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MORE EARLY PEAKON MODEL THAN CAMASSA-HOLM EQUATION: BIFURCATIONS AND DYNAMICAL BEHAVIORS OF TRAVELING WAVE SOLUTIONS FOR KUPERSHMIDT'S COUPLED KDV SYSTEM*

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Abstract This paper considers the traveling wave solutions of Kupershmidt's multicomponent Korteweg-de Vries system derived in 1985. Exploiting the bifurcation theory of planar dynamical systems, we analyze the dynamical behaviors and the bifurcations, and also give all the explicit parametric expressions of solutions when parameters vary. We find that Kupershmidt's model has peakon solutions. This implies that this model is the more early peakon one than Camassa-Holm equation.

Keywords Bifurcation, peakon, solitary wave solution, multicomponent Korteweg-de Vries system.

MSC(2010) 34C23, 35Q51, 35Q53.

1. Introduction

Kupershmidt [11] in 1985 proposed a multicomponent Korteweg-de Vries system with dispersion, which is described as

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x + 2v^T v_x + C^T v_{xx}, \\ v_t &= (2uv)_x - u_{xx}C \end{aligned} \tag{1.1}$$

for $u \in \mathbf{R}$ and $v = (v_1, v_2, \cdots, v_n)^T \in \mathbf{R}^n$ where $C = (c_1, c_2, \cdots, c_n)^T \in \mathbf{R}^n$ is a constant (column) vector. Kupershmidt [11] also proved that system (1.1) is a bi-Hamiltonian system with an infinite number of conservation laws.

If taking $n = 1$, $C = 0$ and applying the transformation $(u, t) \to -(u, t)$, then we can transform system (1.1) into

$$u_t = u_{xxx} + 6uu_x + 2vv_x, \qquad v_t = 2(uv)_x, \tag{1.2}$$

which was posed by Ito [10] from the interaction process of two internal long waves. Using a recursion operator, Ito [10] pointed out that system (1.2) has infinitely

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many symmetries and constants of motion. By a priori estimate, Guo and Tan [9] investigated the existence of global smooth solutions with the initial values. The dynamical behaviours of traveling wave solutions for system (1.2) were also discussed in [16].

It is worth mentioning that after this there is a famous model

$$u_t + 2\kappa u_x + \gamma u_{xxx} - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.3}$$

called Camassa-Holm equation, which was presented by Camassa and Holm [4] in 1993 when the incompressible Euler equations with small amplitude shallow-water waves were considered. With the aid of the Lax pair, [5] proved that equation (1.3) is completely integrable and has a bi-Hamiltonian structure. [6] dealt with the dispersionless case ($\kappa \to 0$) and found that equation (1.3) has unusual nonsmooth solutions called peaked solitons or peakons. For example, the single peakon has a form $u(x,t) = ce^{-|x-ct|}$.

Both systems (1.1) and (1.3) are derived from the water wave problems. Does the more early Kupershmidt's coupled KdV integrable model (1.1) have peakon solutions? In this paper, we will use the dynamical theory to rigorously prove this.

Assume that the traveling wave solutions of system (1.1) have the following form of

$$u(x,t) = \phi(x-ct) = \phi(\xi), \quad v_i(x,t) = v_i(x-ct) = v_i(\xi), \quad \xi = x - ct, \tag{1.4}$$

where c is an arbitrary constant. Plugging (1.4) into the second equation of system (1.1) and differentiating it, we have

$$v_i = \frac{c_i \phi_\xi}{c+2\phi}, \quad (v_i)_\xi = \frac{c_i\phi_{\xi\xi}(c+2\phi) - 2c_i\phi_\xi^2}{(c+2\phi)^2}, \quad i = 1, 2, \cdots, n, \tag{1.5}$$

where the integral constant is set equal to zero.

Inserting (1.5) into the first equation of system (1.1) and again integrating it, we have

$$(c+2\phi)(c+2\phi-a^2)\phi_{\xi\xi} = -a^2\phi_\xi^2 + (c+2\phi)^2(3\phi^2 + c\phi + g), \tag{1.6}$$

where $a^2 = c_1^2 + c_2^2 + \cdots + c_n^2$ and g is the integral constant. It is clear that system (1.6) is equivalent to the following system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-a^2y^2 + (c+2\phi)^2(3\phi^2 + c\phi + g)}{(c+2\phi)(c+2\phi-a^2)}, \tag{1.7}$$

whose first integral is

$$H(\phi, y) = \frac{(c+2\phi-a^2)y^2}{c+2\phi} - (2g\phi + c\phi^2 + 2\phi^3) = h, \tag{1.8}$$

where h is a constant.

