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PULLBACK EXPONENTIAL ATTRACTORS FOR NON-AUTONOMOUS ABSTRACT RETARDED EVOLUTION EQUATIONS*

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Abstract In this paper, we consider an abstract non-autonomous evolution equation with multiple delays in a Hilbert space H:

$$u'(t) + Au(t) = F(t, u(t), u(t - r_1), \dots, u(t - r_n)),$$

where $A : D(A) \subset H \to H$ is a positive definite selfadjoint operator with compact resolvent, and $F : \mathbb{R} \times D(A^\alpha)^{n+1} \to H(\alpha \in [0, 1/2])$ is a locally Lipschitz continuous mapping. We slightly generalize a theoretical existence result for pullback exponential attractors. Based on our abstract theorem, we prove some existence results of pullback exponential attractor for this delay differential equations and derive estimates on the fractal dimension of the attractors.

Keywords Pullback exponential attractor, retarded evolution equation, delay, fractal dimension.

MSC(2010) 37L30, 37B55, 35B41, 34D45, 37L25.

1. Introduction

In this paper we consider the following non-autonomous abstract evolution equation with multiple delays in a Hilbert space H:

$$u'(t) + Au(t) = F\left(t, u(t), u(t - r_1), \dots, u(t - r_n)\right), \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a positive definite selfadjoint operator,$r_1, \dots, r_n$ are positive constants.

Our aim is to investigate the existence of pullback exponential attractor of abstract evolution equation (1.1), which represents a class of parabolic equations arising in mathematical biology, (see, [11,12,17] and the references therein). For non-retarded evolution equations, there have appeared many nice results on pullback exponential attractor,(see, [3–7,13,14]). In contrast, the situation in the case of retarded equations seems to be more complicated. Recently, the existence of pullback exponential attractors for evolution processes generated by non-autonomous delayed ordinary differential equations has been obtained in [8]. They showed how existence results for pullback exponential attractors can be applied to non-autonomous delay

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differential equations with time-varying delays. In 2021, Yang, Wang and Kloeden [20] presented some sufficient conditions for the existence of pullback exponential attractors for non-autonomous delayed dynamical systems in the phase space $C([-r,0],\, X)$, where X is an infinite dimensional Banach space. The proofs in [8] and [20] were based on the general existence theorems for pullback exponential attractors in [3]. This method is grounded in the compact embedding of an auxiliary space into the phase space. They have used the compactness of the embedding $\mathcal{C}^1 \hookrightarrow\hookrightarrow \mathcal{C}$ and $C^1([-r,\, 0],\, X_{\mathcal{N}}) \hookrightarrow\hookrightarrow C([-r,\, 0],\, X)$, respectively .

For applications with our delay problem, we slightly generalize the theoretical existence result in [3,8]. First, We define an evolution process in a Banach space X. Let W and V be two auxiliary normed spaces such that the embedding $V \hookrightarrow\hookrightarrow W$ is compact and assume that the embedding $V \hookrightarrow X \hookrightarrow W$ is continuous. Motivated by [3,8,20], we obtain the existence result of pullback exponential attractor in phase space X. Obviously, if $X = W$, the result is Theorem 4.1 in [8]; if $X = V$, the result is Corollary 2 in [3]. For non-retarded evolution equations, this modification is unnecessary. But it provide more convenience for the situation in the case of retarded equation. Since our process is generated by PDE with delays, the phase space is $W = C([-r,\, 0], D(A^\alpha))$. If $\alpha < \beta$, the imbedding $D(A^\beta) \hookrightarrow\hookrightarrow D(A^\alpha)$ is compact. But $C([-r,0],\, D(A^\beta)) \hookrightarrow C([-r,0],\, D(A^\alpha))$ is not compact. Only $V = C^{0,\gamma}([-r,0],\, D(A^\beta))$ that ensure the embedding $V \hookrightarrow\hookrightarrow W$ is compact by Ascoli-Arzela Theorem. If we can't obtain the process $\{U(t,\, s),\, t \geq s\}$ is Lipschitz in V, then we have to choose phase space is $C([-r,0],\, D(A^\alpha))$. Using our method, we can choose $X = C([-r,0],\, D(A^\beta))$ as phase space and obtain pullback exponential attractors. So we establish some new results in more regular spaces under weaker assumptions.

Then, we apply our theoretical existence result to the abstract equation (1.1). We present essential conditions on the nonlinearity F to guarantee the dissipation of the equation, and construct absorbing sets, which the corresponding absorbing times are bounded in the past. Then we prove the existence of pullback exponential attractors for evolution processes generated by problem (4.1) and derive explicit estimates for their fractal dimension.

The rest of the paper is organized as follows. In section 2, we provide some preliminaries; In section 3, we slightly generalize previous existence results for pullback exponential attractors for application with delay equation; In section 4, we consider the dissipation of the Eq. (1.1) and establish the existence of pullback exponential attractor for the system; Section 5 is devoted to some examples that show the applicability of our results.

2. Preliminaries

For convenience we use the following notation throughout this paper. We denote by $\|\cdot\|_E$ the norm of a Banach space E, by $\|\cdot\|$ the operator norm . Let

$$\mathfrak{B}_E(a,\, \rho) = \{x \in E \,|\, \|x - a\|_E < \rho\},$$

which is the ball of radius $\rho > 0$ and center $a \in E$ in Banach space E .

We recall some basic facts on analytic semigroups of linear operator and fractional powers space, which are needed to prove our main results.

Throughout this paper, we assume that H be a Hilbert space with inner product $(\cdot,\, \cdot)$ and with norm $\|\cdot\|$, $A:\, D(A) \subset H \to H$ be a positive definite selfadjoint operator and with compact resolvent. If A has compact resolvent, by the spectral resolution theorem of selfadjoint operator, the spectrum $\sigma(A)$ consists of real eigenvalues and it can be arrayed in sequences as

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \to \infty \quad (k \to \infty).$$

By the positive definite property of A, the first eigenvalue $\lambda_1 > 0$. It is well known in [9, 19] $-A$ generates an analytic operator semigroup $T(t)(t \geq 0)$ in H, which satisfies

$$\|T(t)\| \leq e^{-\lambda_1 t}, \qquad \forall t \geq 0.$$

We recall some concepts and conclusions on the fractional powers of A in [9,19]. The fractional power A^α of the operator A is defined to be $A^\alpha = (A^{-\alpha})^{-1}$ and the domain $D(A^\alpha) := H_\alpha$ is a Hilbert space with inner product $(\cdot, \cdot)_\alpha = (A^\alpha \cdot, A^\alpha \cdot)$ and associated norm $|\cdot|_\alpha$. Especially, $H_0 = H$ and $H_1 = D(A)$.

