Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1544–1557

Website:http://www.jaac-online.com
DOI:10.11948/20210328

DARBOUX TRANSFORMATION, EXACT SOLUTIONS OF THE VARIABLE COEFFICIENT NONLOCAL FOKAS-LENELLS EQUATION*

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Abstract In this paper, a (1+1)-dimensional integrable variable coefficient nonlocal Fokas-Lenells (NFL) equation is studied. On the basis of the Lax pair, the Darboux transformation of the variable coefficient NFL equation is constructed at the first time and an explicit form of the N-fold Darboux transformation is given. The exact solutions of the variable coefficient NFL equation are derived using the zero seed solution and the nonzero seed solution according to the Darboux transformation. Subsequently, one-soliton solution, two-soliton solution, and kink solution with periodic waves are obtained by choosing the proper parameters and plotting the corresponding figures. With the help of figures, the behaviors of the obtained solutions are revealed and it is possible to find that the interaction between solitons is elastic no matter the coefficient function is constant or arbitrary variable. In addition, this paper also indicates that the exact solutions of the variable coefficient NFL equation are more general than its constant coefficient form.

Keywords Lax pair, soliton solutions, nonlocal equation, Darboux transformation method.

MSC(2010) 35Q51, 35Q55.

1. Introduction

Nonlinear evolution equations have received much attention because of their indispensable role in the study of nonlinear phenomena in nature, especially in the fields of nonlinear optics, plasma physics and fluid mechanics [8,11,12,17]. However, these nonlinear evolution equations are local and the development of their solutions are only depend on time and space [2,22,28].

Ablowitz and Musslimani gave the nonlocal nonlinear Schrödinger equation [1],

$$iq_t\left(x,t\right)+q_{xx}\left(x,t\right)+2q^2\left(x,t\right)q^*\left(-x,t\right)=0, \tag{1.1}$$

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*The authors were supported by National Natural Science Foundation of China(11505090), Research Award Foundation for Outstanding Young Scientists of Shandong Province(BS2015SF009), The doctoral foundation of Liaocheng University under Grant(318051413), Liaocheng University level science and technology research fund(318012018).

which was derived from the AKNS system in 2013. The nonlocal equations have attracted the attention of more and more scientists and become one of the important topics of research. The nonlocal nonlinear term of Eq. (1.1) depends not only on the value of q at x but also at $-x$. After studying , it is found that the nonlocal nonlinear Schrödinger equation is integrable in the sense of Lax pair. It also has an infinite number of conservation laws and the exact solutions are obtained using the inverse scattering method, while some properties exist that are not present in the classical Schrödinger equation. These motivated scholars to study other nonlocal equations and their exact solutions as well as related properties with some remarkable results in the recent past. In [29], the authors proposed the localized wave solutions of the nonlocal integrable Lakshmanan-Porsezian-Daniel equation and analyzed their wave structures and dynamic properties. In [5], the authors studied a (2+1)-dimensional nonautonomous coupled nonlinear Schrödinger equation, which includes partial nonlocal nonlinearity under the linear and harmonic potentials, and derived diversified exact solutions using the Darboux transformation method and the bilinear method. In [13], the authors derived the soliton and rational solutions of the nonlocal Mel'nikov equation on the periodic background. Furthermore, the Darboux transformation method is extended to find one-soliton solutions, breather solutions, rogue-wave solutions and other types of interaction solutions for nonlocal equations [3, 30–32]. The nonlocal equations with variable coefficients have attracted little attention, but it is a very valuable direction for research [4, 26]. As the generalization of the coefficient aspect of the constant coefficient equations, the variable coefficient equations can be used not only to describe more detailed natural phenomena but also contain richer physical meanings [14, 18, 21, 27]. The study of nonlocal equations with variable coefficients has contributed to the development of integrable system.

In this paper, we study a nonlocal Fokas-Lenells equation with time-partial variable coefficient,

$$q(x,t)_{xt} + \delta(t)q(x,t) - 2i\delta(t)q(x,t)_x q(x,t)\, q(-x,-t) = 0, \tag{1.2}$$

where $\delta(t)$ is an any function with respect to t and the Eq. (1.2) is the constant coefficient NFL equation when $\delta(t) = 1$. The nonlocal equations generally include four types of inverse time $(x,-t)$, inverse space $(-x,t)$, inverse time-space $(-x,-t)$ and PT symmetry, which are used to describe the relationship between interconnected/entangled events at different times and places. The well-known FL equation was proposed by Fokas to describe the propagation of femtosecond pulse in single mode optical silica fiber [7,9,23]. Furthermore, the corresponding nonlocal FL model can be used to predict the characteristics at point $(-x,-t)$ from the pulse data that are measured at point (x,t), and combine the experimental data to study the propagation features of laser pulse in different materials. Several scholars already made some studies on the NFL equation. In the literature [33], the authors derived two-bright soliton solution, kink and soliton mixed type solution and some other mixed type solutions for the NFL equation, but did not give the specific expressions for the exact solutions that were obtained using the 2-fold Darboux transformation. In the literature [24], the authors analyzed the breather positions as well as the rogue-wave solution of the NFL equation, however, the dynamic properties of the breather position solutions were not further studied and the trajectories of the breather positions were not found. In the literature [15], the authors derived the N-soliton solution of the NFL equation based on the scattering relationship using the Riemann-Hilbert method, provided a new method for solving the nonlocal integrable systems, but did not investigate the mathematical structure and physical properties of the nonlocal systems in depth. We will use the Darboux transformation method to study the variable coefficient nonlocal Fokas-Lenells equation, and the novelty of this paper shows two aspects. First, the variable coefficient NFL equation is presented for the first time. Second, we give the N-fold Darboux transformation and some exact analytical solutions of the variable coefficient NFL equation, including the one-soliton solution, the two-soliton interaction solution, the kink solution with periodic waves, and analyze the behavior of these solutions with the help of figures. The results of the variable coefficient NFL equation contribute to the development of femtosecond pulse spectroscopy and ultrafast optical communication.

