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ON A SUPERLINEAR SECOND ORDER ELLIPTIC PROBLEM AT RESONANCE*

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Abstract We show the existence of solutions of the superlinear problem

$$\begin{aligned} &-\Delta u=\lambda_1 u+f(u^+)+h(x), && \text{in } \Omega,\\ &u=0, && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega\subset\mathbb{R}^N$ be a bounded domain whose boundary is a $C^{2,\alpha}$ manifold, f satisfies some superlinear growth conditions and h satisfies a one-sided Landesman-Lazer condition. A priori bounds for the solutions of the equation is obtained by using Hardy-Sobolev type inequalities. Existence of solutions is then obtained by using topological degree arguments.

Keywords Elliptic equations, superlinear nonlinearity, a priori bounds, topological degree.

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1. Introduction

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain whose boundary is a $C^{2,\alpha}$ manifold. Denote by λ_1 the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Existence of solutions for semilinear elliptic Dirichlet problems

$$\begin{aligned} &-\Delta u=g(x,u), && \text{in } \Omega,\\ &u=0, && \text{on } \partial\Omega \end{aligned}$$

with distinct behaviours of $\frac{g(x,s)}{s}$ as $s\to\infty$ has been difficult to establish in the case when

(i) $g(x,0)\neq 0$;

(ii) there is resonance in one direction;

(iii) the problem is superlinear in the other.

This work is dedicated to present results for a class of nonlinear elliptic problem with superlinear asymmetric nonlinearities and resonant in the first eigenvalue

$$\begin{aligned} &-\Delta u=\lambda_1 u+f(u^+)+h(x), && \text{in } \Omega,\\ &u=0, && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

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where $u^+ = \max\{u, 0\}$, f and h satisfy the following

(H1) There are nonnegative constants A, B, A_1, B_1 and p with $B > A$, $1 \leq p < \frac{N+1}{N-1}$ for $N \geq 3$ such that

$$As^p - A_1 \leq f(s) \leq Bs^p + B_1 \quad \forall\ s \in [0, \infty).$$

(H2) There exists $s_0 > 0$, such that $f \in C^1[0, s_0]$ and $f \in C[0, \infty)$,

$$\begin{aligned} &f(s) > 0 \quad \text{for} \quad s > 0, \\ &\lim_{s \to 0^+} f'(s) = 0, \\ &f'(s) > 0 \quad \text{for } s \in (0, s_0]. \end{aligned}$$

(H3) $h \in L^r$ with some $r > N$, and

$$\int_\Omega h\phi_1 dx < 0,$$

where ϕ_1 is the positive eigenfunction associated to λ_1 and normalized to have L^2-norm equal to 1.

The motivation for this work is the paper M. Cuesta, D. G. De Figueiredo and P. N. Srikanth [4], in which the authors showed the following resonant Dirichlet problems

$$\begin{aligned} &-\Delta u = \lambda_1 u + (u^+)^p + h(x), && \text{in } \Omega, \\ &u = 0, && \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

has at least one solution in $W^{2,r}(\Omega) \cap H_0^1(\Omega)$ under the assumptions $h \in L^r(\Omega)$, $1 < p < \frac{N+1}{N-1}$ and

$$\int_\Omega h(x)\phi_1(x)dx < 0. \tag{1.3}$$

The proof of the main result in [4] uses the technique introduced in [2]. The method consists in getting *a priori* bounds, using Hardy-Sobolev type inequalities, with topological degree arguments. Similar problems, under Dirichlet and Neumann boundary condition, can be found in D. Arcoya and S. Villegas [1], M. Cuesta and C. De Coster [3], F. M. Ferreira, F. O. de Paiva [6], R. Kannan and R. Ortega [7, 8], S. Kyritsi and N. S. Papageorgiou [9], D. Motreanu, V. Motreanu, N. S. Papageorgiou [10], K. Perera [12], N. S. Papageorgiou and V. D. Radulescu [11], F. O. de Paiva and A. E. Presoto [5], L. Recova and A. Rumbos [13], J. R. Ward [14].