Note that system (1.7) is a singular one of the first class (see [7, 8, 12, 13]) and the singular straight lines are $\phi = \phi_{s1} = -\frac{1}{2}c$ and $\phi = \phi_{s2} = \frac{1}{2}(a^2 - c)$. We will show that when the periodic solutions of (1.7) in a periodic annulus have segments close to $\phi = \phi_{s2}$, their limits are a periodic cusp wave solution (called periodic peakon). Meanwhile, the boundary curve of this periodic annulus corresponds to a heteroclinic triangle orbit of system (2.1), which yields a solitary cusp wave solution (called peakon). Note that the singular system (1.7) also has pseudopeakon and compacton solutions besides peakons and periodic peakons. As pointed out in [14,15], a periodic peakon consists of a classical solution related to two time scales in a singular traveling wave system while a peakon is a limiting solution of periodic peakons or pseudopeakons.

The main result of this paper is as follows.

Theorem 1.1. *Assume that the parameter pair (a^2, c) of system (1.7) is fixed. Then,*

(1) *By varying the parameter $g \in (-\infty, \infty)$, system (1.7) has the bifurcations of phase portraits given in (a)-(h) of Figure 1.*

(2) *For $g = g_s = \frac{ca^2}{4} - \frac{3a^4}{16}$, corresponding to the triangle orbits in (d) of Figure 1 and (b) of Figure 2, system (1.1) has a peakon solution $u(x,t) = \phi(x-ct) = \phi_{s1} + (\phi_1 - \phi_{s1})\mathrm{ctnh}^2_{q_0}(\omega_1|x-ct|)$ defined by (3.4).*

In addition, system (1.1) has a solitary wave solution defined by (3.3).

(3) *For $g_s < g < \infty$, system (1.1) has a periodic peakon wave solution defined by (3.2).*

For $g_m = \frac{1}{4}(g_s - c^2) < g < g_s$, system (1.1) has a periodic peakon wave solution defined by (3.6).

(4) *For $g = \frac{1}{12}c^2$, system (1.1) has a solitary wave solution defined by (3.7).*

For $g_s < g < \frac{1}{12}c^2$, system (1.1) has two solitary wave solutions defined by (3.8) and (3.9).

For $-\frac{1}{4}c^2 < g < g_s$, system (1.1) has a solitary wave solution defined by (3.10).

(5) *For $g_m < g < \frac{1}{12}c^2$, system (1.1) has a periodic wave solution defined by (3.11).*

For $g = g_m$, system (1.1) has a periodic wave solution defined by (3.12).

For $-\infty < g < g_m$, system (1.1) has a periodic wave solution defined by (3.13).

The proof of this theorem will be provided in next sections.

This paper is organized as follows. In section 2, we consider the bifurcations of phase portraits of system (1.7). Section 3 gives all possible explicit expressions of the solution $\phi(\xi)$ when the parameters vary.

2. Bifurcations of phase portraits of system (1.7)

It is clear that the singular system (1.7) can be changed into

$$\frac{d\phi}{d\zeta} = y(c+2\phi)(c+2\phi-a^2), \quad \frac{dy}{d\zeta} = -a^2y^2 + (c+2\phi)^2(3\phi^2 + c\phi + g), \tag{2.1}$$

where $d\xi = (c+2\phi)(c+2\phi-a^2)d\zeta$, for $\phi \neq \phi_{s1} = -\frac{1}{2}c$ and $\phi \neq \phi_{s2} = \frac{a^2-c}{2}$. Systems (2.1) and (1.7) have the same first integral, but in the phase plane, they give different vector fields on two sides of the singular straight lines.

Obviously, system (2.1) has the equilibrium points $E_1(\phi_1, 0)$, $E_2(\phi_2, 0)$ and $E_s(\phi_{s1}, 0)$ where $\phi_{1,2} = \frac{1}{6}(-c \mp \sqrt{\Delta})$ and $\Delta = c^2 - 12g > 0$. When $F_s = 3a^4 - 4ca^2 + c^2 + 4g > 0$, on the singular line $\phi = \phi_{s2} = \frac{a^2-c}{2}$, there are two more equilibrium points $S_{\mp}(\phi_{s2}, \mp y_s)$, where $y_s = \frac{a}{2}\sqrt{F_s}$. Clearly, when $\Delta < 0$, we have $F_s > 0$.

It is easy to check that $E_s\left(-\frac{1}{2}c,0\right)$ is degenerate. To justify the directions that the orbits of system (2.1) approach E_s as $\zeta \to \pm\infty$, we see that at E_s, system (2.1) has the form

$$\frac{d\psi}{d\zeta} = -2a^2y\psi + h.o.t., \quad \frac{dy}{d\zeta} = (4g+c^2)\psi^2 - a^2y^2 + h.o.t., \tag{2.2}$$

