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ABELIAN INTEGRALS FOR A KIND OF QUADRATIC REVERSIBLE CENTERS OF GENUS ONE ($R7$)*

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Abstract For the quadratic reversible centers of genus one ($r7$), its all periodic orbits are quartic curves. Using the method of Picard-Fuchs equation and Riccati equation, we study that the upper bound of the number of zeros for Abelian integrals of system ($r7$) under arbitrary polynomial perturbations of degree n, and obtain that the upper bound of the number is $45n-72$ when $n \geq 2$, 5 when $n=1$, and 0 when $n=0$, which depends linearly on n.

Keywords Abelian integral, limit cycle, quadratic reversible center, weakened Hilbert's 16th problem.

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1. Introduction and Main Result

Consider nearly integrable system

$$\dot{x} = \frac{H_y(x,y)}{M(x,y)} + \varepsilon f(x,y), \quad \dot{y} = -\frac{H_x(x,y)}{M(x,y)} + \varepsilon g(x,y), \tag{1.1$_\varepsilon$}$$

where ε $(0 < \varepsilon \ll 1)$ is a real parameter, $H_y(x,y)/M(x,y)$, $H_x(x,y)/M(x,y)$, $f(x,y)$, $g(x,y)$ are all polynomials of x and y, with $\max\{\deg(f(x,y)), \deg(g(x,y))\} = n$ and $\max\{\deg(H_y(x,y)/M(x,y)), \deg(H_x(x,y)/M(x,y))\} = m$. For system $(1.1)_0$, we suppose that it is an integrable system, function $H(x,y)$ is its a first integral with an integrating factor $M(x,y)$. And it has at least one center, that is, we can define a continuous family of periodic orbits

$$\{\Gamma_h\} \subset \left\{(x,y) \in \mathbb{R}^2 : H(x,y) = h, h \in \Delta\right\},$$

which are defined on a maximal open interval $\Delta = (h_1, h_2)$. The problem to be studied in this paper is: for any small number ε, how many limit cycles in system $(1.1)_\varepsilon$ can be bifurcated from periodic orbits $\{\Gamma_h\}$. It is well known that in any compact region of periodic annulus, the number of limit cycles of system $(1.1)_\varepsilon$ is no more than the number of isolated zeros for the following Abelian integrals $A(h)$,

$$A(h) = \oint_{\Gamma_h} M(x,y)\,[g(x,y)\,dx - f(x,y)\,dy]\,, \quad h \in \Delta. \tag{1.2}$$

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A) If $(1.1)_0$ is a Hamiltonian system, i.e., $M(x,y)$ is a constant, then $H(x,y)$ is a polynomial of x and y with $\deg(H(x,y)) = m+1$. Finding the least upper bound $Z(m,n)$ of the number of isolated zeros for Abelian integrals $A(h)$ is a significative and difficult problem, where $Z(m,n)$ only depends on m, n, and does not depend on the specific forms of $H(x,y)$, $f(x,y)$, and $g(x,y)$. This problem is called the weakened Hilbert's 16th problem [1], it is also called the Hilbert-Arnold problem, for some specially planar systems, researchers obtain plentiful important results [2,3,5,15,18–20,22,24–26], and more specific situations can be found in the books [4,8], the review article [16], and the references therein.

B) If $(1.1)_0$ is an integrable non-Hamiltonian system, i.e., $M(x,y)$ is not a constant. When $H(x,y)$ or $M(x,y)$ are not polynomials, the research work of the associated Abelian integrals $A(h)$ becomes much more difficult. Thus, researchers consider this problem by starting from the simplest case, namely m is low. For the specific case of $m=2$, people conjecture that the upper bound $Z(2,n)$ of the number of zeros for associated Abelian integrals $A(h)$ depends linearly on n, and it is positive correlation to the degree of the periodic orbital curves. Researchers studied some special systems and got some very good results [6,21,23,27]. Horozov and Iliev [14] first used the method of Picard-Fuchs equation and Riccati equation to study the upper bound of the number of zeros, and obtained an upper bound $Z(3,n) \leq 5n+15$, References [21,27] also used this method. Unfortunately, this conjecture is still far from being solved. For quadratic reversible centers of genus one, in reference [7], Gautier et al showed that there are essentially 22 types in the classification, divided into $(r1)$–$(r22)$ specifically. For the linear dependence of the upper bound of the number of zeros, three special cases of system $(r1)$ was studied in [28]; $(r2)$ is a Hamiltonian system; some special cases of systems $(r3)$–$(r6)$ were studied in [17]; systems $(r9)$, $(r13)$, $(r17)$, and $(r19)$ were studied in [13]; systems $(r11)$, $(r16)$, $(r18)$, and $(r20)$ were studied in [12]; systems $(r12)$ and $(r21)$ were studied in [11]; system $(r10)$ was studied in [10]; system $(r22)$ was studied in [9]. All of these upper bounds depend linearly on n. In this paper, we also study system $(r7)$ using the method of Picard-Fuchs equation and Riccati equation, and obtain that the upper bound is $45n-72$ when $n \geq 2$, 5 when $n=1$, and 0 when $n=0$. Our result shows that the upper bound depends linearly on n.