Denote the natural norm of $L^r(\Omega)$ by $||\cdot||_r$, that is,

$$||u||_r = (\int_\Omega |u|^r dx)^{1/r}.$$

Denote the natural norm of $H_0^1(\Omega)$ by $||\cdot||$, that is,

$$||u|| = (\int_\Omega |\nabla u|^2 dx)^{1/2}.$$

The space X is defined as $X = \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ which is a Banach space with norm

$$||u||_X = \max_{x\in\bar{\Omega}}|u(x)| + \max_{x\in\bar{\Omega}}|\nabla u(x)|.$$

The main results of this paper is the following

Theorem 1.1. *Assume that* $1 < p < \frac{N+1}{N-1}$. *Under assumptions (H1)-(H3) the Dirichlet problem* (1.1) *has a weak solution* $u \in W^{2,r}(\Omega) \cap H_0^1(\Omega)$.

Since we will use topological arguments to prove Theorem 1.1, we shall need *a priori* bounds on the solutions of (1.1). This is the content of the next result. Notice that from regularity theory all weak solutions of (1.1) belong to $W^{2,r}(\Omega)$, and recall that $W^{2,r}(\Omega) \subset C^1(\bar{\Omega})$ because $r > N$.

Theorem 1.2. *Assume that* $1 < p < \frac{N+1}{N-1}$. *Let (H1)-(H3) hold. Let* $u \in H_0^1(\Omega)$ *be a solution of problem* (1.1). *Then there exists a constant* $C > 0$ *such that*

$$||u||_X \leq C. \tag{1.4}$$

Remark 1.1. For the special nonlinearity $(u^+)^p$, M. Cuesta, D. G. De Figueiredo and P. N. Srikanth [4, Theorem 1.2] obtained *a priori* estimates of form

$$||u||_X \leq \rho(||h||_r) \tag{1.5}$$

for all solutions of (1.2), where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function, depending only on p and Ω, such that

$$\rho(0) = 0.$$

Our *a priori* estimates (1.4) for (1.1) is weaker than (1.5) and is not enough to guarantee that all solution are non-degenerate solution of Morse index equal to 1. To overcome this difficulty, we need to introduce hypothesis (H2) in order to prove the following

For any $\epsilon \in (0, s_0)$, *there exists* $\delta > 0$ *such that any solution* u *of* (1.1) *with* $||h||_r < \delta$ *satisfies*

$$||u^+||_X < \epsilon.$$

See Proposition 3.1 below.

Remark 1.2. It is worth remarking that, if (H2) holds, the necessary condition for the existence of solutions of (1.1) is (H3). In fact, if u is a solution of (1.1), then

$$\int_\Omega (-\Delta u - \lambda_1 u)\phi_1 dx = \int_\Omega f(u^+)\phi_1 dx + \int_\Omega h\phi_1 dx.$$

2. Proof of Theorem 1.2

Let us first introduce the following lemma based on the Hardy-Sobolev inequality (c.f. for instance [2, 4]).

Lemma 2.1 ([4]). *Let $1 < p < \frac{N+1}{N-1}$. Then there exists a constant $C = C(p, \Omega)$ such that, for all $u, v \in H_0^1(\Omega)$ with $|u| \leq v$ a.e., it holds*

$$\int_\Omega |u|^p v dx \leq C\big(\int_\Omega |u|^p \phi_1 dx\big)^\alpha \big(\int_\Omega |\nabla v|^2 dx\big)^{\delta/2}, \tag{2.1}$$

where

$$\alpha = 1 - \frac{N}{2 + 2N - (N-2)p} \in (0,1), \tag{2.2}$$

$$\delta = 1 + \frac{Np}{2 + 2N - (N-2)p} \in (1,2). \tag{2.3}$$

Throughout the rest of this section, we use the same letter C to denote distinct constants. In addition, we remark that all of them are independent of u.

