

Using Lemma 3.2 n times, we conclude

$$\begin{aligned}
\mathcal{H}^s_{\beta_n}\left(B_{nN}(x,\delta)\right) &\le \mathcal{H}^s_{b_n(x)\rho}\left(B_{nN}(x,\delta)\right)\\
&\le C_0 \exp\left\{-\varphi^s\left(x, f^N\right)\right\}\cdot \mathcal{H}^s_{b_{n-1}(x)\rho}\left(f^N\left(B_{nN}(x,\delta)\right)\right)\\
&\le C_0 \exp\left\{-\varphi^s\left(x, f^N\right)\right\}\cdot \mathcal{H}^s_{b_{n-1}(x)\rho}\left(B_{(n-1)N}\left(f^N(x),\delta\right)\right)\\
&\le C_0^2 \exp\left\{-\varphi^s\left(x, f^N\right)\right\}\cdot \exp\left\{-\varphi^s\left(f^N(x), f^N\right)\right\}\cdot\\
&\quad \mathcal{H}^s_{b_{n-2}(x)\rho}\left(f^N\left(B_{(n-1)N}\left(f^N(x),\delta\right)\right)\right)\\
&\le \cdots\\
&\le C_0^n \exp\left[-\sum_{i=0}^{n-1}\varphi^s\left(f^{iN}(x), f^N\right)\right]\cdot \mathcal{H}^s_{\rho}\left(B\left(f^{nN}(x),\delta\right)\right)\\
&\le C_0^n C_1\cdot \exp\left[-\sum_{i=0}^{n-1}\varphi^s\left(f^{iN}(x), f^N\right)\right],
\end{aligned}$$

where $C_1 = \sup_{x\in M}\mathcal{H}^s_\rho(B(x,\delta))$. It yields that

$$\begin{aligned}
\mathcal{H}^s_{\beta_n}\left(A_n\right) &\le \sum_{x_j\in E}\mathcal{H}^s_{\beta_n}\left(B_{nN}\left(x_j,\delta\right)\right)\\
&\le \sum_{x_j\in E} C_0^n C_1\cdot \exp\left[-\sum_{i=0}^{n-1}\varphi^s\left(f^{iN}\left(x_j\right), f^N\right)\right]\\
&\le C_0^n C_1\cdot \sum_{x_j\in E}\exp\left[-\sum_{i=0}^{nN-1}\varphi^s\left(f^{i}\left(x_j\right), f\right)\right]\\
&\le C_0^n C_1\cdot \exp\left[nN\left(h_\mu(f)+\varepsilon\right)\right]\cdot \exp\left\{nN\left[-\int\varphi^s(x,f)\,d\mu+\varepsilon\right]\right\}\\
&= C_0^n C_1\cdot \exp\left\{nN\left[h_\mu(f)-\int\varphi^s(x,f)\,d\mu+2\varepsilon\right]\right\}\\
&= C_0^n C_1\cdot e^{-nN\varepsilon}\\
&= \left(C_0 e^{-N\varepsilon}\right)^n\cdot C_1.
\end{aligned}$$

Since N satisfies $C_0 e^{-N\varepsilon} < 1$, we have that

$$\lim_{n\to\infty}\mathcal{H}^s_{\beta_n}\left(A_n\right) = 0.$$

The definition of β_n and (3.9) tell us that $\beta_n \to 0$ if $n \to \infty$. Combining with Lemma 2.1, we obtain $\dim_H \mu \le s$. Thus $\dim_H \mu \le t_1$. □

Now we are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. For any $n > 1$, the measure μ is f-invariant ergodic, but it may be not ergodic for f^n although μ is still f^n-invariant. It is well known that there exists an f^n-ergodic probability measure ν such that

$$\mu = \frac{1}{m}\left[\nu + f_*(\nu)+\cdots+f_*^{m-1}(\nu)\right],$$

where $m\in\mathbb{N}\backslash\{0\}$ divides n (see Theorem 2.1 in [18] for proofs). By Proposition 2.7 and Lemma 3.5 in [22], we have $\dim_H\nu = \dim_H\mu$. Since ν is expanding and ergodic for f^n, then we apply Lemma 3.3 to get

$$\dim_H \mu = \dim_H \nu \le t_n,$$

where t_n is the unique root of the equation $P_\mu\left(f^n, -\varphi^s\left(\cdot, f^n\right)\right) = 0$. By the subadditivity of $\{-\varphi^s(\cdot, f^n)\}_{n\ge 1}$, we obtain

$$\frac{1}{2^{k+1}}\left[h_\mu(f^{2^{k+1}}) - \int \varphi^s(x, f^{2^{k+1}})\, d\mu\right] \le \frac{1}{2^k}\left[h_\mu(f^{2^k}) - \int \varphi^s(x, f^{2^k})\, d\mu\right].$$

Hence

$$\frac{1}{2^{k+1}} P_\mu\left(f^{2^{k+1}}, -\varphi^s(\cdot, f^{2^{k+1}})\right) \le \frac{1}{2^k} P_\mu\left(f^{2^k}, -\varphi^s(\cdot, f^{2^k})\right).$$

Then we have $t_{2^{k+1}} \le t_{2^k}$ and hence the limit $t^* = \lim_{k\to\infty} t_{2^k}$ exists. Therefore $\dim_H \mu \le t^*$. Combining with

$$\begin{aligned} P_\mu\left(f, \left\{-\varphi^t\left(\cdot, f^n\right)\right\}\right) &= h_\mu(f) - \lim_{n\to\infty} \int \frac{1}{n}\varphi^t\left(x, f^n\right)\, d\mu \\ &= h_\mu(f) - \lim_{k\to\infty} \frac{1}{2^k} \int \varphi^t(x, f^{2^k})\, d\mu \\ &= \lim_{k\to\infty} \frac{1}{2^k} P_\mu\left(f^{2^k}, -\varphi^t(\cdot, f^{2^k})\right), \end{aligned}$$

we conclude $P_\mu\left(f, \left\{-\varphi^{t^*}\left(\cdot, f^n\right)\right\}\right) = 0$. The proof of Theorem 1.2 is completed. $\square$

Acknowledgements

The authors are grateful for Professor Yongluo Cao's suggestions which lead to a great improvement of the manuscript.

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