For $0 \leq \alpha \leq \beta$, one has H_β is continuous embedded into H_α and

$$\|v\|_\alpha^2 \leq \lambda_1^{2(\alpha-\beta)} \|v\|_\beta^2, \qquad \text{for all } v \in H_\beta. \tag{2.1}$$

See [19], Page.93. From Theorem 37.5 in [19], the following statements hold:
For any $\alpha \geq 0$, there exist constants $M_\alpha > 0$, $a > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{-at}, \qquad \text{for all } t > 0. \tag{2.2}$$

For $0 < \alpha \leq 1$, there exists a constant $K_\alpha > 0$ such that

$$\|(T(t) - I)x\| \leq K_\alpha t^\alpha \|A^\alpha x\|, \qquad \text{for all } t \geq 0 \text{ and } x \in H_\alpha. \tag{2.3}$$

3. General existence theorems for pullback exponential attractors

We recall some basic definitions and facts in the theory of non-autonomous dynamical systems for evolution process on a Banach space $(X, \|\cdot\|_X)$. Given any subsets A, B of X, define the **Hausdorff semi-distance** $dist_{\mathrm{H}}(A, B)$ of A and B as

$$dist_{\mathrm{H}}(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X,$$

Definition 3.1. Let $t, s, \tau \in \mathbb{R}$. The two-parameter family of operators $U(t, s) : X \to X, t \geq s$, is called an evolution process in X if it satisfies the following properties:

(1) $U(t, s) \circ U(s, \tau) = U(t, \tau), \quad \forall t \geq s \geq \tau;$

(2) $U(t, t) = Id, \quad \forall t \in \mathbb{R}$;

(3) $(t, s, x) \mapsto U(t, s)x$ is continuous.

Evolution processes extend the definition of semigroups.There are different approaches to generalize the notion of global attractors of semigroups to non-autonomous evolution processes (cf. [1, 2, 10, 12, 15, 16, 18]). In this paper we use the notion of so-called pullback attractors.

Definition 3.2. The family of nonempty subsets $\{\mathcal{A}(t)|t \in \mathbb{R}\}$ of X is called a pullback attractor for the process $\{U(t,s)|t \geq s\}$ if $\mathcal{A}(t)$ is compact for all $t \in R$, the family $\{\mathcal{A}(t)|t \in \mathbb{R}\}$ is strictly invariant, that is

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t) \qquad \text{for all } t \geq s,$$

it pullback attracts all bounded subsets of X, that is for every bounded $D \subset X$ and $t \in \mathbb{R}$,

$$\lim_{s\to\infty} dist_H(U(t,t-s)D, \mathcal{A}(t)) = 0,$$

and the family is minimal within the families of closed subsets that pullback attract all bounded subsets of X.

Applying the pullback approach and generalizing the concept of exponential attractors for evolution processes, we obtain the following definition (cf. [6]).

Definition 3.3. Let $\{U(t,\, s)|\, t \geq s\}$ be an evolution process in X. The family of non-empty compact subsets $\mathcal{M} = \{\mathcal{M}(t)|t \in \mathbb{R}\}$ is called a pullback exponential attractor for the evolution process U if

(1) $\mathcal{M}$ is positively invariant, i.e.,

$$U(t,\, s)\mathcal{M}(s) \subset \mathcal{M}(t), \quad \forall t \geq s;$$

(2) the fractal dimension of the sections $\mathcal{M}(t)$, $t \in \mathbb{R}$, is uniformly bounded,

$$\sup_{t\in\mathbb{R}}\{dim_f^X(\mathcal{M}(t))\} < \infty;$$

(3) $\mathcal{M}$ exponentially pullback attracts all bounded sets, i.e., there exists a constant $\omega > 0$ such that for every bounded subset $D \subset X$ and every $t \in \mathbb{R}$

$$\lim_{s\to\infty} e^{\omega s} dist_H(U(t,\, t-s)D,\, \mathcal{M}(t)) = 0.$$

We recall that the fractal dimension of a pre-compact subset $A \subset X$ is defined as

$$dim_f^X(A) = \limsup_{\epsilon\to 0^+} \frac{\ln(N_\epsilon^X(A))}{\ln(\frac{1}{\epsilon})},$$

where $N_\epsilon^X(A)$ denotes the minimal number of ϵ-balls in X with centers in A needed to cover A.

Definition 3.4. A family of nonautonomous sets $\mathcal{B} = \{\mathcal{B}(t) \subset X|t \in \mathbb{R}\}$ is said to be bounded if $\mathcal{B}(t)$ is bounded in X for all $t \in \mathbb{R}$. We say that $\{\mathcal{B}(t)\}$ grows at most sub-exponentially in the past if

$$\lim_{t\to-\infty} diam(\mathcal{B}(t))e^{\gamma t} = 0, \qquad \forall \gamma > 0,$$

where $diam(A)$ denotes the diameter of a subset $A \subset X$.

We slightly generalize the theoretical existence results in [3, 8], where the construction of the exponential attractor was based on the compact embedding of the phase space and an auxiliary normed space. To include the two cases in [3, 8], we require two auxiliary normed spaces satisfying the following property:

(A0) Let W and V be two auxiliary normed spaces such that the embedding $V \hookrightarrow\hookrightarrow W$ is dense and compact and assume that the embedding $V \hookrightarrow X \hookrightarrow W$ is continuous.

Then we don't have to prove that the evolution process satisfies the Lipschitz property in V, which is difficult to verify in some delay problems. The Lipschitz property is only required for the weaker space X, (but which is a more regular space in comparison with W). This generalization is essential when we apply the following abstract result to prove the existence of pullback exponential attractors for some non-autonomous retarded differential equations.

Theorem 3.1. *Let $\{U(t,\, s)|\, t \geq s\}$ be an evolution process in X and* (A0) *be satisfied. We assume that for some $t_0 \in \mathbb{R}$ the following properties are satisfied:*

(A1) *For the process $\{U(t,\, s)|\, t \geq s\}$ there exists a family of bounded pullback absorbing sets $\mathcal{B} = \{\mathcal{B}(t)\}_{t\in\mathbb{R}}$, i.e., for every bounded set $D \subset X$ and $t \leq t_0$, there exists $T_D \geq 0$ such that*

$$U(t,\, t-\tau)D \subset \mathcal{B}(t), \quad \forall \tau \geq T_D.$$

Moreover, there exists $\tilde{t} > 0$ such that

$$U(t, t-\tilde{t})\mathcal{B}(t-\tilde{t}) \subset \mathcal{B}(t) \qquad \forall\, t \geq t_0,$$

and the diameter of the family of absorbing sets $\mathcal{B} = \{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ grows at most sub-exponentially in the past.