where $\psi = \phi + \frac{1}{2}c$ and $h.o.t.$ denotes the high order terms. We know from (2.2) that $G(\theta) = \cos\theta[(4g+c^2)\cos^2\theta + a^2\sin^2\theta] = 0$ has solutions $\theta_{1,2} = \frac{\pi}{2}, \frac{3\pi}{2}$ and for $g < -\frac{1}{4}c^2$, $\theta_{3,4} = \pm\arctan\left(\frac{1}{a}\sqrt{|4g+c^2|}\right)$. It is easy to prove that for $g > -\frac{1}{4}c^2$, on the both sides of $\phi = -\frac{1}{2}c$, there are two elliptic areas of orbits of system (2.1). In addition, we see from (1.8) that for all $h \in (-\infty,\infty), y^2 = \frac{(h+2g\phi+c\phi^2+2\phi^3)(c+2\phi)}{(c+2\phi-a^2)}$. When $\phi = -\frac{1}{2}c$, we have $y = 0$. It means that for every given h, there is a branch of the level curves given by $H(\phi,y) = h$, which touches the line $\phi = -\frac{1}{2}c$ at $E_s\left(-\frac{1}{2}c,0\right)$. Thus, using the results in [13], we obtain that these uncountably infinitely many elliptic orbits form two families of periodic solutions of (1.7).

Let $M(\phi_j,0)$ be the coefficient matrix of the linearized system of (2.1) at the equilibrium point $(\phi_j,0)$. Obviously,

$$\begin{aligned} J(\phi_j,0) &= \det M(\phi_j,0) = \pm(c+2\phi_j)^3(c+2\phi_j-a^2)\sqrt{\Delta}, \\ J(\phi_{s2},\pm y_s) &= \det M(\phi_{s2},\pm y_s) = -4a^4y_s^2 < 0. \end{aligned}$$

Applying the theory of the planar dynamical systems, we can justify that the equilibrium point is a center point or a saddle point.

For the first integral given by (1.8), we write that as

$$h_1 \triangleq H(\phi_1,0) = \frac{1}{54}(-\Delta^{\frac{3}{2}} - c + 18cg), \quad h_2 \triangleq H(\phi_2,0) = \frac{1}{54}(\Delta^{\frac{3}{2}} - c + 18cg)$$

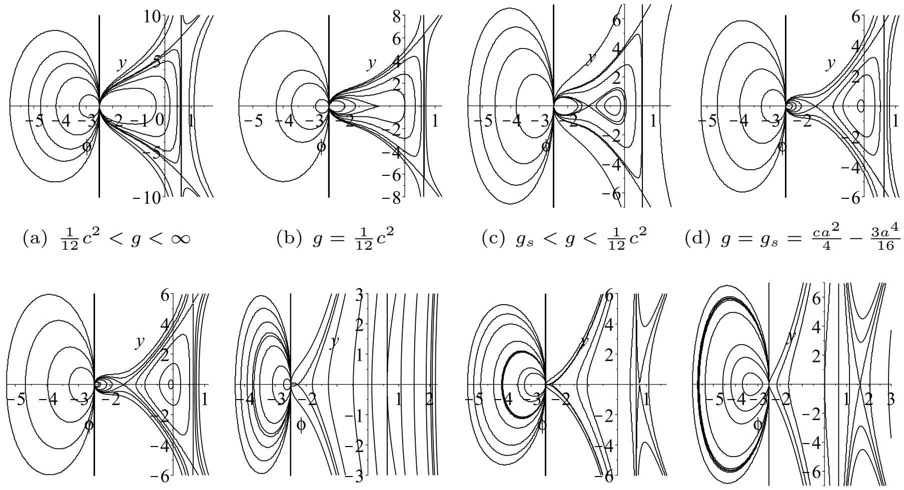
and

$$h_s \triangleq H(\phi_{s2},\mp y_s) = -\frac{1}{4}(a^2-c)(a^4-ca^2+4g).$$

For a fixed pair (a^2,c), when $g = g_s = \frac{1}{4}ca^2 - \frac{3}{16}a^4$, we have $h_2 = h_s$; When $g = g_m = -\frac{1}{4}c^2 - \frac{3}{4}a^4 + ca^2$, we have $\phi_2 = \phi_{s2}$ and $h_2 = h_s$; When $g = -\frac{1}{4}c^2$, we have $\phi_1 = \phi_{s1}$.

For the fixed parameter pair (a^2,c) of the three-parameter system (1.7) with $c > 0$, by changing the parameter g from ∞ to $-\infty$, i.e. by varying the corresponding positions of the points $(\phi_j,0)$, $(\phi_{s1},0)$ and $(\phi_{s2},0)$, we obtain the bifurcations of the phase portraits of system (1.7) as (a)-(h) of Figure 1.

We see from Figure 1 that for all $g \in (-\infty,\infty)$, there is a global closed orbit family of system (1.7) which touches the line $\phi = \phi_{s1}$ from the left. It means that system (1.7) always has a global family of periodic solutions.



