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PERIODIC SOLUTIONS FOR 1-DIMENSIONAL P-SUPERLINEAR LAPLACIAN EQUATION*

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Abstract Existence and multiplicity of periodic solutions for 1-dimensional p-Laplacian equation with partial p-superlinear are proved. Proofs are based on a geometric approach and the Poincaré-Birkhoff twist theorem. Result generalizes the classical results of Jacobowitz and Hartman.

Keywords Hamiltonian systems, p-Superlinear Laplacian equation, periodic solution, Poincaré-Birkhoff twist theorem.

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1. Introduction

In this paper, we are interested in the existence and multiplicity of periodic solutions for the 1-dimensional p-Laplacian equation

$$(|x'|^{p-2}x')' + f(t,x) = 0. \tag{1.1}$$

where $p > 1$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, 2π-least periodic in its first variable, p-superlinear in the sense

$$(f_s^p) \quad \lim_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} = +\infty, \quad \text{uniformly in } t \in \mathbb{R}.$$

Notice that, when $p = 2$, we have a second order equation $x'' + f(t,x) = 0$.

Superlinear differential equation is one of the typical models both in ODE and forced vibrations. There are many interesting results on the existence and multiplicity of periodic solutions of superlinear second order differential equations. The used methods range from the Poincaré-Birkhoff twist theorem [6, 9, 10, 12, 17–19, 21, 25], variational method [1, 2, 20], to Leray-Schauder continuation method of topological degree [3, 4].

The case with superlinear nonlinearity appears to be the most delicate to treat. In the application of topological degree, it is not easy to find a priori estimate for possible periodic solutions. In the application of the Poincaré-Birkhoff twist theorem, the Poincaré map may not be well defined. In fact, Coffmann and Ulrich [5] gave an example of a positive $q(t) \in C^0([0, 2\pi])$ such that $x'' + q(t)x^3 = 0$ has a solution which does not exist on $[0, 2\pi]$. To avoid this problem, Jacobowitz [18] and Hartman [17] gave a priori estimates for possible periodic solutions with given

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zeros based on the application of Sturm comparison theorem. Then They applied Poincaré-Birkhoff twist theorem on the corresponding truncation equations and obtained the existence of infinitely many periodic solutions for superlinear second order equations.

In case superlinear second order equations is nonconservative, Capietto, Mawhin, and Zanolin [3, 4] proved a continuation theorem where the new ingredient is the use of a functional which is proper on the set of possible periodic solutions of the homotopic family of equations. The property of proper is, roughly speaking, the elastic property of the solution which associated to the global existence of the solution.

In [6], Ding and Zanolin considered the case of forced type, that is $f(t,x) = g(x) - p(t)$, where $f(t,x)$ can be considered as a global time perturbation of $g(x)$. In this case, the behavior of the perturbed system can be estimated by the energy function of the autonomous system, so that the global existence of the solution of the equation can be easily obtained. Therefore, the Poincaré map of the equation is defined. We refer to [21–23, 26] and the references therein for the related research.

Recently, Fonda and Sfecci [13] use so-called admissible spiral method to prove the existence of infinitely many periodic solutions for weakly coupled superlinear second order systems by using a higher dimensional version of the Poincaré-Birkhoff theorem recently obtained by Fonda and Ureña [14]. In [13], superlinear condition (f_s^2) is also used to construct admissible spiral curves.

This paper is the further research on the above topic. We consider the partial p-superlinear Laplacian equation. We assume

$$(f_p^p)\ sgn(x)f(t,x) \geq 0 \text{ for } |x| \gg 1, \quad \lim_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} = +\infty \text{ uniformly in } t \in I,$$

where $I \subset [0, 2\pi]$ is a set of positive measure. Motivated by [10, 12, 13] and the early papers [7, 11], we use phase-plane analysis to show the rapidly spiral property of the solution in time interval $[0, 2\pi]$ under partial p-superlinear condition. So we can consider the suitable truncation of the equation instead such that all solutions of the new equation exist globally. It allows us to apply the Poincaré-Birkhoff twist theorem on an annulus using rapidly spiral properties of solutions. Finally, the nodal properties of periodic solutions corresponding to fixed points of the Poincaré-Birkhoff twist theorem ensure that these 2π-periodic solutions are exactly the 2π-periodic solutions of the original equation.

To use the Poincaré-Birkhoff twist theorem, we transform the p-Laplacian equation (1.1) into an equivalent Hamiltonian system of the form

$$\begin{cases} x' = |y|^{q-2}y = \dfrac{\partial H}{\partial y}(t,x,y), \\ y' = -f(t,x) = -\dfrac{\partial H}{\partial x}(t,x,y) \end{cases} \tag{1.2}$$

where q is a positive integer conjugate to p, that is, $1/p + 1/q = 1$, and $H(t,x,y) = \frac{1}{q}|y|^q + \int_0^x f(t,s)ds$.