(A2) *The evolution process $\{U(t,\, s)|\, t \geq s\}$ satisfies the smoothing property in $\mathcal{B}$, i.e., there exist positive constants $\tilde{t}$ and κ such that*

$$\|U(t, t-\tilde{t})u - U(t, t-\tilde{t})v\|_V \leq \kappa \|u - v\|_W \quad \forall u,\, v \in \mathcal{B}(t-\tilde{t}),\, t \leq t_0.$$

(A3) *The evolution process $\{U(t,\, s)|\, t \geq s\}$ is Lipschitz continuous in $\mathcal{B}$, i.e., for all $t \in \mathbb{R}, t \leq s \leq t+\tilde{t}$, there exists $L_{t,\, s} \geq 0$ such that*

$$\|U(s,t)u - U(s,t)v\|_X \leq L_{t,\, s}\|u - v\|_X \quad \forall u,\, v \in \mathcal{B}(t).$$

Then, for every $\nu \in (0,\, 1/2)$ there exists a pullback exponential attractor $\mathcal{M}^\nu = \{\mathcal{M}^\nu(t)\}$ in X, and the fractal dimension of its section is bounded by

$$dim_f^X(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}}(N^W_{\frac{\nu}{\kappa}}(\mathfrak{B}_V(0,\, 1))), \quad \forall t \in \mathbb{R}.$$

Proof. See Theorem 3.2 and Theorem 3.3 in [3]. Our conditions (A0), (A1) and (A2) immediately imply hypothesis (H0),($\mathcal{H}$1),($\mathcal{H}$2),($\mathcal{H}$3), (A1) and (A2) in [3] for all $t \leq t_0$. In this case, the construction of the pullback exponential attractor is valid for all $t \leq t_0$. Firstly, let $\{U(n\tilde{t}, m\tilde{t})|n \geq m, n, m \in \mathbb{Z}\}$ be the discrete evolution process in X. We can obtain a pullback attractor $\{\mathcal{M}^\nu(k\tilde{t})|t \in \mathbb{Z}\}$, and the fractal dimension of its sections can be estimated by

$$dim_f^X(\mathcal{M}^\nu(k\tilde{t})) \leq \log_{\frac{1}{2\nu}}(N^W_{\frac{\nu}{\kappa}}(\mathfrak{B}_V(0,\, 1))), \quad \text{for all } k \in \mathbb{Z}.$$

Next, using the Lipschitz continuity of the prooerty (A3), we can construct pullback exponential attractor $\{\mathcal{M}^\nu(t)|t \leq t_0\}$ for time continuous processes in X. The proof of above result is similar to that of [3, Theorem 3.2 and Theorem 3.3], and we omit the details here. Finally, thanks to Lipschitz continuity of the prooerty (A3), using similar arguments (see Theorem 4.1 in [8]), the section $\mathcal{M}^{\nu}(t)$ for $t > t_0$ can be defined as

$$\mathcal{M}^{\nu}(t) = U(t, t_0)\mathcal{M}^{\nu}(t_0).$$

This ends the proof. □

Remark 3.1. Theorem 3.1 includes the two cases of [3, 8]. Obviously, if $X = V$, the result is Corollary 2 in [3]; if $X = W$, the result is Theorem 4.1. in [8].

An immediate consequence is the existence and finite dimensionality of the pullback attractor.

Corollary 3.1. *Let $\{U(t,\, s)|\, t \geq s\}$ be an evolution process in a Banach space X. If the hypotheses (A0),(A1),(A2) and (A3) are satisfied, then the evolution process $\{U(t,\, s)|\, t \geq s\}$ possesses a global pullback attractor $\mathcal{A} = \{\mathcal{A}(t)|t \in \mathbb{R}\}$, and the fractal dimension of its sections is bounded by*

$$dim_f^X(\mathcal{A}(t)) \leq \inf_{\nu\in(0,\,\frac{1}{2})}\{\log_{\frac{1}{2\nu}}(N^W_{\frac{\nu}{\kappa}}(\mathfrak{B}_V(0,\,1)))\}, \quad \forall t \in \mathbb{R}.$$

4. Dissipative and pullback exponential attractor

In this section, we present essential conditions on the nonlinearity F to guarantee the dissipation of the equation. Then we consider the existence of pullback exponential attractor. For convenience, we list the following assumption:

(H1) $\|F(t,\, v_0,\, v_1,\, \dots,\, v_n)\| \leq \sum_{i=0}^{n} \beta_i |v_i|_\alpha + K$, $t \in \mathbb{R}$, $(v_0,\, \dots,\, v_n) \in H_\alpha^{n+1}$;

(H2) $\sum_{i=0}^{n} \beta_i < \lambda_1^{1-\alpha}$;

(H3) F is a locally Lipschitz continuous mapping from $\mathbb{R} \times H_\alpha^{n+1} \to H$ for some $0 \leq \alpha < 1$, i.e., $\forall M > 0$, $\exists L_{M,i} \geq 0$ such that for all $t \in \mathbb{R}$ and $v_i,\, w_i \in \mathfrak{B}_{H_\alpha}(0,\, M)$,

$$\|F(t,\, v_0,\, \dots,\, v_n) - F(t,\, w_0,\, \dots,\, w_n)\| \leq \sum_{i=0}^{n} L_{M,i} |v_i - w_i|_\alpha.$$

4.1. well-posedness of the initial value problem

We first discuss the well-posedness of the initial value problem of the nonlinear delay evolution (1.1). Let $r = max\{r_1,\, \dots,\, r_n\}$ and $C_{H_\alpha} = C([-r,\, 0],\, H_\alpha)$ denote the Banach space of continuous functions from $[-r,\, 0]$ into H_α equipped with the maximum norm

$$\|u\|_{C_{H_\alpha}} = \max_{t\in[-r,\,0]} |u(t)|_\alpha.$$

For $u \in C([-r,\, T),\, H_\alpha)$ and $t \in [0,\, T)$, we define $u_t \in C_{H_\alpha}$ by

$$u_t(s) = u(t+s), \qquad s \in [-r,\, 0].$$

For convenience in statement, the function u_t will be referred to as the lifting of u in C_{H_α}.

Consider the initial value problem of the evolution equation with delays

$$\begin{cases} \dfrac{d}{dt}u(t) + Au(t) = F(t,\, u(t),\, u(t-r_1), \ldots,\, u(t-r_n)),\, t \in \mathbb{R}, \\ u_\tau = \varphi, \end{cases} \tag{4.1}$$

where $\varphi \in C_{H_\alpha}$.

Theorem 4.1. *Suppose that $F : \mathbb{R} \times H_\alpha^{n+1} \to H$ be continuous and satisfy condition (H3). Then for any $\varphi \in C_{H_\alpha}$ and $\tau \in \mathbb{R}$, the problem (4.1) has a unique mild solution $u(t) = u(t;\, \tau, \varphi)$ on a maximal interval $[-r, T_\varphi)$, and that is*

$$u(t) \in C([-r+\tau,\, T_\varphi);\, H_\alpha) \cap L^2_{loc}(\tau,\, T_\varphi;\, H_1) \cap C^{0,\,\gamma}(\tau,\, T_\varphi;\, H_\beta),$$

for all $0 \le \beta < 1$, $0 < \gamma < 1$, which can be expressed by

$$u(t) = T(t-\tau)u_\tau + \int_\tau^t T(t-s)F(s,\, u(s),\, u(s-r_1),\, \ldots,\, u(s-r_n))ds. \tag{4.2}$$

Proof. This result can be obtained by combining the proofs of Theorem 3.1 in [17], Lemma 47.1 in [19] and Theorem 42.12 in [19]. Here we omit the proof details and the interested readers can be referred to Theorem 5 in [11]. □

4.2. Decay estimate

Then, we can obtain the following estimate about the problem (4.1) under dissipative conditions $(H1)$ and $(H2)$.