(a) $\frac{1}{12}c^2 < g < \infty$ (b) $g = \frac{1}{12}c^2$ (c) $g_s < g < \frac{1}{12}c^2$ (d) $g = g_s = \frac{ca^2}{4} - \frac{3a^4}{16}$

(e) $g_m < g < g_s$ (f) $g = g_m = \frac{1}{4}(g_s - c^2)$ (g) $-\frac{1}{4}c^2 < g < g_m$ (h) $-\infty < g < -\frac{1}{4}c^2$

Figure 1. Bifurcations of phase portraits of system (1.7) for a fixed pair (a^2, c). (a) Bifurcation parameter group $(a^2, c) = (6.25, 5)$.

3. Explicit peakons, periodic peakons, solitary wave solutions and periodic wave solutions of system (1.1)

Notice that (1.8) gives $y^2 = \frac{(h+2g\phi+c\phi^2+2\phi^3)(c+2\phi)}{(c+2\phi-a^2)}$. The first equation of (1.7) implies that

$$\sqrt{2}\xi = \int_{\phi_0}^{\phi} \frac{(\phi_{s2}-\tau)d\tau}{\sqrt{-\left(\frac{1}{2}h + g\tau + \frac{1}{2}c\tau^2 + \tau^3\right)(\phi_{s2}-\tau)(\tau-\phi_{s1})}} \triangleq \int_{\phi_0}^{\phi} \frac{(\phi_{s2}-\tau)d\tau}{\sqrt{F(\tau)}}. \tag{3.1}$$

Generally, the function $F(\phi)$ is a fifth polynomial, and we can not calculate this integral. For some spacial level curves defined in (1.8), by using (3.1), we can get some explicit expressions of traveling wave solutions of system (1.1).

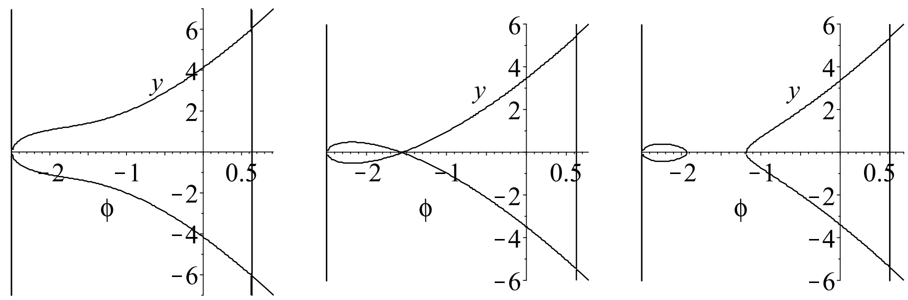
3.1. Explicit solutions determined by $H(\phi, y) = h_s$

When $h = h_s$, the level set $H(\phi, y) = h_s$ is given in (a)-(c) of Figure 2.

(i) Consider $H(\phi, y) = h_s$ in (a) of Figure 2. The polynomial $F(\phi)$ in (3.1) can be written as $F(\phi) = (\phi_{s2}-\phi)^2(\phi-\phi_{s1})(\phi-b)(\phi-\bar{b}) = (\phi_{s2}-\phi)^2(\phi-\phi_{s1})[(\phi-b_1)^2+a_1^2]$, where $a_1^2 = (\mathrm{Im}(b))^2, b_1 = \mathrm{Re}(b)$ and

$$a_1^2 = \frac{3a^4}{16} - \frac{a^2c}{4} + g, \quad b_1 = -\frac{a^2}{4}.$$

Now, (3.1) becomes that $\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{d\tau}{\sqrt{(\tau-\phi_{s1})[(\tau-b_1)^2+a_1^2]}}$.



(a) $g_s < g < \infty$ (b) $g = g_s$ (c) $g_m < g < g_s$

Figure 2. The level curve $H(\phi, y) = h_s$ for system (1.7)

Thus, (3.1) gives rise to the following explicit expression of a periodic peakon solution of system (1.1) (see (a) of Figure 3)

$$\phi(\xi) = (\phi_{s1} - A_1) + \frac{2A_1}{1 + \mathrm{cn}(\sqrt{2A_1}\xi, k))}, \quad \xi \in (-\xi_{01}, \xi_{01}), \tag{3.2}$$

where $A_1^2 = (b_1 - \phi_{s1})^2 + a_1^2, k^2 = \frac{A_1 + b_1 - \phi_{s1}}{2A_1}, \xi_{01} = \frac{1}{\sqrt{2A_1}}\mathrm{cn}^{-1}\left(\frac{2A_1}{\phi_{s2} - \phi_{s1} + A_1} - 1\right)$, and $\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$ are Jacobian elliptic functions given in [3].

(ii) Consider $H(\phi, y) = h_s$ in (b) of Figure 2. We obtain a homoclinic orbit tending to $E_1(\phi_1, 0)$ and a heteroclinic triangle surrounding the center point $E_2(\phi_2, 0)$. The polynomial $F(\phi)$ in (3.1) has a form $F(\phi) = (\phi_{s2} - \phi)^2(\phi - \phi_1)^2(\phi - \phi_{s1})$. By (3.1), the homoclinic orbit has a form $\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{d\tau}{(\phi_1 - \tau)\sqrt{\tau - \phi_{s1}}}$. Hence, we obtain its expression

$$\phi(\xi) = \phi_{s1} + (\phi_1 - \phi_{s1}) \tanh^2\left(\sqrt{2(\phi_1 - \phi_{s1})}\xi\right), \tag{3.3}$$

which yields a solitary wave solution of system (1.1) (see (c) of Figure 3).