The main result of this paper is the following.

Theorem 1.1. *Assume $f(t,x)$ is a continuous function, 2π-least periodic to t and there is uniqueness for the solutions of the Cauchy problems associated with (1.1).*

Moreover, assume (f_p^p) *and*

(f_0) *There are* $c_0,\ \alpha > 0$ *such that* $|f(t,x)| \leq \alpha |x|^{p-1},$ *for* $|x| \leq c_0,\ \forall t \in \mathbb{R}.$

Then the p-Laplacian equation (1.1) has a sequence $\{x_k(t)\}$ *of* 2π*-periodic solutions, such that* $x_k(t)$ *has exactly* $2k$ *simple zeros in the interval* $[0, 2\pi)$ *and satisfying*

$$\lim_{k\to+\infty} \sup_{t\in[0,2\pi]} \{|x_k(t)| + |y_k(t)|\} = +\infty.$$

Moreover, for any given $m \in \mathbb{N}$*, equation (1.1) has a sequence* $\{x_{m,k}(t)\}$ *of m-th subharmonic solutions, such that*

$$\lim_{k\to+\infty} \sup_{t\in[0,2m\pi]} \{|x_{m,k}(t)| + |y_{m,k}(t)|\} = +\infty.$$

Remark 1.1. When $p = 2$ the assumption (f_0) is just the local Lipschitz condition of f at $x = 0$. In [17, 18], Jacobowitz and Hartman considered the second equation $x'' + f(t,x) = 0$, where $f \in C^1$, $f(t,0) \equiv 0$ and satisfies the superlinear condition (f_s^2). They applied Poincaré-Birkhoff twist theorem and obtained the existence of infinitely many periodic solutions for superlinear second order equations. So our result generalizes the classical results of Jacobowitz and Hartman.

Remark 1.2. Our theorem is valid to typical equation $x'' + a(t)x^3 = 0$, where $a(t)$ is a 2π-periodic continuous function, $a(t) \geq 0$ and $\int_0^{2\pi} a(t)ds > 0$.

The rest of the paper is organized as follows. Section 2 is devoted to study rapidly spiral property of the solutions by using phase-plane analysis for the solution. In section 3, existence and multiplicity of periodic solutions are obtained via a generalized version of the Poincaré-Birkhoff twist theorem.

2. Rapidly spiral property of the solutions with large amplitude

In what follows, we perform a phase-plane analysis for the first order Hamilton system (1.2). Notice that for $p > 2 (q < 2)$ the term $|y|^{q-2}y$ is only Hölder continuous, so a theorem of existence, uniqueness and continuous dependence on the initial data for the initial value problem is required. Using analogous arguments to Lemma 2.2 [26], we can prove the above conclusion.

From the uniqueness of the initial value problem associated to system (1.2) and condition (f_0) , we know that the solution starting form $(x_0, y_0) \neq (0,0)$ does not attain the origin. So, we can use polar coordinates to express system (1.2) as the form of

$$\begin{cases} r' = \mathcal{R}(t,r,\theta) = \dfrac{x|y|^{q-2}y - yf(t,x)}{r}, \\ \theta' = \Theta(t,r,\theta) = -\dfrac{xf(t,x) + |y|^p}{r^2}. \end{cases} \tag{2.1}$$

We can describe the twist properties of solutions by polar coordinates system (2.1) in the phase-plane.

For the superlinear equations, the large amplitude solutions have the rapid oscillatory property. In other words, the solutions passing through points in the phase plane which are farther from the origin, oscillate more in a fixed time interval. We consider the Poincaré map

$$\Psi: \ (x(t_0), y(t_0)) \mapsto (x(t_0+2\pi), y(t_0+2\pi)),$$

where $(x(t), y(t))$ is a solution of (1.2). From the rapidly oscillating property, Poincaré map Ψ and its $m-$th iterates associated to system (1.2) have strong twist property. We show that if the amplitude of the solution is large enough, then its rotation angle of the corresponding trajectory is quite large. Moreover, such rotation has some monotonicity of the time.

To simplify the notation, we denote $(x(t;t_0,x_0,y_0), y(t;t_0,x_0,y_0))$ by $(x(t), y(t))$ and its polar form $(r(t;t_0,r_0,\theta_0), \theta(t;t_0,r_0,\theta_0))$ by $(r(t), \theta(t))$. Let $\mathcal{A}[\rho,\rho'] = \{(r,\theta) : \rho \le r \le \rho'\}$, $\mathcal{A}(\rho,\rho') = \{(r,\theta) : \rho < r < \rho'\}$, $\mathcal{A}(\Gamma,\Gamma')$ and $\mathcal{A}[\Gamma,\Gamma']$ expresses an open annulus and a closed annulus bounded by two star-shaped closed curves Γ and Γ', respectively.