Lemma 4.1. *Assume that $0 \le \alpha \le 1/2$. Let $F : \mathbb{R} \times H_\alpha^{n+1} \to H$ be continuous and satisfy $(H1)$-$(H3)$. Let $u(t) \in C([-r+\tau,\, +\infty);\, H_\alpha) \cap L^2_{loc}(\tau,\, +\infty;\, H_1)$ be a solution of (4.1), then for all $t \ge \tau$, the following estimate holds:*

$$|u(t)|_\alpha^2 \le C_1 e^{-\delta(t-\tau)} \|u_\tau\|^2_{C_{H_\alpha}} + C_2, \tag{4.3}$$

where δ, C_1 and C_2 are positive constants.

Proof. By assumption $(H2)$, we can take $\delta > 0$ small enough such that

$$\lambda_1^{1-\alpha} - \beta_0 - \sum_{i=1}^n e^{\frac{\delta r_i}{2}} \beta_i - \frac{\lambda_1^\alpha + \lambda_1^{-\alpha}}{2}\delta > 0. \tag{4.4}$$

For convenience, denote $l_i = \lambda_1^{-\alpha} e^{\frac{\delta r_i}{2}} (i = 1,\, \ldots,\, n)$. Then we take the inner product in H of the equation (1.1) with $A^{2\alpha}u$, we have

$$\frac{1}{2}\frac{d}{dt}|u(t)|_\alpha^2 + |u(t)|^2_{\frac{1}{2}+\alpha} = (F(t,\, u(t),\, u(t-r_1),\, \ldots,\, u(t-r_n)),\, A^{2\alpha}u(t)).$$

By Hölder's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|_\alpha^2 + |u(t)|^2_{\frac{1}{2}+\alpha} \le \|F(t,\, u(t),\, u(t-r_1),\, \ldots,\, u(t-r_n))\| \cdot \|A^{2\alpha}u(t)\|.$$

Using Poincáre's inequality, $(H1)$ and Young inequality, we have

$$\frac{d}{dt}|u(t)|_\alpha^2 + (2 - 2\lambda_1^{\alpha-1}\beta_0 - \sum_{i=1}^n \beta_i l_i \lambda_1^{2\alpha-1} - \delta\lambda_1^{2\alpha-1})|u(t)|_{\frac{1}{2}+\alpha}^2$$
$$\leq \sum_{i=1}^n \frac{\beta_i}{l_i}|u(t-r_i)|_\alpha^2 + \frac{K^2}{\delta},$$
$$\frac{d}{dt}|u(t)|_\alpha^2 + \lambda_1(2 - 2\lambda_1^{\alpha-1}\beta_0 - \sum_{i=1}^n \beta_i l_i \lambda_1^{2\alpha-1} - \delta\lambda_1^{2\alpha-1})|u(t)|_\alpha^2$$
$$\leq \sum_{i=1}^n \frac{\beta_i}{l_i}|u(t-r_i)|_\alpha^2 + \frac{K^2}{\delta},$$

that is

$$\frac{d}{dt}|u(t)|_\alpha^2 + \delta|u(t)|_\alpha^2 \leq -(2\lambda_1 - 2\lambda_1^\alpha\beta_0 - \sum_{i=1}^n \beta_i l_i \lambda_1^{2\alpha} - \delta\lambda_1^{2\alpha} - \delta)|u(t)|_\alpha^2 + \sum_{i=1}^n \frac{\beta_i}{l_i}|u(t-r_i)|_\alpha^2 + \frac{K^2}{\delta}.$$

Then

$$\frac{d}{dt}(e^{\delta t}|u(t)|_\alpha^2) \leq -(2\lambda_1 - 2\lambda_1^\alpha\beta_0 - \sum_{i=1}^n \beta_i l_i \lambda_1^{2\alpha} - \delta\lambda_1^{2\alpha} - \delta)|u(t)|_\alpha^2 e^{\delta t} + \sum_{i=1}^n \frac{\beta_i}{l_i}|u(t-r_i)|_\alpha^2 e^{\delta t} + \frac{K^2}{\delta}e^{\delta t}.$$

Integrating from τ to t, we obtain

$$|u(t)|_\alpha^2 \leq -2(\lambda_1 - \lambda_1^\alpha\beta_0 - \sum_{i=1}^n \frac{\beta_i l_i}{2}\lambda_1^{2\alpha} - \frac{1+\lambda_1^{2\alpha}}{2}\delta)\int_\tau^t |u(s)|_\alpha^2 e^{-\delta(t-s)}ds + e^{-\delta(t-\tau)}|u_\tau(0)|_\alpha^2 + \sum_{i=1}^n \frac{\beta_i}{l_i}\int_\tau^t |u(s-r_i)|_\alpha^2 e^{-\delta(t-s)}ds + \frac{K^2}{\delta^2}.$$

As

$$\int_\tau^t |u(s-r_i)|_\alpha^2 e^{-\delta(t-s)}ds = \int_{\tau-r_i}^{t-r_i} |u(s)|_\alpha^2 e^{-\delta(t-s-r_i)}ds \leq e^{\delta r_i}\int_\tau^t |u(s)|_\alpha^2 e^{-\delta(t-s)}ds + e^{\delta r_i}\int_{\tau-r_i}^\tau |u(s)|_\alpha^2 e^{-\delta(t-s)}ds,$$

we have

$$|u(t)|_\alpha^2 \leq -2\lambda_1^\alpha(\lambda_1^{1-\alpha} - \beta_0 - \sum_{i=1}^n e^{\frac{\delta r_i}{2}}\beta_i - \frac{\lambda_1^\alpha + \lambda_1^{-\alpha}}{2}\delta)\int_\tau^t |u(s)|_\alpha^2 e^{-\delta(t-s)}ds + e^{-\delta(t-\tau)}|u_\tau(0)|_\alpha^2 + \sum_{i=1}^n \frac{\beta_i}{l_i}e^{\delta r_i}\int_{\tau-r_i}^\tau |u(s)|_\alpha^2 e^{-\delta(t-s)}ds + \frac{K^2}{\delta^2}.$$

which, jointly with (4.4), yields that

$$|u(t)|_{\alpha}^{2} \leq e^{-\delta(t-\tau)}(1+\sum_{i=1}^{n}\frac{1}{\delta}\beta_i\lambda_1^{\alpha}e^{\frac{\delta r_i}{2}})\|u_{\tau}\|_{C_{H_\alpha}}^{2}+\frac{K^2}{\delta^2}.$$

Thus, inequality (4.3) is proved. □

4.3. Existence of pullback exponential attractor

By Definition 3.1 and Theorem 4.1, we can define an evolution process $U(t,s)$: $C_{H_\alpha} \to C_{H_\alpha}$, $t \geq s$,i.e.,

$$U(t,\, s)\varphi = u_t, \qquad \forall\, (s,\, \varphi) \in \mathbb{R} \times C_{H_\alpha},$$

where $u_t(\theta) = u(t+\theta;\, s,\, \varphi)$, $\theta \in [-r,\, 0]$, and $u(\cdot\,;\, s,\, \varphi)$ is the solution of (4.1).