Proof of Theorem 1.2. Let $u \in H_0^1(\Omega)$ be a weak solution of (1.1). Let us write $u = t\phi_1 + u_1$ with $\int_\Omega u_1 \phi_1 dx = 0$. By multiplying (1.1) by ϕ_1 we find

$$\int_\Omega f(u^+)\phi_1 dx = -\int_\Omega h\phi_1 dx \leq C||h||_r. \tag{2.4}$$

This together with the first inequality in (H1) imply that

$$\int_\Omega |u^+|^p \phi_1 dx \leq C||h||_r + \frac{A_1}{A}\int_\Omega \phi_1 dx, \tag{2.5}$$

and therefore

$$\begin{aligned} t &= \int_\Omega u^+\phi_1 dx - \int_\Omega u^-\phi_1 dx \\ &\leq C\big(\int_\Omega |u^+|^p \phi_1 dx\big)^{1/p} \leq \big(C||h||_r + \frac{A_1}{A}||\phi_1||_1\big)^{1/p}. \end{aligned} \tag{2.6}$$

We break the proof into two parts, according to $t < 0$ or $t \geq 0$.

Case 1. $t \geq 0$.

From the previous inequality we obtain a bound on $|t|$. In order to get an estimate on u_1 we multiply (1.1) by u_1 to obtain

$$\int_\Omega |\nabla u_1|^2 dx - \lambda_1 \int_\Omega u_1^2 dx = \int_\Omega f(u^+)u_1 dx + \int_\Omega h u_1 dx, \tag{2.7}$$

and using that $\int_\Omega u_1\phi_1 dx = 0$ and the variational characterization of the second eigenvalue λ_2 of $(-\Delta, H_0^1(\Omega))$ we have

$$\begin{aligned} \big(1 - \frac{\lambda_1}{\lambda_2}\big)||u_1||^2 &\leq \int_\Omega f(u^+)u_1 dx + \int_\Omega h u_1 dx \\ &\leq \int_\Omega (B|u^+|^p + B_1)u_1 dx + \int_\Omega h u_1 dx \\ &\leq \int_\Omega B|u^+|^p u_1 dx + \int_\Omega (h + B_1)u_1 dx \\ &\leq C||h + B_1||_r||u_1|| + |\int_\Omega B(u^+)^p u_1 dx|. \end{aligned} \tag{2.8}$$

Next we use the fact that $u_1^+ \le u^+$, and that $u^+ \le t\phi_1$ on $\{u_1 \le 0\}$ to obtain

$$\begin{aligned} |\int_\Omega (u^+)^p u_1 dx| &\le \int_\Omega (u^+)^p u_1^+ dx + \int_\Omega (u^+)^p u_1^- dx \\ &\le \int_\Omega (u^+)^{p+1} dx + t^p \int_\Omega (\phi_1)^p u_1^- dx. \end{aligned} \tag{2.9}$$

By a simple calculation,

$$t^p \int_\Omega (\phi_1)^p u_1^- dx \le C\Big(\|h\|_r + \frac{A_1}{A}\|\phi_1\|_1\Big)\|u_1\|.$$

To estimate the integral in the right side of (2.9) we use (2.4) and Lemma 2.1 with $u = v = u^+$. Hence

$$\begin{aligned} |\int_\Omega (u^+)^{p+1} dx| \le & C\Big(\int_\Omega |u^+|^p \phi_1 dx\Big)^\alpha \Big(\int_\Omega |\nabla u^+|^2 dx\Big)^{\delta/2} \\ \le & C\Big(\|h\|_r^{\alpha+\delta/p} + (\frac{A_1}{A})^{\delta/p}\|h\|_r^\alpha \|\phi_1\|_1^{\delta/p} + \|h\|_r^\alpha \|u_1\|^\delta \\ & + (\frac{A_1}{A})^\alpha \|h\|_r^{\delta/p}\|\phi_1\|_1^\alpha + (\frac{A_1}{A})^{\alpha+\delta/p}\|\phi_1\|_1^{\alpha+\delta/p} + (\frac{A_1}{A})^\alpha \|\phi_1\|_1^\alpha \|u_1\|^\delta\Big). \end{aligned} \tag{2.10}$$