The outline of this paper is as follows: In Section 2, the variable coefficient NFL equation is derived using Lax pair. In Section 3, the Darboux transformation of variable coefficient NFL is constructed and an explicit form of the multi-parameters N-fold Darboux transformation is given. In Section 4, based on the Darboux transformation in the previous section, the exact solutions of the variable coefficient NFL equation are determined using the zero seed solution and the nonzero seed solution. These exact solutions contain the one-soliton solution, the two-soliton solution, and the kink solution with periodic waves solution. The conclusions of this paper can be found in Section 5.

2. Variable coefficient NFL equation

First, we construct the variable coefficient NFL equation based on the Lax pair of the traditional FL equation [25]. The Lax pair of Eq. (1.2) is the following

$$\phi_x = U\phi, \qquad \phi_t = V\phi, \tag{2.1}$$

where $\phi = (\phi_1(x,t), \phi_2(x,t))^T$ and U, V are the two matrices determined by $q(x,t), r(x,t)$ and the spectral parameter λ,

$$U = \begin{pmatrix} \frac{i}{2}\lambda^2 & \lambda q_x \\ \lambda r_x & -\frac{i}{2}\lambda^2 \end{pmatrix}, V = \begin{pmatrix} \delta(t)\left(\frac{i}{2\lambda^2} + iqr\right) & -\delta(t)\frac{i}{\lambda}q \\ \delta(t)\frac{i}{\lambda}r & -\delta(t)\left(\frac{i}{2\lambda^2} + iqr\right) \end{pmatrix}, \tag{2.2}$$

with $\delta(t)$ is an arbitrary function about t.

By the compatibility condition $U_t - V_x + [U, V] = 0$ of Eq. (1.2) to get the coupled variable coefficient FL equation are

$$\begin{aligned} &q(x,t)_{xt} + \delta(t)\left(q(x,t) - 2iq(x,t)_x q(x,t) r(x,t)\right) = 0, \\ &r(x,t)_{xt} + \delta(t)\left(r(x,t) + 2ir(x,t)_x r(x,t) q(x,t)\right) = 0, \end{aligned} \tag{2.3}$$

and in order to obtain the Eq. (1.2), we assume that a symmetry reduction is

$$r(x,t) = q(-x,-t), \tag{2.4}$$

using this reduction we are able to get

$$\begin{aligned} &q(x,t)_{xt} + \delta(t)\left(q(x,t) - 2iq(x,t)_x q(x,t) q(-x,-t)\right) = 0, \\ &q(x,t)_{xt} + \delta(-t)\left(q(x,t) - 2iq(x,t)_x q(x,t) q(-x,-t)\right) = 0. \end{aligned} \tag{2.5}$$

By observing Eqs. (2.5) we can easily find that $\delta(t)$ is an even function is a necessary condition for the above two equations to be equal, i.e. $\delta(t) = \delta(-t)$. Under this condition, we are able to derive the variable coefficient NFL equation on the basis of the Eq. (2.5) and it is integrable in the sense of Lax pair.

3. Darboux transformation of variable coefficient NFL equation

In soliton theory, the Darboux transformation is one of the more effective methods to find exact solutions of partial differential equations (PDEs) [6, 10, 16]. It can be used to construct the soliton solutions of local PDEs as well as nonlocal PDEs. In this section, we are going to construct the Darboux transformation with variable coefficients NFL equation, and the procedures are analogous to those of the construction for FL equation. The first step, we take the gauge transformation which is

$$\phi^{[1]} = T^{[1]}\phi. \tag{3.1}$$

Subsequently, Eqs. (2.1) are transformed into

$$\phi^{[1]}_x = U^{[1]}\phi^{[1]}, \qquad \phi^{[1]}_t = V^{[1]}\phi^{[1]}, \tag{3.2}$$

with

$$\begin{aligned} U^{[1]} &= \left(T^{[1]}_x + T^{[1]}U\right)\left(T^{[1]}\right)^{-1}, \\ V^{[1]} &= \left(T^{[1]}_t + T^{[1]}V\right)\left(T^{[1]}\right)^{-1}. \end{aligned} \tag{3.3}$$

In the following, we construct the matrix $T^{[1]}$ which is used to ensure $U^{[1]}, V^{[1]}$ have the same form with U, V, at the same time mapping the old potentials q, r to the new potentials $q^{[1]}, r^{[1]}$ respectively. We assume

$$T^{[1]} = \begin{pmatrix} \dfrac{1}{\lambda} + \lambda b_{11}(x,t) & b_{12}(x,t) \\ b_{21}(x,t) & \dfrac{1}{\lambda} + \lambda b_{22}(x,t) \end{pmatrix}, \tag{3.4}$$

with $b_{ij}, (i,j = 1,2)$ are arbitrary functions with respect to x, t.