Lemma 4.2. *Let $0 \leq \alpha \leq 1/2$. Assume that (H1)-(H3) hold. Then there exists a bounded uniformly pullback absorbing set $B_1 \subset C_{H_\alpha}$.*

Proof. From Lemma 4.1, we can take

$$\rho_\alpha^2 = 1 + C_2. \tag{4.5}$$

Given any bounded $D \in C_{H_\alpha}$, by (4.3), there exists $T_{H_\alpha}(D) > 0$ such that for any $s \in \mathbb{R}$, $\varphi \in D$,

$$|u(s;\, s-\tau,\, \varphi)|_\alpha^2 \leq \rho_\alpha^2, \qquad \text{for all } \tau > r + T_{H_\alpha}(D).$$

Then for any $t \in \mathbb{R}$ and $\tau > T_{H_\alpha(D)} + r$, we have that for all $\varphi \in D$,

$$\|U(t,\, t-\tau)\varphi\|_{C_{H_\alpha}}^2 = \max_{\theta\in[-r,\,0]}|u_t(\theta)|_\alpha^2 \leq \max_{\theta\in[-r,\,0]}|u(t+\theta;\, t-\tau,\, \varphi)|_\alpha^2 \leq \rho_\alpha^2. \tag{4.6}$$

This means that the closed ball $B_1 = \overline{\mathfrak{B}}_{C_{H_\alpha}}(0,\, \rho_\alpha)$ forms an uniformly absorbing set in C_{H_α}. □

Theorem 4.2. *Let $0 \leq \alpha \leq 1/2$. Assume that $\alpha < \beta < 1$. If F satisfies conditions (H1)-(H3). Then, for every $\nu \in (0,\, \frac{1}{2})$ and $\eta \in [\alpha,\, \beta]$, there exists a pullback exponential attractor $\mathcal{M} = \{\mathcal{M}^\nu(t)\}$ for process $\{U(t,\, s)|t \geq s\}$ in X and the fractal dimension of its section is bounded by*

$$dim_f^X(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}}(N_{\frac{\nu}{\kappa}}^{W}(\mathfrak{B}_V(0,\, 1))), \quad \forall t \in \mathbb{R},$$

where $W = C_{H_\alpha}$, $X = C_{H_\eta}$ and $V = C_{H_\beta}^{0,\,\gamma}$.

Proof. Assume that $W = C_{H_\alpha}$, $X = C_{H_\eta}$ and $V = C^{0,\,\gamma}([-r, 0],\, H_\beta) := C_{H_\beta}^{0,\,\gamma}$. For any $u \in C_{H_\beta}^{0,\,\gamma}$, define its norm by

$$\|u\|_{C_{H_\beta}^{0,\,\gamma}} = \|u\|_{C_{H_\beta}} + [u]_{C_{H_\beta}^{0,\,\gamma}}\,,$$

where $[u]_{C_{H_\beta}^{0,\,\gamma}}$ is $\gamma^{th}-$ Hölder seminorm of u. Since $\alpha < \beta$ and $\alpha \leq \eta \leq \beta$, (A0) is satisfied due to the compact embedding

$$V \hookrightarrow\hookrightarrow W,$$

and the continuously embedding

$$V \hookrightarrow X \hookrightarrow W.$$

(i) Constructing pullback absorbing set (A1): let $t_0 \in \mathbb{R}$ be arbitrary and B_1 forms an uniformly absorbing set in W in Lemma 4.2. A family of bounded pullback absorbing sets is given by

$$\mathcal{B}(t) := \begin{cases} \bigcup_{\tau \geq T_{H_\alpha}(B_1)+3r} U(t,\, t-\tau)(B_1), & \text{for } t \leq t_0, \\ U(t,\, t_0)\mathcal{B}(t_0), & \text{for } t \geq t_0. \end{cases} \tag{4.7}$$

Moreover, the family $\mathcal{B} = \{\mathcal{B}(t) | t \in \mathbb{R}\}$ is positively semi-invariant for the evolution process $\{U(t, s) | t \geq s\}$. For any $u_t \in \mathcal{B}(t)$, there exist $\tau \geq T_{H_\alpha}(B_1) + 3r$ and $\varphi \in B_1$ such that

$$u_t = U(t,\, t-\tau)\varphi.$$

We denote by $u(s) = u(s;\, t-\tau, \varphi)$ the solution of (4.1) with initial value φ, then

$$u(s) = T(s-(t-\tau))\varphi + \int_{t-\tau}^{s} T(s-\sigma)F(\sigma,\, u(\sigma),\, u(\sigma - r_1), \ldots,\, u(\sigma - r_n))d\sigma.$$

If $s - (t-\tau) \geq T_{H_\alpha}(B_1)$, by Lemma 4.2 we have

$$|u(s)|_\alpha \leq \rho_\alpha,$$

then

$$\mathcal{B}(t) \subset B_1. \tag{4.8}$$

Jointly with condition (H_1) and (H_2), for any $s \in [t-2r,\, t]$ we have

$$\begin{aligned} \|F(s,\, u(s),\, u(s-r_1),\, \ldots,\, u(s-r_n))\| &\leq \beta_0 |u(s)|_\alpha + \sum_{i=1}^{n} \beta_i |u(s-r_i)|_\alpha + K \\ &\leq \left(\sum_{i=0}^{n} \beta_i \right) \rho_\alpha + K = \lambda_1^{1-\alpha} \rho_\alpha + K. \end{aligned}$$

Let $u_{t-2r}(0) = u(t-2r)$ be initial value, we have

$$u(s) = T(s-(t-2r))u_{t-2r}(0) + \int_{t-2r}^{s} T(s-\sigma)F(\sigma,\, u(\sigma),\, u(\sigma - r_1), \ldots,\, u(\sigma - r_n))d\sigma.$$

Using (2.2) and (2.3), we can obtain, for any $s \in [t-r,\, t]$,

$$\begin{aligned} &|u(s)|_\beta \\ \leq& \|A^{\beta-\alpha}T(s-t+2r)\|\, |u_{t-2r}(0)|_\alpha \\ &+ \int_{t-2r}^{s} \|A^\beta T(s-\sigma)\| \|F(\sigma,\, u(\sigma,\, u(\sigma),\, u(\sigma - r_1),\, \ldots,\, u(\sigma - r_n))\| d\sigma \\ \leq& M_{\beta-\alpha}(s-t+2r)^{-(\beta-\alpha)} e^{-a(s-t+2r)} \rho_\alpha + (\lambda_1^{1-\alpha}\rho_\alpha + K) \int_{t-2r}^{s} M_\beta (s-\sigma)^{-\beta} d\sigma \\ \leq& M_{\beta-\alpha}\rho_\alpha \sup_{h\in[r,\, 2r]} \{h^{-(\beta-\alpha)} e^{-ah}\} + (\lambda_1^{1-\alpha}\rho_\alpha + K) M_\beta (1-\beta)^{-1} (2r)^{1-\beta} := \rho_\beta. \end{aligned}$$

Hence, $\|u_t\|_{C_{H_\beta}} \le \rho_\beta$, for all $t \in \mathbb{R}$.