For the heteroclinic triangle, using $\sqrt{2}\xi = \int_{\phi}^{\phi_{s2}} \frac{d\tau}{(\phi_1 - \tau)\sqrt{\tau - \phi_{s1}}}$, we obtain the explicit expression of the peakon solution (see (b) of Figure 3)

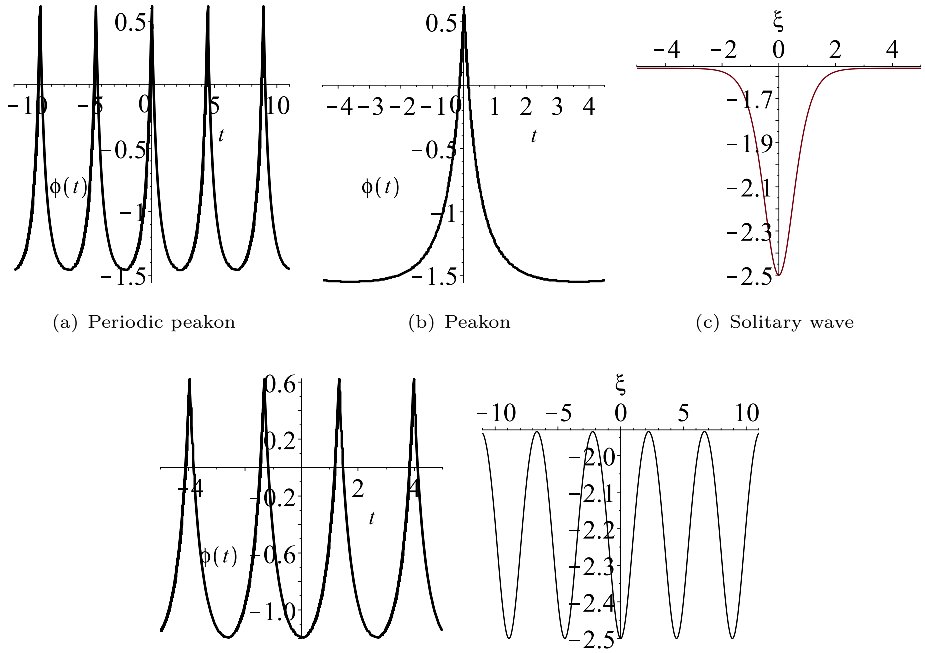
$$\begin{aligned} \phi(\xi) &= \phi_{s1} + (\phi_1 - \phi_{s1})\left(\frac{e^{\omega_1|\xi|} + q_0 e^{-\omega_1|\xi|}}{e^{\omega_1|\xi|} - q_0 e^{-\omega_1|\xi|}}\right)^2 \equiv \phi_{s1} + \frac{\phi_1 - \phi_{s1}}{\tanh_{q_0}^2(\omega_1|\xi|)} \\ &= \phi_{s1} + (\phi_1 - \phi_{s1})\mathrm{ctnh}_{q_0}^2(\omega_1|\xi|), \end{aligned} \tag{3.4}$$

where $q_0 = \frac{\sqrt{\phi_{s2} - \phi_{s1}} - \sqrt{\phi_1 - \phi_{s1}}}{\sqrt{\phi_{s2} - \phi_{s1}} + \sqrt{\phi_1 - \phi_{s1}})}, \omega_1 = \sqrt{\frac{\phi_1 - \phi_{s1}}{2}}, \tanh_{q_0}(\xi), \mathrm{ctnh}_{q_0}^2(\xi)$ are the Arai q-deformed functions (see [1, 2]).

(iii) Consider $H(\phi, y) = h_s$ in (c) of Figure 2. We get a periodic orbit touching $E_s(\phi_{s1}, 0)$ and a heteroclinic arch surrounding the center point $E_2(\phi_2, 0)$. The polynomial $F(\phi)$ in (3.1) becomes $F(\phi) = (\phi_{s2} - \phi)^2(\phi - \phi_r)(\phi - \phi_l)(\phi - \phi_{s1})$ where

$$\phi_l = \frac{1}{4}\left(-a^2 - \sqrt{-3a^4 + 4a^2c - 16g}\right), \quad \phi_r = \frac{1}{4}\left(-a^2 + \sqrt{-3a^4 + 4a^2c - 16g}\right).$$

Therefore, the periodic orbit has a form $\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{d\tau}{\sqrt{(\phi_r - \tau)(\phi_l - \tau)(\tau - \phi_{s1})}}$. Thus, the expression of the periodic wave solution for system (1.1) is the following (see (e) of



(a) Periodic peakon (b) Peakon (c) Solitary wave

(d) Periodic peakon (e) Periodic wave

Figure 3. The wave profiles related to the level curves in Figure 2

Figure 3):

$$\phi(\xi) = \phi_{s1} + (\phi_l - \phi_{s1})\mathrm{sn}^2(\omega_2\xi, k), \tag{3.5}$$

where $\omega_2 = \sqrt{\frac{\phi_r-\phi_{s1}}{2}}, k^2 = \frac{\phi_l-\phi_{s1}}{\phi_r-\phi_{s1}}$.