Replacing (2.9) and (2.10) in (2.8) we obtain

$$\begin{aligned} ||u_1||^2 \le & \int_\Omega f(u^+)u_1 dx + \int_\Omega h u_1 dx \\ \le & C||h + B_1||_r ||u_1|| + |\int_\Omega B(u^+)^p u_1 dx| \\ \le & C\|h + B_1\|_r \|u_1\| + BC\Big(\|h\|_r^{\alpha+\delta/p} + (\frac{A_1}{A})^{\delta/p}\|h\|_r^\alpha \|\phi_1\|_1^{\delta/p} + \|h\|_r^\alpha \|u_1\|^\delta \\ & + (\frac{A_1}{A})^\alpha \|h\|_r^{\delta/p}\|\phi_1\|_1^\alpha + (\frac{A_1}{A})^{\alpha+\delta/p}\|\phi_1\|_1^{\alpha+\delta/p} + (\frac{A_1}{A})^\alpha \|\phi_1\|_1^\alpha \|u_1\|^\delta\Big) \\ & + BC(\|h\|_r + \frac{A_1}{A}\|\phi_1\|_1)\|u_1\|. \end{aligned} \tag{2.11}$$

By Young's inequality we deduce that

$$\|u_1\|^2 \le C\Big(\|h + B_1\|_r^2 + \|h\|_r^{\alpha+\delta/p} + \|h\|_r^\alpha + \|h\|_r^{\frac{2\alpha}{2-\delta}} + \|h\|_r^{\delta/p} + \|h\|_r^2 + 1\Big).$$

This imply that there exists a constant $C > 0$ such that

$$||u|| \le C. \tag{2.12}$$

Case 2. $t < 0$**.**

First, we find an a-*priori* bound of $||u_1||_X$.

Similarly to the previous case we have also (2.8). But in this case $|\int_\Omega (u^+)^p u_1 dx|$ can be estimated directly by using Lemma 2.1. Indeed, notice that we have $u^+ \le u_1$, since $t < 0$. Hence, by Lemma 2.1 and (2.5), we have

$$|\int_\Omega (u^+)^p u_1 dx| \le C(||h||_r + \frac{A_1}{A}\|\phi_1\|_1)^\alpha ||u_1||^\delta, \tag{2.13}$$

and replacing this estimate in (2.8) we find

$$\begin{aligned} ||u_1||^2 &\le C\Big(\int_\Omega f(u^+)u_1dx + \int_\Omega hu_1dx\Big) \\ &\le C\Big(\int_\Omega |f(u^+)u_1|dx + \int_\Omega hu_1dx\Big) \\ &\le C||h+B_1||_r||u_1|| + |\int_\Omega B(u^+)^pu_1dx| \\ &\le C\Big(||h+B_1||_r||u_1|| + ||h||_r^\alpha\|u_1\|^\delta + (\frac{A_1}{A})^\alpha\|\phi_1\|_1^\alpha||u_1||^\delta\Big). \end{aligned} \tag{2.14}$$

By Young's inequality we deduce that

$$\begin{aligned} ||u_1||^2 \le& C(||h+B_1||_r^2 + \frac{1}{2}\|u_1\|^2 + \frac{2-\delta}{2}||h||_r^{\frac{2\alpha}{2-\delta}} + \frac{2-\delta}{2}\|\phi_1\|_1^{\frac{2\alpha}{2-\delta}} + \delta\|u_1\|^2) \\ \le& C(||h+B_1||_r^2 + ||h||_r^{\frac{2\alpha}{2-\delta}} + 1). \end{aligned}$$