There remains to show that u_t is Hölder continuous. Let $0 < \gamma \le \min\{\beta - \alpha,\, 1-\beta\}$. Assume that $s,\, s+\delta \in [t-r,\, t]$, by (4.2), we obtain

$$\begin{aligned}
&|u(s+\delta) - u(s)|_\beta \\
\le &|(T(\delta) - I)T(s-t+2r)u_{t-2r}(0)|_\beta \\
&+ \int_{t-2r}^{s} |(T(\delta)-I)T(s-\sigma)F(\sigma,\, u(\sigma,\, u(\sigma),\, u(\sigma - r_1),\, \ldots,\, u(\sigma - r_n))|_\beta d\sigma \\
&+ \int_{s}^{s+\delta} |T(s+\delta-\sigma)F(\sigma,\, u(\sigma,\, u(\sigma),\, u(\sigma - r_1),\, \ldots,\, u(\sigma - r_n))|_\beta ds \\
:= &J_1 + J_2 + J_3.
\end{aligned}$$

Next we estimate the three terms on the right hand of inequality. we have

$$\begin{aligned}
J_1 &\le \|(T(\delta) - I)T(s-t+2r)A^\beta u_{t-2r}\| \\
&\le K_\gamma \delta^\gamma \|A^{\beta-\alpha+\gamma}T(s-t+2r)\| |u_{t-2r}(0)|_\alpha \\
&\le K_\gamma \delta^\gamma M_{\beta-\alpha+\gamma}(s-t+2r)^{-(\beta-\alpha+\gamma)} e^{-a(s-t+2r)} \rho_\alpha \le C_{J_1}\delta^\gamma, \\
J_2 &\le \int_{t-2r}^{s} K_\gamma \delta^\gamma \|A^{\gamma+\beta}T(s-\sigma)\| d\sigma (\left(\sum_{i=0}^{n} \beta_i\right) \rho_\alpha + K) \\
&\le K_\gamma \delta^\gamma M_{\beta+\gamma} \int_{t-2r}^{s} (s-\sigma)^{-(\gamma+\beta)} e^{-a(s-\sigma)} d\sigma (\left(\sum_{i=0}^{n} \beta_i\right) \rho_\alpha + K) \\
&\le C_{J_2}\delta^\gamma,
\end{aligned}$$

and

$$\begin{aligned}
J_3 &\le \int_{s}^{s+\delta} \|A^\beta T(s+\delta-\sigma)\| d\sigma (\left(\sum_{i=0}^{n} \beta_i\right) \rho_\alpha + K) \\
&\le M_\beta (1-\beta)^{-1} (\left(\sum_{i=0}^{n} \beta_i\right) \rho_\alpha + K) \delta^{1-\beta} \le C_{J_3}\delta^\gamma.
\end{aligned}$$

Comprehensively,

$$|u(t+\delta) - u(t)|_\beta \le (C_{J_1} + C_{J_2} + C_{J_3})\delta^\gamma, \tag{4.9}$$

which implies u_t is γ^{th}–Hölder continuous and

$$[u_t]_{C^{0,\gamma}_{H_\beta}} \le (C_{J_1} + C_{J_2} + C_{J_3}) := \rho_\gamma.$$

Then

$$\|u_t\|_V \le \rho_\beta + \rho_\gamma.$$

For any $t \in \mathbb{R}$, we obtain

$$\mathcal{B}(t) \subset \mathfrak{B}_V(0,\, \rho_\beta + \rho_\gamma) \subset V.$$

(ii) Smoothing property (A2): We show the smoothing property with respect to the space W and V for $\tilde{t} = 2r$. Let $t \le t_0$, for any $\varphi,\, \psi \in \mathcal{B}(t-2r)$, denote by

$$u(s) = u(s;\, t-2r,\, \varphi), \qquad v(s) = v(s;\, t-2r,\, \psi)$$

the solutions of (4.1)with initial value φ, $\psi \in \mathcal{B}(t-2r)$.

Let $w(s) = u(s) - v(s)$, we have

$$w'(s)+Aw(s) = F(s, u(s), u(s-r_1), \ldots, u(s-r_n)) - F(s, v(s), v(s-r_1), \ldots, v(s-r_n)).$$

Taking inner product with $A^{2\alpha}w(s)$ and using Hölder's inequality, (H3), Poincáre's inequality and Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}|w(s)|_\alpha^2 + |w(s)|^2_{\frac{1}{2}+\alpha}\\
\leq&(L_0|w(s)|_\alpha + \sum_{i=1}^n L_i|w(s-r_i)|_\alpha)\cdot|w(s)|_{2\alpha}\\
\leq&\frac{\lambda_1^\alpha}{2}L_0|w(s)|_\alpha^2 + \frac{\lambda_1^\alpha}{2}\sum_{i=1}^n L_i|w(s-r_i)|_\alpha^2 + \frac{\lambda_1^{-\alpha}}{2}\sum_{i=0}^n L_i|w(s)|^2_{2\alpha},
\end{aligned}$$

where L_i is the Lipschitz coefficients of F in the set B_1, which yields that

$$\frac{d}{ds}|w(s)|_\alpha^2 \leq \lambda_1^\alpha(-2\lambda_1^{1-\alpha} + 2L_0 + \sum_{i=1}^n L_i)|w(s)|_\alpha^2 + \sum_{i=1}^n \lambda_1^\alpha L_i|w(s-r_i)|_\alpha^2.$$

Integrating from $t-2r$ to s, we obtain

$$\begin{aligned}
&|w(s)|_\alpha^2 - |w(t-2r)|_\alpha^2\\
\leq&-2\lambda_1^\alpha(\lambda_1^{1-\alpha} - \sum_{i=0}^n L_i)\int_{t-2r}^s |w(\sigma)|_\alpha^2 d\sigma + \sum_{i=1}^n \lambda_1^\alpha L_i \int_{t-2r-r_i}^{t-2r}|w(\sigma)|_\alpha^2 d\sigma,
\end{aligned}$$

that is

$$|w(s)|_\alpha^2 + 2\lambda_1^\alpha(\lambda_1^{1-\alpha} - \sum_{i=0}^n L_i)\int_{t-2r}^s |w(\sigma)|_\alpha^2 d\sigma \leq (1+\sum_{i=1}^n \lambda_1^\alpha L_i r_i)\|\varphi-\psi\|^2_{C_{H_\alpha}}.$$

Then by Gronwall's lemma we have

$$|u(s) - v(s)|_\alpha \leq \gamma_1 e^{-\beta'(s-(t-2r))}\|\varphi - \psi\|_{C_{H_\alpha}} \tag{4.10}$$

where

$$\gamma_1 = (1+\sum_{i=1}^n \lambda_1^\alpha L_i r_i)^{\frac{1}{2}}, \qquad \beta' = \lambda_1^\alpha(\lambda_1^{1-\alpha} - \sum_{i=0}^n L_i).$$

Let

$$L = \max_{h\in[0,\,2r]} \gamma_1 e^{-\beta' h}, \tag{4.11}$$

then

$$|u(s) - v(s)|_\alpha \leq L|\varphi - \psi|_{C_{H_\alpha}}, \qquad \forall\, t-2r \leq s \leq t. \tag{4.12}$$