Hypothesis(f_p^p) implies sign condition

(f_s) $\quad \mathrm{sgn}(x)f(t,x) \ge 0$, $\quad$ provided $|x|$ large enough, $\forall t \in \mathbb{R}$.

Then, taking into account the expression of $\theta'(t)$ and condition (f_s), we have $\theta'(t) \le 0$ when $r(t)$ sufficiently large. In other words, large solutions will rotate clockwise. Moreover, $\theta' = 0 \iff r'(t) = 0$.

The following lemma considers the time and scope of the solution passing through the vertical strip $E_a = \{(x,y) \in \mathbb{R}^2 : \ |x| \le a\}$ for any parameter $a > 0$, provided y_0 large enough.

Lemma 2.1. *For any given $a > 0, \delta > 0$, there is $\lambda = \lambda(a,\delta) > 0$, such that if the solution $z(t) = (x(t), y(t))$ with initial value $x_0 = 0, y_0 \ge \lambda$ satisfies $x(t_1) = a$ and $0 \le x(t) \le a, \ \forall t \in [t_0, t_1]$, then $t_1 - t_0 < \delta$. In this manner, the time Δt in which the solution $z(t;x_0,y_0)$ with $x_0 = 0$ or $|x_0| = a$ passes through E_a satisfies $\Delta t < \delta$ if $|y_0| \ge \lambda$.*

Proof. We choose

$$\lambda = \max\{4\pi M_0, \left(\frac{2^{q-1}a}{\delta}\right)^{\frac{1}{q-1}}, 2\left(\frac{a}{2\pi}\right)^{\frac{1}{q-1}}\},$$

where $M_0 = \max_{t\in[0,2\pi], x\in[-a,a]}\{|f(t,x)|\}$.

Assume $y_0 \ge \lambda$. Then for $t \in (t_0, t_1]$, it follows from $|y'| = |-f(t,x)| \le M_0$ that

$$|y(t) - y_0| \le M_0(t-t_0) \le 2\pi M_0 \le \frac{\lambda}{2M_0}M_0 = \frac{1}{2}\lambda.$$

Thus,

$$\frac{\lambda}{2} \le \frac{y_0}{2} \le y(t) \le \frac{3y_0}{2}, \quad \text{for} \quad t \in [t_0, t_1].$$

Since $x' = |y|^{q-2}y$, an integration from t_0 to t yields

$$a = \int_{t_0}^{t_1} x'(t)dt \ge \left(\frac{\lambda}{2}\right)^{q-1}(t_1 - t_0).$$

Therefore,

$$t_1 - t_0 \le \frac{2^{q-1}a}{\lambda^{q-1}} \le \delta < 2\pi.$$

The lemma is proved. □

Next, we will discuss the rapid spiral property of those large solutions.

Lemma 2.2 (spiral property)**.** *For $\forall k \in \mathbb{N}$, there exist $r_k,\ r_k',\ r_k''$, with $r_k' < r_k < r_k''$, such that the solution $(x(t), y(t))$ starting from $r_0 = |(x_0, y_0)| = r_k$, we have : either $(x(t), y(t)) \in \mathcal{A}(r_k', r_k'')$ for $t \in [t_0, t_0 + 2\pi]$ and its polar angle satisfies*

$$\theta(t_0 + 2\pi) - \theta_0 < -2k\pi;$$

or there exists $t_0' \in [t_0, t_0 + 2\pi)$, such that $(x(t), y(t)) \subset \mathcal{A}(r_k', r_k'')$ for $t \in [t_0, t_0')$, intersects the boundary of $\mathcal{A}(r_k', r_k'')$ at the time of t_0', and

$$\theta(t_0') - \theta_0 < -(2k+1)\pi.$$

Moreover, $r_k \to +\infty \iff r_k', r_k'' \to +\infty$.