The arch orbit has a form $\sqrt{2}\xi = \int_{\phi_r}^{\phi} \frac{d\tau}{\sqrt{(\tau-\phi_r)(\tau-\phi_l)(\tau-\phi_{s1})}}$. It yields the periodic peakon solution of system (1.1) (see (d) of Figure 3)

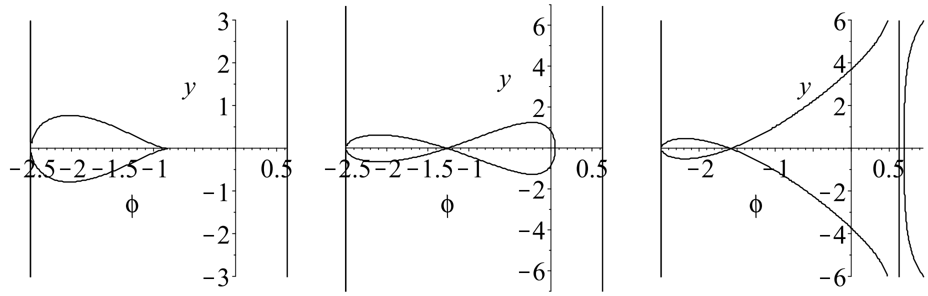
$$\phi(\xi) = \phi_r + \frac{(\phi_r - \phi_l)\mathrm{sn}^2(\omega_2\xi, k)}{1 - \mathrm{sn}^2(\omega_2\xi, k)}, \quad \xi \in (-\xi_{02}, \xi_{02}), \tag{3.6}$$

where $\xi_{02} = \frac{1}{\omega_2}\mathrm{sn}^{-1}\left(\sqrt{\frac{\phi_{s2}-\phi_r}{\phi_{s2}-\phi_l}}, k\right)$ and k, ω_2 are the same as in (3.5).

3.2. Explicit solutions determined by $H(\phi, y) = h_1$

When $h = h_1$, the level set $H(\phi, y) = h_1$ is shown in (a)-(c) of Figure 4.

(i) Consider $H(\phi, y) = h_1$ in (a) of Figure 4. We have a homoclinic orbit tending to the cusp point $E_{12}(\phi_{12}, 0)$, where $\phi_1 = \phi_2 = \phi_{12}$. The polynomial $F(\phi)$ in (3.1) is $F(\phi) = (\phi_{s2} - \phi)(\phi_1 - \phi)^3(\phi - \phi_{s1})$. Now, (3.1) becomes that $\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{(\phi_{s2}-\tau)d\tau}{(\phi_1-\tau)\sqrt{(\phi_{s2}-\tau)(\phi_1-\tau)(\tau-\phi_{s1})}}$. Thus, it gives the parametric expression of



(a) $g = \frac{1}{12}c^2$ (b) $g_s < g < \frac{1}{12}c^2$ (c) $-\frac{1}{4}c^2 < g < g_s$

Figure 4. The level curve $H(\phi, y) = h_1$ for system (1.4)

a solitary wave solution of system (1.1) (similar to (c) of Figure 3):

$$\begin{aligned}
\phi(s) &= \phi_{s1} + (\phi_1 - \phi_{s1})\mathrm{sn}^2(s,k),\\
\xi(s) &= \sqrt{\frac{2}{\phi_{s2}-\phi_{s1}}}\Big[(1+\frac{\phi_{s2}-\phi_1}{\phi_1-\phi_{s1}})s + \left(\frac{\phi_{s2}-\phi_{s1}}{\phi_{s2}-\phi_1}\right)\big(\mathrm{dn}(s,k)\mathrm{tn}(s,k)\\
&\quad - E(\arcsin(\mathrm{sn}(s,k)),k)\big)\Big],
\end{aligned} \tag{3.7}$$

where $k^2 = \frac{\phi_1-\phi_{s1}}{\phi_{s2}-\phi_{s1}}, E(\cdot,k)$ is the normal elliptic integral of the second kind.