In view of (4.2), we have

$$\begin{aligned}
&|u(s) - v(s)|_\beta\\
\leq&|T(s-t+2r)(\varphi(0) - \psi(0))|_\beta\\
&+\int_{t-2r}^s |T(s-\sigma)(F(\sigma,\, u(\sigma), \ldots,\, u(\sigma - r_n)) - F(\sigma,\, v(\sigma), \ldots,\, v(\sigma - r_n)))|_\beta d\sigma
\end{aligned}$$

$$:= J_4 + J_5. \tag{4.13}$$

Since F is locally Lipshitz, by (4.12),

$$\begin{aligned}&\|F(\sigma, u(\sigma), u(\sigma-r_1), \ldots, u(\sigma-r_n)) - F(\sigma, v(\sigma), v(\sigma-r_1), \ldots, v(\sigma-r_n)))\| \\ \le&(\sum_{i=0}^{n} L_i)|u(\sigma)-v(\sigma)|_\alpha \le (\sum_{i=0}^{n} L_i)L\|\varphi-\psi\|_{C_{H_\alpha}}.\end{aligned}$$

Next we estimate the two terms on the right hand of inequality. Jointly with (2.2) and (2.3), we have

$$\begin{aligned}J_4 &\le \|A^{\beta-\alpha}T(s-t+2r)\|\,\|\varphi-\psi\|_{C_{H_\alpha}}, \\ &\le M_{\beta-\alpha}(s-t+2r)^{-(\beta-\alpha)}e^{-a(s-t+2r)}\|\varphi-\psi\|_{C_{H_\alpha}},\end{aligned} \tag{4.14}$$

$$\begin{aligned}J_5 &\le (\sum_{i=0}^{n} L_i)L\int_{t-2r}^{s}\|A^\beta T(s-\sigma)\|d\sigma\|\varphi-\psi\|_{C_{H_\alpha}} \\ &\le (\sum_{i=0}^{n} L_i)L\int_{0}^{s-t+2r} M_\beta\sigma^{-\beta}d\sigma\|\varphi-\psi\|_{C_{H_\alpha}}.\end{aligned} \tag{4.15}$$

Let

$$\kappa_1 = \max_{h\in[r,\,2r]} M_{\beta-\alpha}h^{-(\beta-\alpha)}e^{-ah} + (\sum_{i=0}^{n} L_i)L\max_{h\in[r,\,2r]}\int_0^h M_\beta\sigma^{-\beta}d\sigma, \tag{4.16}$$

then

$$\|u_t - v_t\|_{C_{H_\beta}} \le \kappa_1\|\varphi-\psi\|_{C_{H_\alpha}} \tag{4.17}$$

There remains to show that $[u_t - v_t]_{C^{0,\gamma}_{H_\beta}}$ Hölder semi-norm is bounded. Assume that $s,\, s+\delta\in[t-r,\, t]$, we have

$$\begin{aligned}&|(u(s+\delta)-v(s+\delta))-(u(s)-v(s)))|_\beta \\ \le&|(T(\delta)-I)T(s-t+2r)(\varphi(0)-\psi(0)|_\beta \\ &+\int_{t-2r}^{s}|(T(\delta)-I)T(s-\sigma)(F(\sigma,\ldots,u(\sigma-r_n))-F(\sigma,\ldots,v(\sigma-r_n)))|_\beta d\sigma \\ &+\int_{s}^{s+\delta}|T(s+\delta-\sigma)(F(\sigma,u(\sigma),\ldots,u(\sigma-r_n))-F(\sigma,v(\sigma),\ldots,v(\sigma-r_n)))|_\beta d\sigma \\ :=&J_6+J_7+J_8.\end{aligned}$$

Next we estimate the three terms on the right hand of inequality. Using (2.2) and (2.3), we have

$$\begin{aligned}J_6 &\le \|(T(\delta)-I)A^{\beta-\alpha+\gamma}T(s-t+2r)\|\,\|\varphi-\psi\|_{C_{H_\alpha}} \\ &\le K_\gamma\delta^\gamma M_{\beta-\alpha+\gamma}(s-t+2r)^{-(\beta-\alpha+\gamma)}e^{-a(s-t+2r)}\|\varphi-\psi\|_{C_{H_\alpha}}, \\ J_7 &\le (\sum_{i=0}^{n} L_i)L\int_{t-2r}^{s}\|(T(\delta)-I)A^\beta T(s-\sigma)\|d\sigma\|\varphi-\psi\|_{C_{H_\alpha}}\end{aligned}$$

$$\leq K_\gamma M_{\beta+\gamma}(\sum_{i=0}^n L_i)L\int_{t-2r}^s (s-\sigma)^{-(\beta+\gamma)}e^{-a(s-\sigma)}d\sigma\delta^\gamma\|\varphi-\psi\|_{C_{H_\alpha}},$$

$$J_8 \leq (\sum_{i=0}^n L_i)L\int_s^{s+\delta}\|A^\beta T(s+\delta-\sigma)\|d\sigma\|\varphi-\psi\|_{C_{H_\alpha}}$$

$$\leq M_\beta(\sum_{i=0}^n L_i)L(1-\beta)^{-1}r^{1-\beta-\gamma}\delta^\gamma\|\varphi-\psi\|_{C_{H_\alpha}}.$$

Let

$$\kappa_2 = K_\gamma M_{\beta-\alpha+\gamma}r^{-(\beta-\alpha+\gamma)} + K_\gamma M_\beta(\sum_{i=0}^n L_i)L(2r)^{1-\beta} + M_\beta(\sum_{i=0}^n L_i)Lr^{1-\beta-\gamma},$$

then

$$\sup_{\theta,\,\theta+\delta\in[-r,0],\,\delta>0}\frac{|(u_t(\theta+\delta)-v_t(\theta+\delta))-(u_t(\theta)-v_t(\theta))|_\beta}{\delta^\gamma} \leq \kappa_2\|\varphi-\psi\|_{C_{H_\alpha}}.$$

Let $\kappa=\kappa_1+\kappa_2$, we have the smoothing property, i.e.,

$$\|U(t,\,t-2r)\varphi-U(t,\,t-2r)\psi\|_V \leq \kappa\|\varphi-\psi\|_W, \qquad \forall\,\varphi,\,\psi\in B(t-2r),\; t\leq t_0.$$

(iii) Lipschitz continuity (A3): By Theorem 4.1 and the proof of (i), $\{U(t,\,s)|t\geq s\}$ is an evolution process in C_{H_η}, too. For any bounded set$D\subset C_{H_\eta}$, D is bounded in C_{H_α}. Consequently, $\mathcal{B}$ is absorbing set for $\{U(t,\,s),\,t\geq s\}$ in C_{H_η} from the proof of (i). For any $t\in\mathbb{R}$ and $s\in[t,\,t+2r]$, denote by

$$u(s)=u(s;\,t,\,u_t), \qquad v(s)=v(s;\,t,\,v_t)$$

the solutions of (4.1)with initial value $u_t,\,v_t\in\mathcal{B}(t)$.