Proof. Let us define

$$\mathcal{D}_1 = \{(x,y) \in \mathbb{R}^2 : 0 \le x < a,\ y > 0\}, \qquad \mathcal{D}_2 = \{(x,y) \in \mathbb{R}^2 : a \le x,\ y > 0\},$$
$$\mathcal{D}_3 = \{(x,y) \in \mathbb{R}^2 : x > a,\ y \le 0\}, \qquad \mathcal{D}_4 = \{(x,y) \in \mathbb{R}^2 : 0 < x \le a,\ y < 0\},$$
$$\mathcal{D}_5 = \{(x,y) \in \mathbb{R}^2 : -a < x < 0,\ y < 0\}, \qquad \mathcal{D}_6 = \{(x,y) \in \mathbb{R}^2 : x \le -a,\ y < 0\},$$
$$\mathcal{D}_7 = \{(x,y) \in \mathbb{R}^2 : x \le -a,\ y \ge 0\}, \qquad \mathcal{D}_8 = \{(x,y) \in \mathbb{R}^2 : -a < x \le 0,\ y > 0\}.$$

We consider the solution $z(t) = (x(t), y(t))$ starting from (x_0, y_0). Without loss of generality, assume $(x_0, y_0) = (0, y_0)$, its polar form is $(r_0, \theta_0 = \pi/2)$. For r_0 is sufficiently large, we will prove that there exist $r_0',\ r_0''$, with $r_0' < r_0 < r_0''$ and

$$r_0 \to +\infty \iff r_0', r_0'' \to +\infty,$$

such that, either $(x(t), y(t)) \in \mathcal{A}(r_0', r_0'')$, $t \in [t_0, t_0 + 2\pi]$, or there exists $t_0' \in [t_0, t_0 + 2\pi)$, such that $(x(t), y(t)) \subset \mathcal{A}(r_0', r_0'')$ for $t \in [t_0, t_0')$, intersects the boundary of $\mathcal{A}(r_0', r_0'')$ at the time of t_0', and satisfies

$$\theta(t_0') - \theta_0 < -2\pi.$$

Moreover, we suppose that $\theta'(t) \le 0$ for $r(t) = |z(t)| \ge r_\star$. In this case, the solution $(x(t), y(t))$ will rotates as follows

$$\mathcal{D}_1 \to \mathcal{D}_2 \to \mathcal{D}_3 \to \mathcal{D}_4 \to \mathcal{D}_5 \to \mathcal{D}_6 \to \mathcal{D}_7 \to \mathcal{D}_8.$$

Let $[t_{i-1}, t_i] \subset [t_0, t_0 + 2\pi)$, such that $(x(t), y(t)) \in \mathcal{D}_i$, for $t_{i-1} \le t \le t_i$, $i = 1, \cdots, 8$.

We divide our proof into four steps.

Step 1 Assume $(x(t), y(t)) \in \mathcal{D}_1$. From Lemma 2.1 we have $y_0/2 \le y(t) \le 3y_0/2$. Then

$$\eta_1(r_0) = \frac{r_0}{2} \le r(t) \le \sqrt{9r_0^2/4 + a^2} = \zeta_1(r_0), \quad \text{for} \quad t \in [t_0, t_1].$$

Moreover, $x'(t_1) > 0$ which implies that $(x(t), y(t)) \in \mathcal{D}_2$ for $t > t_1$.

Step 2 Assume $(x(t), y(t)) \in \mathcal{D}_2$. Define $g_+(x) = \max_{t \in [0, 2\pi]}\{|f(t,x)|, 1\}$. Then

$$G_+(x) = \int_a^x g_+(s)ds \to +\infty \iff x \to +\infty.$$

Let $H_+(t) = \frac{|y(t)|^q}{q} + G_+(x(t))$. We have

$$H_+'(t) = |y|^{q-2}y(g_+(x) - f(t,x)) \ge 0, \quad \text{for} \quad (x(t), y(t)) \in \mathcal{D}_2,$$

which follows that

$$H_+(t) \geq H_+(t_1) = \frac{|y(t_1)|^q}{q} + G_+(a), \quad \text{for} \quad t \in [t_1, t_2].$$

Denote

$$\eta_2(r_0) = \min\{\sqrt{x^2+y^2} : \frac{|y|^q}{q} + G_+(x) = \frac{|y(t_1)|^q}{q} + G_+(a)\}.$$

We have

$$\eta_2(r_0) \leq r(t), \quad \text{for} \quad t \in [t_1, t_2].$$

On the other hand, set $H_-(t) = \frac{y^q(t)}{q} + \frac{x^p(t)}{p}$, then for $t \in [t_1, t_2]$,

$$H'_-(t) = y^{q-1}(t)(-f(t, x(t))) + x^{p-1}(t)y^{q-1}(t) \leq x^{p-1}(t)y^{q-1}(t) \leq H_-(t)$$

which follows that

$$H_-(t) \leq \mathrm{e}^{2\pi} H_-(t_1), \quad \text{for} \quad t \in [t_1, t_2] \subset [t_0, t_0 + 2\pi].$$

Thus

$$r(t) \leq \zeta_2(r_0), \quad \text{for} \quad t \in [t_1, t_2],$$

where $\zeta_2(r_0) = \max\{\sqrt{x^2+y^2} : \frac{y^q}{q} + \frac{x^p}{p} = \mathrm{e}^{2\pi} H_-(t_1)\}$.