The form of quadratic reversible centers of genus one as follows:

$$\dot{x} = -xy, \quad \dot{y} = -\frac{a+b+2}{2(a-b)}y^2 + \frac{a+b-2}{8(a-b)^3}x^2 - \frac{b-1}{2(a-b)^3}x - \frac{a-3b+2}{8(a-b)^3}. \tag{1.3}$$

From (1.3), when $(a,b) = (5/2, -1/2)$, we can get system $(r7)$ as follows:

$$(r7) \qquad \dot{x} = -xy, \quad \dot{y} = -\frac{2}{3}y^2 + \frac{1}{36}x - \frac{1}{36}. \tag{1.4}$$

$(r7)$ is an integrable non-Hamiltonian quadratic system. It has a center $(1,0)$, an integral curve $x=0$, a periodic orbital family $\{\Gamma_h\}$ $(-1/16 < h < 0)$ (see Figure 1), the value range of x for the $\{\Gamma_h\}$ is $(1/4, +\infty)$, and all of the periodic orbits are formed by quartic curve. A first integral of system (1.4) as follows:

$$H(x,y) = x^{-\frac{4}{3}}\left(\frac{1}{2}y^2 - \frac{1}{12}x + \frac{1}{48}\right) = h, \quad h \in \left(-\frac{1}{16}, 0\right), \tag{1.5}$$

with an integrating factor $M(x,y) = x^{-7/3}$ and almost all of the orbits are formed by quartic curve.

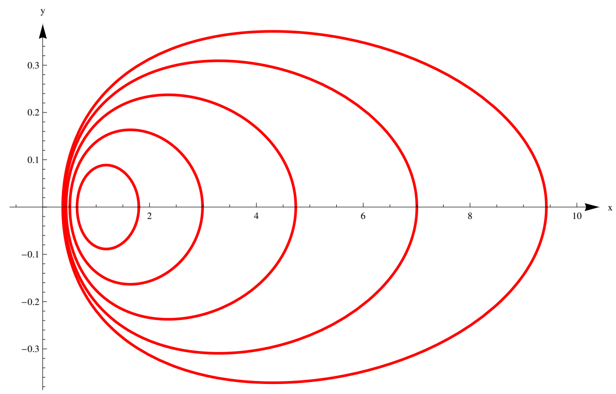

Figure 1. Periodic orbital images of system $(r7)$.

In this paper, our main result is the following theorem.

Theorem 1.1. *If $f(x,y)$ and $g(x,y)$ are any polynomials of x and y, and the maximum value of $\deg(f(x,y))$ and $\deg(g(x,y))$ is n, then the upper bound of the number of zeros of Abelian integrals $A(h)$ for system $(r7)$ depends linearly on n. Concretely, the upper bound is $45n-72$ for $n \geq 2$; the upper bound is 5 for $n=1$; and the upper bound is 0 for $n=0$.*

The rest part of this paper is structured as follows. In Section 2, we seek a simple expression of equivalent function $K(h)$ for Abelian integrals $A(h)$, prove Proposition 2.1. In Section 3, we study relation among functions $J_m(h)$ and their derivatives $J'_m(h)$ for $m=-1/3, 0, 1/3, 2/3$; relation among functions $J'_m(h)$ and their derivatives $J''_m(h)$ for $m=-1/3, 0, 1/3$; relation between $J'_{-1/3}(h)$ and $J'_{1/3}(h)$, obtain two Picard-Fuchs equations and a Riccati equation. In Section 4, we obtain three variable coefficient first order linear ordinary differential equations. For the relation between $V(h)$ and $J'_{1/3}(h)$, obtain a Riccati equation. Finally, we prove Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation. In Section 5, we give a short conclusion.