Then

$$||u_1|| \le C(||h+B_1||_r + ||h||_r^{\frac{\alpha}{2-\delta}} + 1). \tag{2.15}$$

In order to get an estimate for $||u_1||_X$ from the estimate of $\|u_1\|$ we now use the fact that u_1 solves the problem

$$\begin{aligned} -\Delta u_1 = \lambda_1 u_1 + f(u^+) + h, &\qquad \text{in } \Omega, \\ u_1 = 0, &\qquad \text{on } \partial\Omega. \end{aligned} \tag{2.16}$$

Since $u^+ \le u_1$ and $f(u^+) \le B(u^+)^p + B_1$, (2.14) and (2.16) and a bootstrap argument yield that

$$||u_1||_X \le C, \tag{2.17}$$

where $C > 0$ is a constant.

By Hopf's Maximum Principle the first eigenfunction of $(-\Delta, H_0^1(\Omega))$, $\phi_1 > 0$ lies in the interior of the cone of positive functions in the space X.

Assume on the contrary that there exist $\{u_n\}$ with

$$u_n = t_n\phi_1 + u_{1n} \tag{2.18}$$

and

$$t_n \to -\infty. \tag{2.19}$$

Then it follows from (2.19), (2.18) and (2.17) that

$$u_n(x) \to -\infty \qquad \text{for } x \in \mathcal{D},$$

where $\mathcal{D}$ be any compact subset of Ω. We have from the fact

$$\int_\Omega [f(u_n^+(x)) + h(x)]\phi_1 dx = 0 \tag{2.20}$$

that

$$\int_\Omega h\phi_1 dx = 0. \tag{2.21}$$

However, this contradicts (H3). □

3. Proof of Theorem 1.1

In this proof we use the propositions below. For that matter, let us introduce the following fixed point formulation of problem (1.1). Let $T_h : X \to X$ be the map

$$T_h(u) = (-\Delta)^{-1}(\lambda_1 u + f(u^+) + h). \tag{3.1}$$

T_h is a compact continuous map and

$$T_h(u) = u \;\Leftrightarrow\; u \text{ solves } (1.1).$$

In what follows $d(\cdot,\cdot,\cdot)$ denotes the Leray-Schauder degree.

Proposition 3.1. *Assume that* $1 < p < \frac{N+1}{N-1}$. *Let (H1)-(H3) hold. For any* $\epsilon \in (0, s_0)$, *there exists* $\delta > 0$ *such that any solution* u *of* (1.1) *with* $||h||_r < \delta$ *satisfies*

$$||u^+||_X < \epsilon.$$

Proof. Assume on the contrary that there exist $\epsilon_0 \in (0, s_0)$ and $\{h_n\}$ with $||h_n||_r < \frac{1}{n}$, such that (1.1) with $h = h_n$ has solution $\{u_n\}$ satisfying

$$||u_n^+||_X \geq \epsilon_0. \tag{3.2}$$

Then

$$\begin{aligned} -\Delta u_n &= \lambda_1 u_n + f(u_n^+) + h_n(x), && \text{in } \Omega,\\ u_n &= 0, && \text{on } \partial\Omega. \end{aligned}$$

By a simple calculation,

$$\int_\Omega f(u_n^+)\phi_1 dx = \int_\Omega -h_n(x)\phi_1 dx \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