Step 3 Assume $(x(t), y(t)) \in \mathcal{D}_3$. In this case we have

$$H'_+(t) = |y|^{q-2} y(g_+(x) - f(t, x)) \leq 0, \quad \text{for} \quad (x(t), y(t)) \in \mathcal{D}_3, \tag{2.2}$$

which follows that

$$H_+(t) \leq H_+(t_2) = G_+(x(t_2)), \quad \text{for} \quad t \in [t_2, t_3].$$

On the other hand, for $t \in [t_2, t_3]$, we get

$$\begin{aligned} H'_-(t) &= |y(t)|^{q-2} y(t)(-f(t, x(t))) + x^{p-1}(t)|y(t)|^{q-2} y(t) \\ &\geq x^{p-1}(t)|y(t)|^{q-2} y(t) \geq -H_-(t) \end{aligned}$$

which follows that

$$H_-(t) \geq \mathrm{e}^{-2\pi} H_-(t_2), \quad \text{for} \quad t \in [t_2, t_3] \subset [t_0, t_0 + 2\pi].$$

Denote

$$\zeta_3(r_0) = \max\{\sqrt{x^2+y^2} : \frac{|y|^q}{q} + G_+(x) = G_+(x(t_2))\}$$

and

$$\eta_3(r_0) = \min\{\sqrt{x^2+y^2} : \frac{y^q}{q} + \frac{x^p}{p} = \mathrm{e}^{-2\pi} H_-(t_2)\}.$$

We obtain

$$\eta_3(r_0) \leq r(t) \leq \zeta_3(r_0), \quad \text{for} \quad t \in [t_2, t_3].$$

Step 4 The same argument, with minor changes, can be repeated in the remaining cases $(x(t), y(t)) \in \mathcal{D}_i$, $i = 4, 5, 6, 7, 8$. Therefore, there exists $\eta_i(r_0)$, $\zeta_i(r_0)$, such that

$$\eta_i(r_0) \le r(t) \le \zeta_i(r_0), \quad \text{for} \quad t \in [t_{i-1}, t_i], \quad i = 4, 5, 6, 7, 8.$$

Moreover,

$$\eta_i(r_0), \zeta_i(r_0) \to +\infty \iff r_0 \to +\infty, \quad i = 1, \cdots, 8.$$

Let

$$r_0' = \min\{\eta_i(r_0),\ i = 1, \cdots, 8\} - 1, \quad r_0'' = \max\{\zeta_i(r_0),\ i = 1, \cdots, 8\} + 1,$$

we have: either $(x(t), y(t)) \in \mathcal{A}(r_0', r_0'')$, for $t \in [t_0, t_0 + 2\pi]$, or there exists $t_0' \in (t_8, t_0 + 2\pi)$, such that $(x(t), y(t)) \subset \mathcal{A}(r_0', r_0'')$ for $t \in [t_0, t_0')$, intersects the boundary of $\mathcal{A}(r_0', r_0'')$ at the time of t_0', and satisfies

$$\theta(t_0') - \theta_0 < -2\pi.$$

Repeating above arguments k times we can obtain the conclusion of Lemma 2.2. □

The next lemma says that the length of a time interval in which a solution $(x(t), y(t))$ of (1.2), for $t \in I$, completes one clockwise turn around the origin, tends to zero as $r(t) \to +\infty$.

Lemma 2.3 (rapid rotation)**.** *For any $\delta > 0$, there exists $r = r_\delta > 0$, such that, for $r(t) \ge r_\delta$, $t \in [t_1, t_2] \subset I$, and $\theta(t_2) - \theta(t_1) \ge -2\pi$, we have $t_2 - t_1 < c\delta$, where c is a positive constant independent of δ.*

Proof. Let $\delta > 0$ be an arbitrary but fixed constant. According to the hypothesis (f_p^p), there exists $a = a(\delta) > 0$, such that

$$\frac{f(t,x)}{|x|^{p-2}x} > \delta^{-p}, \qquad \forall t \in I,\ |x| > a. \tag{2.3}$$

Set $r_\delta = \sqrt{\lambda^2 + a^2}$, where $\lambda = \lambda(a, \delta)$ is defined in Lemma 2.1. For $r(t) \ge r_\delta$, we have the following two cases:

(i) $|x(t)| \le a$, $|y(t)| \ge \lambda$ for $t \in [t_1', t_2']$. Using Lemma 2.1 we obtain

$$|t_2' - t_1'| \le \delta. \tag{2.4}$$

(ii) $|x| \ge a$ for $t \in [t_1', t_2']$ and $\theta(t_2') - \theta(t_1') \ge -\pi$. It is convenient to introduce another angle variable $\widehat{\theta}$ by

$$\cos\widehat{\theta} = \frac{M^{\frac{1}{2}}|x|^{\frac{p}{2}-1}x}{(M|x|^p + |y|^q)^{\frac{1}{2}}}, \quad \sin\widehat{\theta} = \frac{|y|^{\frac{q}{2}-1}y}{(M|x|^p + |y|^q)^{\frac{1}{2}}}, \tag{2.5}$$

where $M = \delta^{-p}$. (2.5) admits that $\cos\widehat{\theta}$ and $\cos\theta$, respectively, $\sin\widehat{\theta}$ and $\sin\theta$, have the same sign. Hence the two angles $\widehat{\theta}$ and θ always lie in the same quarter. Using (2.1) and (2.3) we have

$$\frac{d\widehat{\theta}}{dt} = M^{\frac{1}{2}} \frac{|x|^{\frac{p}{2}-1}x\frac{d}{dt}(|y|^{\frac{q}{2}-1}y) - |y|^{\frac{q}{2}-1}y\frac{d}{dt}(|x|^{\frac{p}{2}-1}x)}{M|x|^p + |y|^q}$$

$$
\begin{aligned}
&= -\frac{1}{2}M^{\frac{1}{2}}\frac{q|y|^{\frac{q}{2}-1}|x|^{\frac{p}{2}-1}xf(t,x)+p|y|^{\frac{q}{2}+q-1}|x|^{\frac{p}{2}-1}}{M|x|^p+|y|^q}\\
&= -\frac{1}{2}M^{\frac{1}{2}}\frac{|y|^{\frac{q}{2}-1}|x|^{\frac{p}{2}-1}(qxf(t,x)+p|y|^q)}{M|x|^p+|y|^q}\\
&\le -\frac{1}{2}M^{\frac{1}{2}}\frac{|y|^{\frac{q}{2}-1}|x|^{\frac{p}{2}-1}(qM|x|^p+p|y|^q)}{M|x|^p+|y|^q}\\
&\le -c_1M^{\frac{1}{2}}|y|^{\frac{q}{2}-1}|x|^{\frac{p}{2}-1} = -c_1M^{\frac{1}{p}}|\tan\widehat{\theta}|^{-\frac{p-2}{p}},
\end{aligned}
$$

where c_1 is a positive constant independent of M. Moreover, $|t_2'-t_1'|$ is less than the time for $\widehat{\theta}$ passing through half phase-plane which can be estimated by

$$
T(\delta) \le \frac{1}{c_1}M^{-\frac{1}{p}}\int_0^{\pi}|\tan\widehat{\theta}|^{\frac{p-2}{p}}d\widehat{\theta} \doteq c_pM^{-\frac{1}{p}} = c_p\delta, \tag{2.6}
$$

where $c_p = \frac{1}{c_1}\int_0^{\pi}|\tan\widehat{\theta}|^{\frac{p-2}{p}}d\widehat{\theta}$ is finite because of $|\frac{p-2}{p}|<1$.

Combining (2.4) and (2.6) we complete the proof of the lemma. □

3. Proof of theorem 1.1

Proof. we define

$$
H_1(t,x,y) = L(t,y) + W(x^2+y^2)F(t,x) + (1-W(x^2+y^2))\frac{k^2x^2}{2},
$$

where $W(x^2+y^2) = W(r^2) \in C^1$ is a truncating function satisfying that

$$
W(r^2) = \begin{cases} 1, & r \le r_k'', \\ \text{smooth connection}, & r_k'' < r < r_k''+1, \\ 0, & r \ge r_k''+1, \end{cases}
$$

and $L(t,y) = \frac{1}{q}|y|^q$, $F(t,x) = \int_0^x f(t,s)ds$. Hence, the modified system

$$
\begin{cases} x' = \dfrac{\partial H_1}{\partial y}(t,x,y), \\ y' = -\dfrac{\partial H_1}{\partial x}(t,x,y) \end{cases} \tag{3.1}
$$

is same as the system (1.2) for $r \le r_k''$. By applying Gronwall inequality, we can prove that solutions of system (3.1) exist for $t \in \mathbb{R}$. The Poincaré map Ψ_1 associated to system (3.1) is well defined. From the assumptions of theorem 1.1 and the fact that the uniqueness of solution implies continuous dependence on initial values [16], we know that if Poincaré map Ψ_1 associated to the system (3.1) exists, then it will be an area-preserving homeomorphism.