By substituting the above matrix $T^{[1]}$ into Eq. (3.3) and balancing the order of λ, the relationship between q, r and $q^{[1]}, r^{[1]}$ can be derived as

$$\begin{aligned} q^{[1]}(x,t) &= b_{12}(x,t) + q(x,t), \\ r^{[1]}(x,t) &= b_{21}(x,t) + r(x,t), \end{aligned} \tag{3.5}$$

and the symmetry reduction (2.4) is applied to Eqs. (3.5) to obtain a new constraint as

$$b_{12}(x,t) = b_{21}(-x,-t). \tag{3.6}$$

Here, we assume that there are two basic solutions to the spectral problem (2.1) are $f(\lambda_j) = (f_1(\lambda_j), f_2(\lambda_j))^T, g(\lambda_j) = (g_1(\lambda_j), g_2(\lambda_j))^T$ in order to determine the specific expression of $b_{12}(x,t)$. Based on the gauge transformation, it exists arbitrary numbers $\gamma_j, (j=1,2)$ that make

$$\frac{1}{\lambda_j}+\lambda_j b_{11}(x,t)+\alpha_j b_{12}(x,t)=0, \left(\frac{1}{\lambda_j}+\lambda_j b_{22}(x,t)\right)\alpha_j+b_{21}(x,t)=0, \quad (3.7)$$

where

$$\alpha_j=\frac{\gamma_j g_2(\lambda_j)+f_2(\lambda_j)}{\gamma_j g_1(\lambda_j)+f_1(\lambda_j)}, (j=1,2). \quad (3.8)$$

Let's choose the appropriate values of γ_j, λ_j $(j=1,2,\lambda_1\neq\lambda_2)$ to make the determinant for the coefficients of Eqs. (3.7) not equal to 0. Then solve the values of $b_{ij},(i,j=1,2)$ and substitute them into the matrix $T^{[1]}$. The Eq. (3.4) can be rewritten as

$$T^{[1]}=\begin{pmatrix}\frac{1}{\lambda} & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix}-\begin{pmatrix}\frac{\lambda(\alpha_1\lambda_1-\alpha_2\lambda_2)}{\lambda_1\lambda_2(\alpha_1\lambda_2-\alpha_2\lambda_1)} & -\frac{\lambda_1^2-\lambda_2^2}{\lambda_1\lambda_2(\alpha_1\lambda_2-\alpha_2\lambda_1)}\\ \frac{\alpha_1\alpha_2(\lambda_1^2-\lambda_2^2)}{\lambda_1\lambda_2(\alpha_1\lambda_1-\alpha_2\lambda_2)} & \frac{\lambda(\alpha_1\lambda_2-\alpha_2\lambda_1)}{\lambda_1\lambda_2(\alpha_1\lambda_1-\alpha_2\lambda_2)}\end{pmatrix}, \quad (3.9)$$

and $\alpha_j,(j=1,2)$ satisfy

$$\alpha_{jx}=-i\alpha_j\lambda_j^2+\left(r_x-\alpha_j^2 q_x\right)\lambda_j, \quad (3.10)$$

$$\alpha_{jt}=i\delta(t)(q\alpha_j+r)\lambda_j^{-1}-i\delta(t)\left(2qr\lambda_j^2+1\right)\alpha_j\lambda_j^{-2}. \quad (3.11)$$

Here, we will use above Eq. (3.10) and Eq. (3.11) to proof the following two propositions.

Proposition 3.1. *The matrix $U^{[1]}$ which is determined by $U^{[1]}=\left(T_x^{[1]}+T^{[1]}U\right)\left(T^{[1]}\right)^{-1}$ has the same form as U, i.e. the expression for $U^{[1]}$ is*

$$U^{[1]}=\begin{pmatrix}\frac{i}{2}\lambda^2 & \lambda q_x^{[1]}\\ \lambda r_x^{[1]} & -\frac{i}{2}\lambda^2\end{pmatrix}, \quad (3.12)$$

with Eq. (3.5) maps q,r to $q^{[1]},r^{[1]}$.