2. Simple Expression of Equivalent Function for $A(h)$

In this section, we give an equivalent function $K(h)$ of Abelian integral $A(h)$ and its simple expression, obtaining Proposition 2.1.

We suppose $f(x,y)=\sum_{0\leq i+j\leq n} a_{i,j}x^iy^j$ and $g(x,y)=\sum_{0\leq i+j\leq n} b_{i,j}x^iy^j$. From (1.2), Abelian integrals $A(h)$ in Theorem 1.1 have the form

$$A(h)=\oint_{\Gamma_h} x^{-\frac{7}{3}}\left(\sum_{0\leq i+j\leq n} b_{i,j}x^iy^j\,dx-\sum_{0\leq i+j\leq n} a_{i,j}x^iy^j\,dy\right),\ h\in\left(-\frac{1}{16},0\right),$$

where $x^{-7/3}$ is an integrating factor.

For conciseness, we introduce functions $I_{i,j}(h)$ as follows:

$$I_{i,j}(h)=\oint_{\Gamma_h} x^{i-\frac{7}{3}}y^j\,dx,$$

where $i = t/3, t = -3, -2, \cdots, 3n-1, 3n$; $j = 0, 1, 2, \cdots, n, n+1$, and $0 \le i+j \le n$. When $j = 1$, we write $I_{i,1}(h)$ as $J_i(h)$.

Note that

$$\oint_{\Gamma_h} x^{i-\frac{7}{3}} y^j dy = \frac{\oint_{\Gamma_h} x^{i-\frac{7}{3}} dy^{j+1}}{j+1} = \frac{\frac{7}{3}-i}{j+1} \oint_{\Gamma_h} x^{i-\frac{7}{3}-1} y^{j+1} dx = \frac{\frac{7}{3}-i}{j+1} I_{i-1,j+1}(h).$$

Thus, $A(h)$ can be written as

$$\begin{aligned} A(h) &= \sum_{0\le i+j\le n} b_{i,j} I_{i,j}(h) + \sum_{0\le i+j\le n} \frac{i-\frac{7}{3}}{j+1} a_{i,j} I_{i-1,j+1}(h) \\ &= \sum_{0\le i+j\le n, -1\le i\le n, 0\le j\le n+1} \tilde{b}_{i,j} I_{i,j}(h), \end{aligned} \tag{2.1}$$

where $\tilde{b}_{i,j} = b_{i,j} + (i+1-7/3)a_{i+1,j-1}/j$ $(j \ne 0)$, $\tilde{b}_{i,0} = b_{i,0} (i = -1, 0, 1, \cdots, n)$, $b_{-1,j} = 0 (j = 0, 1, \cdots, n+1)$.

The following Proposition 2.1 gives an equivalent function $K(h)$ of Abelian integral $A(h)$ and its simple expression.

Proposition 2.1. *Abelian integrals $A(h)$ can be expressed as*

$$h^{3n-5} A(h) = K(h) = \alpha(h) J_{-\frac{1}{3}}(h) + \beta(h) J_0(h) + \gamma(h) J_{\frac{1}{3}}(h) + \delta(h) J_{\frac{2}{3}}(h), \ (n \ge 2), \tag{2.2}$$

$$A(h) = \beta(h) J_0(h) + \gamma(h) J_{\frac{1}{3}}(h), \ \ (n = 1), \tag{2.3}$$

$$A(h) = \zeta(h) J_{-1}(h), \ \ (n = 0), \tag{2.4}$$

where $0 \le \deg(\alpha(h)) \le 3n-6$, $1 \le \deg(\beta(h)) \le 3n-5$, $2 \le \deg(\gamma(h)) \le 3n-4$, $0 \le \deg(\delta(h)) \le 3n-6$, *when* $n \ge 2$; $\deg(\beta(h)) = 0$, $\deg(\gamma(h)) = 1$, *when* $n = 1$; $\deg(\zeta(h)) = 0$, *when* $n = 0$.

Proof. Since periodic orbits Γ_h are symmetric about x-axis, thus, $I_{i,j}(h) = 0$ as j is even, so, we only need to consider the case of j is odd.