The rest of the proof follows from the estimate in (ii) by replacing β with η and $t-2r$ with t. In fact, by (4.13),(4.14) and (4.15), we obtain

$$\begin{aligned}
&|u(s)-v(s)|_\eta\\
\leq&|T(s-t)(u_t(0)-v_t(0))|_\eta\\
&+\int_t^s|T(s-\sigma)(F(\sigma,\,u(\sigma),\ldots,\,u(\sigma-r_n))-F(\sigma,\,v(\sigma),\ldots,\,v(\sigma-r_n)))|_\eta d\sigma\\
:=&J_4'+J_5',\\
&J_4'\leq\|T(s-t)\|\,\|u_t-v_t\|_{C_{H_\eta}},\\
&\leq e^{-\lambda_1(s-t)}\|u_t-v_t\|_{C_{H_\eta}},
\end{aligned}$$

and

$$J_5' \leq (\sum_{i=0}^n L_i)L\int_t^s\|A^\eta T(s-\sigma)\|d\sigma\|u_t-v_t\|_{C_{H_\alpha}}$$

$$\leq (\sum_{i=0}^n L_i)L\lambda_1^{\alpha-\eta}\int_t^s M_\eta(s-\sigma)^{-\eta}e^{-a(s-\sigma)}d\sigma\|u_t-v_t\|_{C_{H_\eta}}$$

$$\leq (\sum_{i=0}^{n} L_i) L\lambda_1^{\alpha-\eta} M_\eta (1-\eta)^{-1} (s-t)^{1-\eta} \|u_t - v_t\|_{C_{H_\eta}}.$$

Let

$$L_{s,t} = 1 + (\sum_{i=0}^{n} L_i) L\lambda_1^{\alpha-\eta} M_\eta (1-\eta)^{-1} \max_{t\leq s\leq t+2r} (s-t)^{1-\eta}.$$

We have

$$|u(s) - v(s)|_\eta \leq L_{s,\,t} \|u_t - v_t\|_{C_{H_\eta}}, \qquad \forall\, t \leq s \leq t + 2r, \tag{4.18}$$

then

$$\|U(s,t)u_t - U(s,\,t)v_t\|_X \leq L_{s,\,t}\|u_t - v_t\|_X, \qquad \forall u_t,\, v_t \in \mathcal{B}(t).$$

(iv)Existence of pullback exponential attractor: By Theorem 3.1, we can obtain the existence of pullback exponential attractor for the process $\{U(t,\,s),\, t \geq s\}$ in $X = C_{H_\eta}$.

Then, for every $\nu \in (0,\, \frac{1}{2})$ there exists a pullback exponential attractor $\mathcal{M} = \{\mathcal{M}^\nu(t)\}$ for $\{U(t,\,s),\, t \geq s\}$ in X and the fractal dimension of its section is bounded by

$$dim_f^X(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}}(N^W_{\frac{\nu}{\kappa}}(\mathfrak{B}_V(0,\,1))), \quad \forall t \in \mathbb{R}.$$

The proof is complete. □

5. An Example

We now give an example to demonstrate how the abstract results in previous sections can be applied to nonautonomous parabolic equations with delays.

Let L be a differential operator on a bounded domain $\Omega \subset \mathbb{R}^N$,

$$Lu := -\sum_{i,\,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u,$$

where $a_{ij}, a_0 \in L^\infty(\Omega)$, and $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq N$. We assume that there exists a constant $\nu > 0$ such that for a.e. $x \in \Omega$,

$$\sum_{i,\,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \qquad \forall \xi = (\xi_1,\, \ldots,\, \xi_N) \in \mathbb{R}^N,$$

and that

$$a_0(x) \geq 0, \qquad a.e.\ x \in \overline{\Omega}.$$

Hence L is uniformly elliptic on Ω.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitz continuous function. Consider the retarded parabolic equation on Ω:

$$\frac{\partial u}{\partial t} + Lu = f\left(u(x,\, t - r_1), \nabla u(x,\, t - r_2)\right) + g(x,\, t) \tag{5.1}$$

associated with the homogeneous Dirichlet boundary condition:

$$u|_{\partial\Omega} = 0, \tag{5.2}$$

where r_1, $r_2 > 0$ denote time lags.

Let $H = L^2(\Omega)$. Define a symmetric bilinear form $B(u, v)$ on $H_0^1(\Omega)$ as follows:

$$B(u, v) = \int_\Omega \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} + a_0(x) uv \right) dx, \qquad u, v \in H_0^1(\Omega).$$

It is clear that $B(u, v)$ is bounded and coercive. Thanks to the Lax-Milgram Theorem, $B(u, v)$ generates a self-adjoint positive-definite operator A on H with compact resolvent. Note that

$$H_0^1(\Omega) = D(A^{1/2}) := H_{1/2}.$$

We assume f satisfies the following linear growth condition:

(F1) There exist positive constants $b_0, \ldots, b_N$ and k , such that

$$|f(z)| \le \sum_{i=0}^N b_i |z_i| + k, \qquad \forall\, z = (z_0, \ldots, z_N) \in \mathbb{R}^{N+1}.$$

Then one easily sees that the mapping $F : {H_{1/2}}^2 \to H$ defined by

$$F(u, v) = f\,(u, \nabla v)\,, \qquad u, v \in H_{1/2}$$

makes sense and is locally Lipschitz. Setting $g(t) = g(\cdot, t)$, the problem (5.1)-(5.2) can be reformulated as an abstract equation in H as follows:

$$\frac{du}{dt} + Au = F(u(t - r_1), \nabla u(t - r_2)) + g(t). \tag{5.3}$$

Now we are in a situation of the equation (1.1).

If we impose on f appropriate conditions, then one can easily verifies that the mapping F satisfies (H1) and (H2) in the previous sections, and hence the abstract results obtained therein apply. In particular, we have

Theorem 5.1. *Let $\frac{1}{2} < \beta < 1$ and $\gamma = \min\{1 - \beta, \beta - \frac{1}{2}\}$. Assume that f is local Lipschitz and satisfies the linear growth condition (F1) with the positive constants b_i's therein satisfying*

$$b_0 \lambda_1^{-1/2} + \sum_{i=1}^N b_i < \lambda_1^{1/2}, \tag{5.4}$$

where λ_1 is the first eigenvalue of A. Let $g \in C_b(\mathbb{R}; H)$.

Then the following assertions hold:

(1) For every $\nu \in (0, \frac{1}{2})$ and $\eta \in [\frac{1}{2}, \beta]$, the equation (5.1) has pullback exponential attractor $\mathcal{M}^\nu = \{\mathcal{M}^\nu(t)\}_{t \in \mathbb{R}}$ in X, and the fractal dimension of its section is bounded by

$$dim_f^X(\mathcal{M}^\nu(t)) \le \log_{\frac{1}{2\nu}}(N^W_{\frac{\nu}{\kappa}}(\mathfrak{B}_V(0, 1))), \quad \forall t \in \mathbb{R},$$

where $W = C_{H_{1/2}}$, $X = C_{H_\eta}$ and $V = C^{0,\gamma}_{H_\beta}$.

(2) The equation (5.1) has pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$, and the fractal dimension of its sections is bounded by

$$dim_f^X(A(t)) \le \inf_{\nu \in (0, \frac{1}{2})} \{\log_{\frac{1}{2\nu}}(N^W_{\frac{\nu}{\kappa}}(\mathfrak{B}_V(0, 1)))\}, \quad \forall t \in \mathbb{R}.$$

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