Proof. First, we note that $T^{-1}=(\det T)^{-1}T^*$, with

$$(T_x+TU)T^*=\begin{pmatrix}f_{11}(\lambda) & f_{12}(\lambda)\\ f_{21}(\lambda) & f_{22}(\lambda)\end{pmatrix}, \quad (3.13)$$

it is not difficult to verify that $f_{12}(\lambda), f_{21}(\lambda)$ are polynomials of λ^3 and $f_{11}(\lambda), f_{22}(\lambda)$ are polynomials of λ^4. Also, since $\alpha_j,(j=1,2)$ satisfy Eq. (3.10), we can know that $\lambda_j,(j=1,2)$ are the roots of $f_{ij}(\lambda),(i,j=1,2)$ by direct calculation, thus Eq. (3.13) can be rewritten as

$$(T_x+TU)T^*=(\det T)P(\lambda), \quad (3.14)$$

where

$$P(\lambda)=\begin{pmatrix}P_{11}^2\lambda^2+P_{11}^1\lambda+P_{11}^0 & P_{12}^1\lambda+P_{12}^0\\ P_{21}^1\lambda+P_{21}^0 & P_{22}^2\lambda^2+P_{22}^1\lambda+P_{22}^0\end{pmatrix}, \quad (3.15)$$

with $p_{ij}^{l},(i,j=1,2,l=0,1,2)$ are not related to λ, so Eq. (3.14) is equivalent to

$$T_x + TU = P(\lambda)T. \tag{3.16}$$

Substituting the expressions of $T, U, P(\lambda)$ into the above equation and comparing the coefficients of $\lambda^k, (k=0,1,2,3)$, we can obtain the system as

$$\begin{aligned}
&\lambda^3 : \frac{i}{2} - P_{11}^2 = 0, -\frac{i}{2} - P_{22}^2 = 0,\\
&\lambda^2 : P_{11}^1 = 0, P_{22}^1 = 0,\\
&\quad q_x b_{11}(x,t) - \frac{i}{2} b_{12}(x,t) - P_{11}^2 b_{12}(x,t) - P_{12}^1 b_{22}(x,t) = 0,\\
&\quad \frac{i}{2} b_{21}(x,t) + r_x b_{22}(x,t) - P_{21}^1 b_{11}(x,t) - P_{22}^2 b_{21}(x,t) = 0,\\
&\lambda^1 : \frac{\partial b_{11}(x,t)}{\partial x} + \frac{i}{2} + r_x b_{12}(x,t) - P_{11}^0 b_{11}(x,t) - P_{11}^2 - P_{12}^1 b_{21}(x,t) = 0,\\
&\quad \frac{\partial b_{22}(x,t)}{\partial x} + q_x b_{21}(x,t) - \frac{i}{2} - P_{21}^1 b_{12}(x,t) - P_{22}^0 b_{22}(x,t) - P_{22}^2 = 0,\\
&\quad -P_{11}^1 b_{12}(x,t) - P_{12}^0 b_{22}(x,t) = 0, -P_{22}^1 b_{21}(x,t) - P_{21}^0 b_{11}(x,t) = 0,\\
&\lambda^0 : \frac{\partial b_{12}(x,t)}{\partial x} - P_{11}^0 b_{12}(x,t) + q_x - P_{12}^1 = 0,\\
&\quad \frac{\partial b_{21}(x,t)}{\partial x} - P_{22}^0 b_{21}(x,t) + r_x - P_{21}^1 = 0,\\
&\quad -P_{12}^0 b_{21}(x,t) - P_{11}^1 = 0, -P_{21}^0 b_{12}(x,t) - P_{22}^1 = 0,\\
&\lambda^{-1} : P_{11}^0 = 0, P_{12}^0 = 0, P_{21}^0 = 0, P_{22}^0 = 0.
\end{aligned}$$

Solving this system and applying the Eq. (3.5), we get the results as

$$\begin{aligned}
&P_{11}^0 = P_{11}^1 = P_{12}^0 = P_{21}^0 = P_{22}^0 = P_{22}^1 = 0,\\
&P_{11}^2 = \tfrac{1}{2}i, P_{12}^1 = q_x^{[1]}, P_{21}^1 = r_x^{[1]}, P_{22}^2 = -\tfrac{1}{2}i.
\end{aligned} \tag{3.17}$$

It is easy to find that $U^{[1]} = P(\lambda)$, and the proof is complete.

Proposition 3.2. *The matrix $V^{[1]}$ which is determined by $V^{[1]}=(T_x^{[1]}+T^{[1]}V)(T^{[1]})^{-1}$ has the same form as V, i.e. the expression for $V^{[1]}$ is*

$$V^{[1]} = \begin{pmatrix} \delta(t)\left(\dfrac{i}{2\lambda^2} + iq^{[1]}r^{[1]}\right) & -\delta(t)\dfrac{i}{\lambda}q^{[1]} \\ \delta(t)\dfrac{i}{\lambda}r^{[1]} & -\delta(t)\left(\dfrac{i}{2\lambda^2} + iq^{[1]}r^{[1]}\right) \end{pmatrix}, \tag{3.18}$$

with Eq. (3.5) maps q, r to $q^{[1]}, r^{[1]}$.

The details of the proof of this proposition are the same as those of Proposition 3.1, and we do not show them here anymore.