On the other hand, Theorem 1.2 yields that there exists a constant $M > 0$ such that

$$||u_n^+||_X \leq M,$$

there exists a subsequence of u_n that is still denoted by u_n such that

$$u_n^+ \to u_*^+ \qquad \text{in } C(\bar{\Omega}).$$

Obviously, (3.2) implies

$$||u_*^+||_\infty \geq \tilde{\epsilon}_0, \tag{3.4}$$

where $\tilde{\epsilon}_0$ is a constant that depends on ϵ_0. Combining (3.4) and (H2), it deduces that

$$\int_\Omega f(u_*^+)\phi_1 dx > 0.$$

However, this contradicts (3.3) if n is large enough. □

Proposition 3.2. *Assume that* $1 < p < \frac{N+1}{N-1}$. *Let (H1)-(H2) hold. There exist* $\delta > 0$ *and* $R_0 > 0$ *such that for all functions* h *satisfying condition (H3) with* $\|h\|_r < \delta$, *and for which problem* (1.1) *possesses at least one solution, it follows that*

$$d(I - T_h, B_E(0,R), 0) \neq 0, \qquad \forall\; R \geq R_0. \tag{3.5}$$

Proof. Choosing a suitable $\delta > 0$. By Proposition 3.1 and (H2), there exists $\epsilon \in (0, s_0)$ is small enough such that any solution u_0 of (1.1) satisfies

$$f'(s) \le f'(||u_0^+||_X) < \lambda_2 - \lambda_1 \quad \text{for any} \quad s \in [0, \epsilon]. \tag{3.6}$$

Obviously, there exists $R_0 \in (\epsilon, s_0]$ such that $||u_0||_X < R_0$.

The linearized problem of (1.1) at u_0 is the following

$$\begin{aligned} & -\Delta v = \lambda_1 v + f'(u_0^+)v, && \text{in } \Omega, \\ & v = 0, && \text{on } \partial\Omega. \end{aligned} \tag{3.7}$$

Denote by $\mu_1(a) < \mu_2(a) < \cdots$ denote the eigenvalues of the following eigenvalue problem of weight a, i.e.,

$$\begin{aligned} & -\Delta v = \mu a(x) v, && \text{in } \Omega, \\ & v = 0, && \text{on } \partial\Omega. \end{aligned}$$

By (3.6)

$$\lambda_1 < a(x) := \lambda_1 + f'(u_0^+) < \lambda_2 \qquad \text{a.e.},$$

so then

$$\mu_1(a) < 1 < \mu_2(a).$$

Hence $v \equiv 0$ is the unique solution of (3.7) and therefore u_0 is a non-degenerate solution of (1.1) of Morse index equal to 1.

By estimate (3.6) the degree above is well defined for all $R \ge R_0$. Moreover, since all possible solutions of $u = T_h(u)$ are non-degenerated, it follows that they are isolated and that there is only a finite number m of them in $B_X(0, R)$. We recall that the index of each solution is equal to $(-1)^\beta$ where β is the Morse index. So

$$d(I - T_h, B_X(0, R), 0) = \sum_{j=1}^{m} (-1) \ne 0.$$

□

Proof of Theorem 1.1. There exists $t_0 > 0$ such that for $0 < t < t_0$, $h_1 := -f(t\phi_1)$ satisfies

$$\|h_1\|_r = \| - f(t\phi_1)\|_r < \delta,$$

where δ is given by Proposition 3.2. We can verify that $u = t\phi_1$ is the solution of problem (1.1) for h_1 and then Proposition 3.2 applies. Thus

$$d(I - T_{h_1}, B_X(0, R), 0) \ne 0$$

for R large enough. Consider the following homotopy

$$\begin{aligned} & -\Delta v = \lambda_1 v + f(v^+) + (1 - \tau)h(x) + \tau h_1(x), && \text{in } \Omega, \\ & v = 0, && \text{on } \partial\Omega, \end{aligned} \tag{3.8}$$

where $0 \le \tau \le 1$. From the a-*priori* estimates of Theorem 1.2, all the solutions of problem (3.8) are uniformly bounded in X by, say, $R_1 := \max\{C, \epsilon\}$, where C and ϵ are defined in Theorem 1.2 and the proof of Proposition 3.2 respectively. Hence, if $R > \max\{R_0, R_1\}$,

$$d(I - T_h, B_X(0, R), 0) = d(I - T_{h_1}, B_X(0, R), 0) \ne 0$$

and the conclusion of the theorem follows. □

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