(ii) Consider $H(\phi, y) = h_1$ in (b) of Figure 4. We get two homoclinic orbits tending to $E_1(\phi_1, 0)$: one touches the point $E_s(\phi_{s1}, 0)$; the other encloses the point $E_2(\phi_2, 0)$. The polynomial $F(\phi)$ in (3.1) becomes $F(\phi) = (\phi_1-\phi)^2(\phi_{s2}-\phi)(\phi_M - \phi)(\phi-\phi_{s1})$ where $\phi_M = \frac{1}{6}\left(2\sqrt{c^2-12g}-c\right)$. Now, for the right homoclinic orbit, (3.1) becomes that $\sqrt{2}\xi = \int_\phi^{\phi_M} \frac{(\phi_{s2}-\tau)d\tau}{(\tau-\phi_1)\sqrt{(\phi_{s2}-\tau)(\phi_M-\tau)(\tau-\phi_{s1})}}$. Hence, we get the following parametric expression of a solitary wave solution of system (1.1)

$$\begin{aligned}
\phi(s) &= \phi_{s2} - \frac{\phi_{s2}-\phi_M}{\mathrm{dn}^2(s,k)}, \quad s \in (-s_{01}, s_{01}),\\
\xi(s) &= \sqrt{\frac{2}{\phi_{s2}-\phi_{s1}}}\left(\frac{\phi_{s2}-\phi_M}{\phi_M-\phi_1}\right)\Pi(\arcsin(\mathrm{sn}(s,k)),\alpha_1^2,k),
\end{aligned} \tag{3.8}$$

where $k^2 = \frac{\phi_M-\phi_{s1}}{\phi_{s2}-\phi_{s1}}, \alpha_1^2 = \frac{k^2(\phi_{s2}-\phi_1)}{\phi_M-\phi_1}, s_{01} = \mathrm{dn}^{-1}\left(\sqrt{\frac{\phi_{s2}-\phi_M}{\phi_{s2}-\phi_1}}, k\right), \Pi(\cdot,\alpha^2,k)$ is the normal elliptic integral of the third kind.

For the left homoclinic orbit, (3.1) can be expressed as

$$\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{(\phi_{s2}-\tau)d\tau}{(\phi_1-\tau)\sqrt{(\phi_{s2}-\tau)(\phi_M-\tau)(\tau-\phi_{s1})}}.$$

It follows the parametric expression of a solitary wave solution of system (1.1)

$$\begin{aligned}
\phi(s) &= \phi_{s1} + (\phi_M-\phi_{s1})\mathrm{sn}^2(s,k), \quad s \in (-s_{02}, s_{02}),\\
\xi(s) &= \sqrt{\frac{2}{\phi_{s2}-\phi_{s1}}}\left[s + \left(\frac{\phi_{s2}-\phi_1}{\phi_1-\phi_{s1}}\right)\Pi(\arcsin(\mathrm{sn}(s,k)),\alpha_2^2,k)\right],
\end{aligned} \tag{3.9}$$

where $k^2 = \frac{\phi_M-\phi_{s1}}{\phi_{s2}-\phi_{s1}}, \alpha_2^2 = \frac{\phi_M-\phi_1 s1}{\phi_1-\phi_{s1}}, s_{02} = \mathrm{sn}^{-1}\left(\sqrt{\frac{\phi_1-\phi_{s1}}{\phi_M-\phi_{s1}}}, k\right)$.

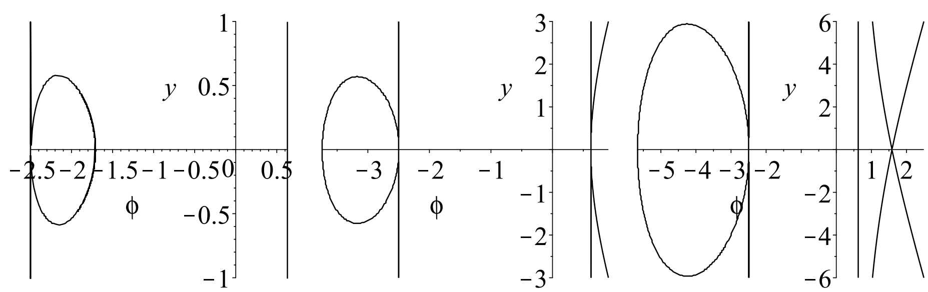
(iii) Consider $H(\phi, y) = h_1$ in (c) of Figure 4. We obtain a homoclinic orbit tending to $E_2(\phi_2, 0)$ and touching the point $E_s(\phi_{s1}, 0)$. Meanwhile, an open curve goes through the point $(\phi_M, 0)$. In this case, (3.1) becomes that $\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{(\phi_{s2}-\tau)d\tau}{(\phi_1-\tau)\sqrt{(\phi_M-\tau)(\phi_{s2}-\tau)(\tau-\phi_{s1})}}$. It gives rise to the parametric expression of a solitary wave solution of system (1.1)