Based on the above facts, we are able to construct the n-fold Darboux transformation of the variable coefficient NFL equation,

$$\phi^{[n]} = T_n(\lambda)\phi = T^{[n]}(\lambda)T^{[n-1]}(\lambda)\cdots T^{[k]}(\lambda)\cdots T^{[1]}(\lambda)\phi, \tag{3.19}$$

where

$$T^{[k]}=\begin{pmatrix}\frac{1}{\lambda} & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix} -\begin{pmatrix}\dfrac{\lambda\left(\alpha_{2k-1}\lambda_{2k-1}-\alpha_{2k}\lambda_{2k}\right)}{\lambda_{2k-1}\lambda_{2k}\left(\alpha_{2k-1}\lambda_{2k}-\alpha_{2k}\lambda_{2k-1}\right)} & -\dfrac{\lambda_{2k-1}^2-\lambda_{2k}^2}{\lambda_{2k-1}\lambda_{2k}(\alpha_{2k-1}\lambda_{2k}-\alpha_{2k}\lambda_{2k-1})}\\ \dfrac{\alpha_{2k-1}\alpha_{2k}(\lambda_{2k-1}^2-\lambda_{2k}^2)}{\lambda_{2k-1}\lambda_{2k}(\alpha_{2k-1}\lambda_{2k-1}-\alpha_{2k}\lambda_{2k})} & \dfrac{\lambda(\alpha_{2k-1}\lambda_{2k}-\alpha_{2k}\lambda_{2k-1})}{\lambda_{2k-1}\lambda_{2k}(\alpha_{2k-1}\lambda_{2k-1}-\alpha_{2k}\lambda_{2k})}\end{pmatrix}, \tag{3.20}$$

with

$$\begin{aligned}\alpha_j&=\frac{\gamma_j g_2^{[k-1]}\left(\lambda_j\right)+f_2^{[k-1]}\left(\lambda_j\right)}{\gamma_j g_1^{[k-1]}\left(\lambda_j\right)+f_1^{[k-1]}\left(\lambda_j\right)},\left(j=2k-1,2k,k=1,2,\cdots,n\right),\\ f^{[k]}\left(\lambda\right)&=\left(f_1^{[k]}\left(\lambda\right),f_2^{[k]}\left(\lambda\right)\right)^T=T^{[k]}\left(\lambda\right)f^{[k-1]}\left(\lambda_1,\lambda_2,\cdots,\lambda_{2k}\right),\\ g^{[k]}\left(\lambda\right)&=\left(g_1^{[k]}\left(\lambda\right),g_2^{[k]}\left(\lambda\right)\right)^T=T^{[k]}\left(\lambda\right)g^{[k-1]}\left(\lambda_1,\lambda_2,\cdots,\lambda_{2k}\right),\end{aligned} \tag{3.21}$$

and the matrix $T^{[k]}$ also has to satisfy a constraint that is

$$b_{12}^{[k]}\left(x,t\right)=b_{21}^{[k]}\left(-x,-t\right). \tag{3.22}$$

By the above analysis, the relationship between the new solution $q^{[n]}$ and the old solution q is obtained to be expressed through

$$q^{[n]}\left(x,t\right)=q\left(x,t\right)+\sum_{k=1}^{n}b_{12}^{[k]}\left(x,t\right). \tag{3.23}$$

The presence of the constraint (3.22) makes the Darboux transformation of the variable coefficient NFL equation have a significant difference from its local form.

4. Exact solutions of variable coefficient NFL equation

In this section, we work on the basis of the Darboux transformation in the previous section to compute the exact solutions of the variable coefficient NFL equation with the zero seed solution $q=r=0$ and the nonzero seed solution $q\left(x,t\right)=ae^{ibx+i\left(2a^2b+1\right)b^{-1}\int\delta(t)dt}, r\left(x,t\right)=ae^{-ibx-i\left(2a^2b+1\right)b^{-1}\int\delta(t)dt}$, $(a,b\in C)$, respectively.

4.1. Exact solutions from zero seed

First, substituting the trivial solution $q=r=0$ into the Eq. (2.1) and solving for it yields

$$\begin{aligned}f\left(x,t;\lambda\right)&=\left(e^{\frac{i}{2}\lambda^2x+\frac{i}{2}\lambda^{-2}\int\delta(t)dt},0\right)^T,\\ g\left(x,t;\lambda\right)&=\left(0,e^{-\frac{i}{2}\lambda^2x-\frac{i}{2}\lambda^{-2}\int\delta(t)dt}\right)^T.\end{aligned} \tag{4.1}$$

Following, substituting the above results into Eq. (3.8) one gets

$$\alpha_j = \gamma_j e^{-i\lambda_j^2 x - i\lambda_j^{-2}\int \delta(t)dt}, j = 1, 2, \tag{4.2}$$

and

$$\begin{aligned} b_{12}(x,t) &= \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1\lambda_2} \cdot \frac{e^{i(a_1+a_2)}}{\gamma_1\lambda_2 e^{i(a_2-a_1)} - \gamma_2\lambda_1 e^{i(a_1-a_2)}}, \\ b_{21}(x,t) &= \frac{\gamma_1\gamma_2\left(\lambda_1^2 - \lambda_2^2\right)}{\lambda_1\lambda_2} \cdot \frac{e^{-i(a_1+a_2)}}{\gamma_1\lambda_1 e^{i(a_2-a_1)} - \gamma_2\lambda_2 e^{i(a_1-a_2)}}, \end{aligned} \tag{4.3}$$

where

$$a_1 = -\frac{1}{2}\left(\lambda_1^2 x + \lambda_1^{-2}\int \delta(t)dt\right), a_2 = -\frac{1}{2}\left(\lambda_2^2 x + \lambda_2^{-2}\int \delta(t)dt\right). \tag{4.4}$$