Furthermore,

$$
y\frac{\partial H_1}{\partial y} = y\frac{\partial L}{\partial y} > 0, \quad \text{for} \quad x=0,\ y\ne 0. \tag{3.2}
$$

We will construct an annulus $\mathcal{A}[\Gamma,\Gamma_+]$, such that Ψ_1 satisfies the assumptions of Poincaré-Birkhoff twist theorem on $\mathcal{A}[\Gamma,\Gamma_+]$ (see the generalized version in [8,23,24] and [15]).

According to $f(t,0)\equiv 0$, we have $(x,y)=(0,0)$ is a solution of (3.1). The continuous dependence on initial data theorem implies that there exists an open neighborhood of $(0,0)$, denoted by U_ε with ε sufficiently small, such that the solutions starting from the boundary $C_\varepsilon=\{(x,y)|x^2+y^2=\varepsilon^2\}$ exist on whole 2π time interval and lie in the neighborhood of $(0,0)$ with radius of 1. From the uniqueness theorem, any solution starting from C_ε cannot meet $(0,0)$. Hence, let $\Gamma=C_\varepsilon$. We take Γ as the inner boundary of $\mathcal{A}$.

Consider the polar form $(r(t;t_0,r_0,\theta_0),\theta(t;t_0,r_0,\theta_0))$ of the solution staring from Γ. Since $\theta(2\pi+t_0;t_0,r_0,\theta_0)-\theta_0$ is a continuous function on $(r_0,\theta_0)\in\widetilde{\Gamma}$ and $\widetilde{\Gamma}$ is compact, $\theta(2\pi+t_0;t_0,r_0,\theta_0)-\theta_0$ is bounded below on $\widetilde{\Gamma}$, where $\widetilde{\Gamma}$ is the polar lifting of Γ. Namely

$$\inf_{(r_0,\theta_0)\in\widetilde{\Gamma}}(\theta(2\pi+t_0;t_0,r_0,\theta_0)-\theta_0)>-2k_0\pi, \tag{3.3}$$

for some $k_0\in\mathbb{N}^+$.

To construct Γ_+, outer boundary of $\mathcal{A}$, we need more detailed analysis. From the continuity of solution for initial value and compactness of $\widetilde{\Gamma}$, we have

$$\sup_{t\in[t_0,t_0+2\pi],(r_0,\theta_0)\in\widetilde{\Gamma}} r(t;t_0,r_0,\theta_0)=r_\Gamma<+\infty.$$

For any $k\geq k_0$, we take r_k large enough, such that $r'_k\geq r_\Gamma$.

Now we choose $\Gamma_+=\{(x,y)|x^2+y^2=r_k^2\}$. Consider the solution $(r(t),\theta(t))$, the polar lifting of $(x(t),y(t))$ starting from Γ_+. From Lemma 2.2 there exist r_k, r'_k, r''_k with $r'_k<r_k<r''_k$, for the solution $(x(t),y(t))$ starting from $r_0=|(x_0,y_0)|=r_k$, it probably satisfies $(x(t),y(t))\in\mathcal{A}(r'_k,r''_k)$ for $t\in[t_0,t_0+2\pi]$. In this case, we choose $\delta<\dfrac{\mathrm{mes}(I)}{k+1}$ and $r'_k>r_\delta$, where r_δ is defined in Lemma 2.3. Without losing the generality, let $I=[a',b']\subset[t_0,t_0+2\pi]$. Then for $t\in[a',b']$, the polar angle $\theta(t)$ gets at least $-2k\pi$-increase, that is $\theta(b')-\theta(a')<-2k\pi$. Notice also $\theta'(t)\leq 0$ for $t\in[t_0,t_0+2\pi]$, which implies that

$$\theta(t_0+2\pi)-\theta_0=(\theta(t_0+2\pi)-\theta(b'))+(\theta(b')-\theta(a'))+(\theta(a')-\theta_0)<-2k\pi. \tag{3.4}$$

Otherwise, there exists $t'_0\in[t_0,t_0+2\pi)$, such that solution $(x(t),y(t))$ intersects the boundary of $\mathcal{A}(r'_k,r''_k)$ at the time of t'_0, then

$$\theta(t'_0)-\theta_0<-(2k+1)\pi. \tag{3.5}$$

On the other hand, the vector field restriction (3.2) shows that the solutions of the modified system(3.1) can never perform counterclockwise rotations at y-axis, that is for any $t>t'_0$, we have $\theta(t)-\theta(t'_0)<\pi$. Then, recalling (3.5), we obtain

$$\begin{aligned}\theta(t_0+2\pi)-\theta_0&=(\theta(t_0+2\pi)-\theta(t'_0))+(\theta(t'_0)-\theta_0),\\&<\pi-(2k+1)\pi=-2k\pi\leq-2k_0\pi.\end{aligned} \tag{3.6}$$