$$
\begin{aligned}
&\phi(s) = \phi_{s1} + (\phi_{s2} - \phi_{s1})\mathrm{sn}^2(s, k), \quad s \in (-s_{03}, s_{03}),\\
&\xi(s) = \sqrt{\frac{2}{\phi_M - \phi_{s1}}}\left[s + \left(\frac{\phi_{s2} - \phi_1}{\phi_1 - \phi_{s1}}\right)\Pi(\arcsin(\mathrm{sn}(s, k)), \alpha_3^2, k)\right],
\end{aligned} \tag{3.10}
$$

where $k^2 = \frac{\phi_{s2}-\phi_{s1}}{\phi_M-\phi_{s1}}, \alpha_3^2 = \frac{\phi_{s2}-\phi_1 s1}{\phi_1-\phi_{s1}}, s_{03} = \mathrm{sn}^{-1}\left(\sqrt{\frac{\phi_1-\phi_{s1}}{\phi_{s2}-\phi_{s1}}}, k\right)$.

3.3. Explicit solutions determined by $H(\phi, y) = h_2$

When $h = h_2$, the level set $H(\phi, y) = h_2$ is shown in (a)-(c) of Figure 5.



(a) $g_m < g < \frac{1}{12}c^2$ (b) $g = g_m$ (c) $-\infty < g < g_m$

Figure 5. The level curve $H(\phi, y) = h_2$ of system (1.7)

(i) Consider $H(\phi, y) = h_2$ in Figure 5(a). We obtain a closed orbit touching $E_s(\phi_{s1}, 0)$ from the right. Now, (3.1) can be written as $\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{(\phi_{s2}-\tau)d\tau}{(\phi_2-\tau)\sqrt{(\phi_{s2}-\tau)(\phi_m-\tau)(\tau-\phi_{s1})}}$ where $\phi_m = \frac{1}{6}\left(-2\sqrt{c^2 - 12g} - c\right)$. It yields the parametric expression of a periodic wave solution of system (1.1)

$$
\begin{aligned}
&\phi(s) = \phi_{s1} + (\phi_m - \phi_{s1})\mathrm{sn}^2(s, k),\\
&\xi(s) = \sqrt{\frac{2}{\phi_{s2} - \phi_{s1}}}\left[s + \left(\frac{\phi_{s2} - \phi_2}{\phi_2 - \phi_{s1}}\right)\Pi(\arcsin(\mathrm{sn}(s, k)), \alpha_4^2, k)\right],
\end{aligned} \tag{3.11}
$$

where $k^2 = \frac{\phi_m-\phi_{s1}}{\phi_{s2}-\phi_{s1}}, \alpha_4^2 = \frac{\phi_m-\phi_{s1}}{\phi_2-\phi_{s1}}$.

(ii) Consider $H(\phi, y) = h_2$ in (b) of Figure 5. We have a closed orbit touching $E_s(\phi_{s1}, 0)$ from the left. Now, $\phi_{s2} = \phi_2, h_s = h_2$. Hence, $F(\phi) = (\phi_2 - \phi)^3(\phi_{s1} - \phi)(\phi - \phi_L)$ where $\phi_L = \frac{1}{6}\left(-3a^2 - \sqrt{c^2 - 12g} + c\right)$. (3.1) becomes $\sqrt{2}\xi =$

$\int_{\phi_L}^{\phi} \frac{d\tau}{\sqrt{(\phi_2-\tau)(\phi_{s1}-\tau)(\tau-\phi_L)}}$. It gives the explicit periodic wave solution of system (1.1)

$$\phi(\xi) = \phi_L + (\phi_{s1} - \phi_L)\mathrm{sn}^2\left(\sqrt{\frac{\phi_2 - \phi_L}{2}}\xi, k\right), \tag{3.12}$$

where $k^2 = \frac{\phi_{s1}-\phi_L}{\phi_2-\phi_L}$.

(iii) Consider $H(\phi, y) = h_2$ in (c) of Figure 5. We have a closed orbit touching $E_s(\phi_{s1}, 0)$ from the left. Now, (3.1) is $\sqrt{2}\xi = \int_{\phi_{s1}}^{\phi} \frac{(\phi_{s2}-\tau)d\tau}{(\phi_2-\tau)\sqrt{(\phi_{s2}-\tau)(\phi_{s1}-\tau)(\tau-\phi_m)}}$. It yields the parametric expression of a periodic wave solution of system (1.1)

$$\begin{aligned} &\phi(s) = \phi_m + (\phi_{s1} - \phi_m)\mathrm{sn}^2(s, k), \\ &\xi(s) = \sqrt{\frac{2}{\phi_{s2} - \phi_m}}\left[s + \left(\frac{\phi_{s2} - \phi_2}{\phi_2 - \phi_m}\right)\Pi(\arcsin(\mathrm{sn}(s, k)), \alpha_5^2, k)\right], \end{aligned} \tag{3.13}$$

where $k^2 = \frac{\phi_{s1}-\phi_m}{\phi_{s2}-\phi_m}, \alpha_5^2 = \frac{\phi_{s1}-\phi_m}{\phi_2-\phi_m}$.

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