Here, under the action of constraint $b_{12}(x,t) = b_{21}(-x,-t)$, we can find that the values of $\gamma_j, (j = 1, 2)$ satisfy the following form as

$$\gamma_1^2 = 1, \qquad \gamma_2^2 = 1. \tag{4.5}$$

Thus, we obtain a new solution of the variable coefficient NFL equation as

$$q^{[1]} = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1\lambda_2} \cdot \frac{e^{i(a_1+a_2)}}{\gamma_1\lambda_2 e^{i(a_2-a_1)} - \gamma_2\lambda_1 e^{i(a_1-a_2)}}. \tag{4.6}$$

When $\delta(t) = 1, \gamma_1 = 1, \gamma_2 = 1$ the solution (4.6) becomes

$$q^{[1]} = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1\lambda_2} \cdot \frac{e^{i(a_1+a_2)}}{\lambda_2 e^{i(a_2-a_1)} - \lambda_1 e^{i(a_1-a_2)}}, \tag{4.7}$$

where $a_1 = -\frac{1}{2}\left(\lambda_1^2 x + \lambda_1^{-2} t\right), a_2 = -\frac{1}{2}\left(\lambda_2^2 x + \lambda_2^{-2} t\right)$.

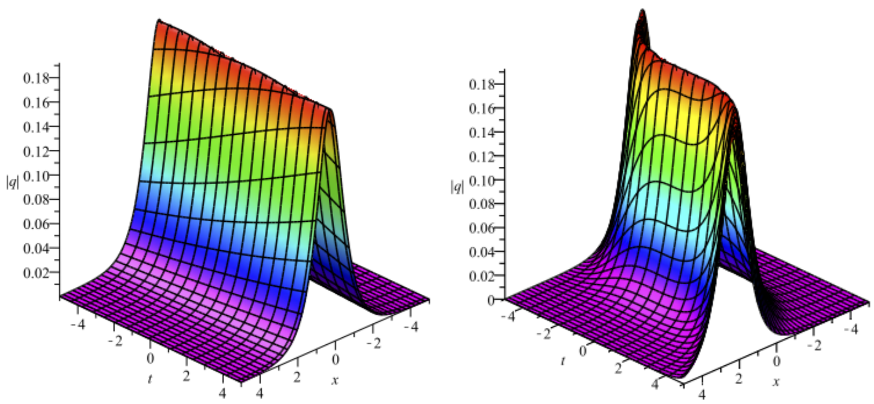
Eq. (4.7) is an exact solution of the constant coefficient NFL equation and we can find this solution in the literature [33]. This also illustrates that the solutions of the variable coefficient NFL equation are more general than its constant coefficient form. To study the properties of the solution (4.6) for the NFL equation with variable coefficient, we plot corresponding figures as follows with $\delta(t)$ taking values of 1 and t^2.

Figures 1 (a)-(b) show the 3-D plots of the absolute values of the wave solution q when $\delta(t)$ is different values. We can find that it is a bright soliton, and this wave propagates steadily along the x-axis. For the coefficient function $\delta(t)$, it only changes the shape of the wave during the propagation and has no effect on the amplitude and velocity [19].

4.2. Exact solutions from nonzero seed

In the following, we compute the exact solution of the variable coefficient NFL equation by the nonzero seed solution

$$q(x,t) = ae^{ibx + i\left(2a^2b+1\right)b^{-1}\int \delta(t)dt}, r(x,t) = ae^{-ibx - i\left(2a^2b+1\right)b^{-1}\int \delta(t)dt}. \tag{4.8}$$



(a) (b)

Figure 1. One-soliton solution (4.6) with parameters that are $\lambda_1 = 0.4 + 2i, \lambda_2 = 0.4 - 2i, \gamma_1 = \gamma_2 = 1$. (a) $\delta(t) = 1$. (b) $\delta(t) = t^2$.

The calculation process is similar to the above. Substituting the trivial solution (4.8) into the Eq. (2.1) and solving it we get

$$
\begin{aligned}
f(x,t;\lambda) &= \begin{pmatrix} -\dfrac{iR+\lambda^2-b}{2ab\lambda}\cdot e^{\dfrac{i\theta - Rx + Rb^{-1}\lambda^{-2}\int \delta(t)dt}{2}} \\ e^{-\dfrac{i\theta + Rx - Rb^{-1}\lambda^{-2}\int \delta(t)dt}{2}} \end{pmatrix}, \\
g(x,t;\lambda) &= \begin{pmatrix} e^{\dfrac{i\theta + Rx - Rb^{-1}\lambda^{-2}\int \delta(t)dt}{2}} \\ -\dfrac{iR+\lambda^2-b}{2ab\lambda}\cdot e^{-\dfrac{i\theta - Rx + Rb^{-1}\lambda^{-2}\int \delta(t)dt}{2}} \end{pmatrix},
\end{aligned} \tag{4.9}
$$

where

$$R = \sqrt{4a^2b^2\lambda^2 + 2b\lambda^2 - \lambda^4 - b^2}, \tag{4.10}$$

$$\theta = bx + 2a^2\int \delta(t)\,dt + b^{-1}\int \delta(t)\,dt. \tag{4.11}$$

We can get $\alpha_j, (j = 1, 2), b_{ij}, (i, j = 1, 2)$ by substituting Eqs. (4.9) into Eq. (3.8) and Eq. (3.9), but their expressions are too wordy, so we do not show them specifically here.