Now the twist condition for inner and outer boundaries of $\mathcal{A}[\Gamma,\Gamma_+]$ is fulfilled by (3.3), (3.4) and (3.6). Therefore, we use the Poincaré-Birkhoff twist theorem to obtain the existence of at least two (geometrically distinct) fixed points $(x_i^{(k)}, y_i^{(k)})$ of Ψ_1, $i = 1, 2$. The solutions $(x(t; t_0, x_i^{(k)}, y_i^{(k)}), y(t; t_0, x_i^{(k)}, y_i^{(k)}))$ of (2.5) are 2π-periodic, $i = 1, 2$. Moreover, their polar forms $(r(t; t_0, r_i^{(k)}, \theta_i^{(k)}), \theta(t; t_0, r_i^{(k)}, \theta_i^{(k)}))$ satisfy

$$\theta(t_0 + 2\pi; t_0, r_i^{(k)}, \theta_i^{(k)}) - \theta_i^{(k)} = -2k\pi, \quad i = 1, 2. \tag{3.7}$$

We will show that these $2\pi-$periodic solutions lie in the region $r < r_k''$. Assuming the contrary, there exists $t_0'' \in [t_0, t_0 + 2\pi]$ satisfying

$$r(t_0'') = r_k''$$

which together with lemma 2.2, we get

$$\theta(t_0'') - \theta_0 < -(2k+1)\pi.$$

Noting the vector field restriction (3.2), we have

$$\theta(t_0 + 2\pi) - \theta_0 = (\theta(t_0 + 2\pi) - \theta(t_0'')) + (\theta(t_0'') - \theta_0) < \pi - (2k+1)\pi = -2k\pi.$$

Thus, we get a contradiction with

$$\theta(t_0 + 2\pi) - \theta_0 = -2k\pi.$$

Therefore, for any fixed $k \in \mathbb{N}^+, k \geq k_0$, we obtain the existence of at least two (geometrically distinct) 2π-periodic solutions $(x(t; t_0, x_i^{(k)}, y_i^{(k)}), y(t; t_0, x_i^{(k)}, y_i^{(k)}))$, $i = 1, 2$, their polar angles satisfying (3.7). Since k can be taken over the positive integers which is not less than k_0 we gain the existence of infinite many periodic solutions.

Note that $\theta(t; t_0, r_i^{(k)}, \theta_i^{(k)})$ satisfies (2.1). If $r(t; t_0, r_i^{(k)}, \theta_i^{(k)})$ are defined on the bounded closed set, then the term on the left hand side of (2.1) is bounded. It follows that $\theta'(t; t_0, r_i^{(k)}, \theta_i^{(k)})$ is bounded, which contradicts to (2.1). Therefore, we have

$$\lim_{k \to +\infty} \sup_{t \in [0, 2\pi]} \{r(t; t_0, r_i^{(k)}, \theta_i^{(k)})\} = +\infty.$$

From Lemma 2.2 and Lemma 2.3, we can also obtain the rapidly spiral property on $[t_0, t_0 + 2m\pi]$, that is, $\forall k \in \mathbb{N}$, there exist $r_k,\ r_k',\ r_k''$, with $r_k' < r_k < r_k''$, for the solution $(x(t), y(t))$ starting from $r_0 = |(x_0, y_0)| = r_k$, we have:

either $(x(t), y(t)) \in \mathcal{A}(r_k', r_k''),\ t \in [t_0, t_0 + 2m\pi]$, such that its polar angle satisfies

$$\theta(t_0 + 2m\pi) - \theta_0 < -2k\pi;$$

or there exists $t_0' \in [t_0, t_0 + 2m\pi)$, such that $(x(t), y(t)) \subset \mathcal{A}(r_k', r_k'')$ for $t \in [t_0, t_0')$, intersects the boundary of $\mathcal{A}$ at the time of t_0', and

$$\theta(t_0') - \theta_0 < -(2k+1)\pi.$$

Moreover, $r_k \to +\infty \iff r_k', r_k'' \to +\infty$.

Then we can use a similar argument on the area-preserving map Ψ^m to obtain fixed points of Ψ^m and obtain the infinitely many $2m\pi$-periodic solutions. Further, using the similar argument with [6], we can obtain the infinitely many m-th subharmonic solutions.

The proof of Theorem 1.1 is thus completed. $\square$

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