In the same way as the zero seed case, under the constraint $b_{12}(x,t) = b_{21}(-x,-t)$, γ_1 and γ_2 still satisfy $\gamma_1^2 = 1, \gamma_2^2 = 1$.

By applying Eq. (3.23) and Eqs. (4.8), we can obtain the exact solution from the nonzero seed solution of the variable coefficient NFL equation, the specific expression is the following

$$q(x,t) = ae^{i\theta} + \frac{F(x,t)}{G(x,t)}, \tag{4.12}$$

with

$$
\begin{aligned}
F(x,t) &= \left(\lambda_1^2-\lambda_2^2\right)\left(\left(iR_2+\lambda_2^2-b\right)e^{\frac{3}{2}\Delta_2}-2ab\gamma_2\lambda_2 e^{\frac{1}{2}\Delta_2}\right)\\
&\quad\times\left(\left(iR_1+\lambda_1^2-b\right)e^{\frac{1}{2}\Delta_1+i\theta}-2ab\gamma_1\lambda_1 e^{-\frac{1}{2}\Delta_1+i\theta}\right),\\
G(x,t) &= \left(\left(-i\lambda_2^2+ib+R_2\right)\gamma_2\lambda_1^2\lambda_2R_1+\left(-i\lambda_1^2+ib\right)\gamma_2\lambda_1^2\lambda_2R_2\right)e^{\frac{1}{2}(\Delta_1+\Delta_2)}\\
&\quad+\left(4a^2b^2\lambda_2^2-\lambda_1^2\lambda_2^2+\lambda_1^2b+\lambda_2^2b-b^2\right)\gamma_2\lambda_1^2\lambda_2 e^{\frac{1}{2}(\Delta_1+\Delta_2)}\\
&\quad-\left(\left(-i\lambda_2^2+ib+R_2\right)R_1\gamma_1\lambda_1\lambda_2^2+\left(-i\lambda_1^2+ib\right)R_2\gamma_1\lambda_1\lambda_2^2\right)e^{-\frac{1}{2}\Delta_1-\frac{3}{2}\Delta_2}\\
&\quad+\left(4a^2b^2\lambda_1^2-\lambda_1^2\lambda_2^2+b\lambda_1^2+b\lambda_2^2-b^2\right)\gamma_1\lambda_1\lambda_2^2 e^{-\frac{1}{2}\Delta_1-\frac{3}{2}\Delta_2}\\
&\quad+2ab\gamma_1\gamma_2\lambda_1\lambda_2\left(iR_1\lambda_2^2-iR_2\lambda_1^2+b\lambda_1^2-b\lambda_2^2\right)e^{\frac{1}{2}(\Delta_2-\Delta_1)}\\
&\quad+2ab\lambda_1^2\lambda_2^2\left(iR_1-iR_2+\lambda_1^2-\lambda_2^2\right)e^{\frac{1}{2}\Delta_1+\frac{3}{2}\Delta_2},
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
\Delta_1 &= -R_1x+R_1b^{-1}\lambda_1^{-2}\int\delta(t)\,dt,\\
\Delta_2 &= -R_2x+R_2b^{-1}\lambda_2^{-2}\int\delta(t)\,dt,\\
R_1 &= \sqrt{4a^2b^2\lambda_1^2+2b\lambda_1^2-\lambda_1^4-b^2},\\
R_2 &= \sqrt{4a^2b^2\lambda_2^2+2b\lambda_2^2-\lambda_2^4-b^2}.
\end{aligned}
\tag{4.14}
$$

Next, we put some restrictions on the parameters of the solution (4.12) to obtain interesting two-soliton solution and kink solution with periodic waves.

To obtain the two-soliton solution, we let $\lambda_{1R}=\lambda_{2R}=0, \lambda_{1I}\to-\lambda_{2I}$, a is an arbitrary complex number and b is an arbitrary real number and $ab\neq 0$.

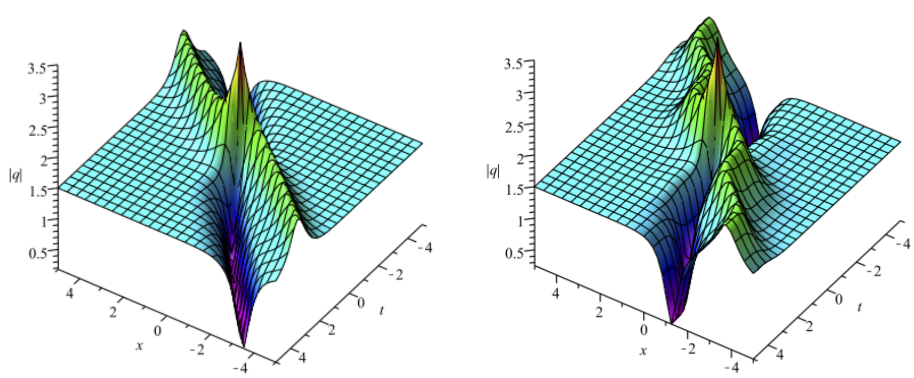


Figure 2. Two-soliton solution with parameters that are $\lambda_1=i, \lambda_2=-1.0001i, a=-1.5i, b=1, \gamma_1=-1, \gamma_2=1$. (a) $\delta(t)=1$. (b) $\delta(t)=\cos^2 t$.

Figure 2 (a)-(b) show the interaction between two soliton waves (a bright soliton and a dark soliton) with different velocities. In addition, the collision is elastic because soliton waves do not affect each other before and after the collision. Obviously, the value of the coefficient function $\delta(t)$ does not affect the amplitude and velocities of the waves during the propagation, but only the shape of the waves [20].

We can also obtain the kink solution with periodic waves by restricting the parameters to $\lambda_{1R} = 0, \lambda_{1I} + \lambda_{2I} = 0$ and a, b are real numbers that are not equal to zero.

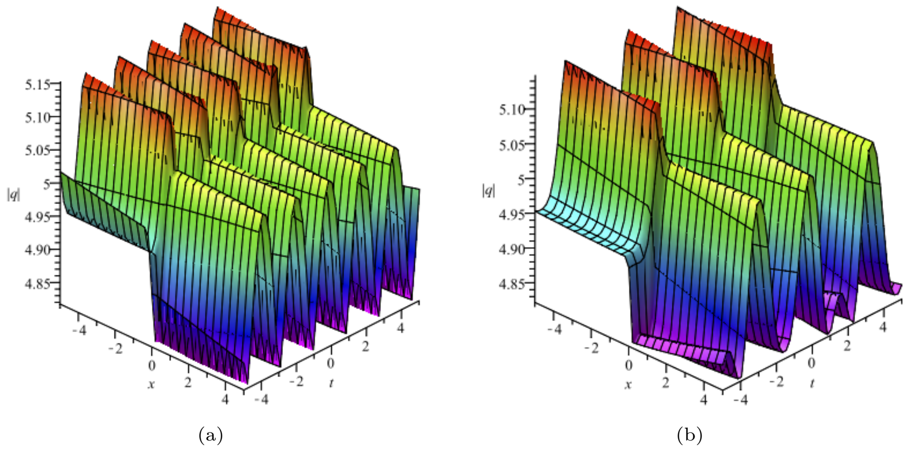


Figure 3. Kink solution with parameters that are $\lambda_1 = 3i, \lambda_2 = 1 - 3i, a = 5, b = 4, \gamma_1 = -1, \gamma_2 = -1$. (a) $\delta(t) = 1$. (b) $\delta(t) = \cos t$.

Figures 3 (a)-(b) demonstrate the interaction of the kink wave with the periodic waves. This solution is composed of two periodic line waves with different background heights. We can observe that the value of the coefficient function $\delta(t)$ can affect the period of the solution in this case. The other discussions are the same as the case of the two-soliton interaction solution, and we do not repeat them again.

Looking at the above figures, it is easy to find that there are huge differences in the exact solutions of the variable coefficient NFL equation for choosing different functions of $\delta(t)$. In other words, there are various choices of $\delta(t)$, for example, exponential functions, trigonometric functions, power functions, etc. Therefore, we can derive more general exact solutions for Eq. (1.2).

5. Conclusions

In this paper, we studied the variable coefficient NFL equation using the Darboux transformation method, and derived the one-soliton solution, the two-soliton interaction solution, and the kink solution with period waves through the zero seed solution and the nonzero seed solution. Subsequently, the behaviors of the solutions were analyzed with the help of figures. The results show that the coefficient function $\delta(t)$ has significant effects on the behaviors of the waves, which can lead to a deeper understanding of physical phenomena in the different scientific fields. The solutions obtained in this paper can contribute to the development of femtosecond pulse spectroscopy and ultrafast optical communication.

There is still a lot of work to be done in the future. First, constructing the nonlocal equation which contains several different variable coefficients, and studying its mathematical structure and physical properties in depth. Second, using other methods to study the new exact solutions of the nonlocal equations. For example, extending the Lie group method to nonlocal equations and constructing group invariant solutions of nonlocal equations. Furthermore, rogue wave solution has gradually developed into one of the hot spots of today's research, using the Darboux transformation method to construct rogue wave solution of variable coefficient nonlocal equations, and then exploring the generating mechanism for deriving the construction of rogue waves, which are valuable for the development of integrable system.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Nos. 11505090), Research Award Foundation for Outstanding Young Scientists of Shandong Province (No. BS2015SF009) and the doctoral foundation of Liaocheng University under Grant No. 318051413, Liaocheng University level science and technology research fund No. 318012018.

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