From (1.5), we obtain

$$-\frac{2}{3} x^{-\frac{7}{3}} y^2 + x^{-\frac{4}{3}} y \frac{\partial y}{\partial x} + \frac{1}{36} x^{-\frac{4}{3}} - \frac{1}{36} x^{-\frac{7}{3}} = 0. \tag{2.5}$$

Multiplied equality (2.5) by $x^i y^{j-2} dx$ and integrated it over Γ_h, we can get

$$\frac{2j+3m-4}{j} I_{m,j}(h) = \frac{1}{12} \left[I_{m+1,j-2}(h) - I_{m,j-2}(h) \right], \tag{2.6}$$

where $j = 1, 3, 5, \cdots, 2[n/2]+1$. We restrict $m = i/3$, $i = -3, -2, -1, 0, \cdots, 3n-3$, and $0 \le m+j \le n$.

(i) When $2j + 3m - 4 = 0$, that is, $(m, j) = (2/3, 1)$, $(-2/3, 3)$, when $(m, j) = (-2/3, 3)$, from (2.6), we obtain

$$J_{\frac{1}{3}}(h) = J_{-\frac{2}{3}}(h). \tag{2.7}$$

(ii) When $2j + 3m - 4 \ne 0$, that is, $(m, j) \ne (2/3, 1), (-2/3, 3)$, from (2.6), we have

$$I_{m,j}(h) = \frac{j}{12(2j+3m-4)} \left[I_{m+1,j-2}(h) - I_{m,j-2}(h) \right], \tag{2.8}$$

which indicates that $I_{m,j}(h)$ can be expressed in terms of $I_{m,j-2}(h)$ and $I_{m+1,j-2}(h)$. Then step by step, since j is a positive odd number, we use $(j-1)/2$ times (2.8) and obtain that $I_{m,j}(h)$ can be written as a linear combination of $J_k(h)(k=-1,0,\cdots)$ with the form

$$I_{m,j}(h)=\sum_{k=0}^{\frac{j-1}{2}} c_{m,k}J_{m+k}(h).$$

From (2.1), we have

$$A(h)=\sum_{0\le m+j\le n,-1\le m\le n-1,j\equiv 1\,mod\,2}\tilde{b}_{m,j}\sum_{k=0}^{\frac{j-1}{2}} c_{m,k}J_{m+k}(h).$$

Because the maximum number of $m+k$ is $m+(j-1)/2=n-1+0=n-1$, and its the minimum number is $-1+0=-1$, therefore $A(h)$ is a linear combination of $J_{-1}(h)$, $J_0(h)$, $J_1(h)$, $\cdots$, $J_{n-1}(h)$. So we can suppose that

$$A(h)=\sum_{k=0}^{n} e_kJ_{k-1}(h), \tag{2.9}$$

where $e_k\in\mathbb{R}$ $(k=0,1,\cdots,n)$.

Again, it follows from (1.5) that

$$\frac{1}{2}x^{-\frac{4}{3}}y^2-\frac{1}{12}x^{-\frac{1}{3}}+\frac{1}{48}x^{-\frac{4}{3}}=h.$$

Multiplying above equality by $x^{m-1}y^{j-2}dx$ and integrated it over Γ_h, we obtain

$$I_{m,j}(h)=\frac{1}{6}I_{m+1,j-2}(h)-\frac{1}{24}I_{m,j-2}(h)+2hI_{m+\frac{4}{3},j-2}(h).$$

By (2.8), the above equality can be written as

$$\frac{j[I_{m+1,j-2}(h)-I_{m,j-2}(h)]}{12(2j+3m-4)}=\frac{1}{6}I_{m+1,j-2}(h)-\frac{1}{24}I_{m,j-2}(h)+2hI_{m+\frac{4}{3},j-2}(h).$$

When $j=3$, the above equality becomes

$$48h(3m+2)J_{m+\frac{4}{3}}(h)=(3m+4)J_m(h)-2(6m+1)J_{m+1}(h). \tag{2.10}$$

$\mathcal{A}$) When $m\ge 1$, assume that $\hbar:=1/h$, then we rewrite (2.10) as

$$\hbar J_m(h)=\frac{3m-8}{48(3m-2)}\hbar^2J_{m-\frac{4}{3}}(h)-\frac{6m-7}{24(3m-2)}\hbar^2J_{m-\frac{1}{3}}(h),$$

which indicates that $\hbar J_m(h)$ can be expressed in terms of $\hbar^2J_{m-4/3}(h)$, $\hbar^2J_{m-1/3}(h)$. Then step by step, $\hbar J_m(h)$ can be written as a linear combination of $J_{-1/3}(h)$, $J_0(h)$, $J_{1/3}(h)$ and $J_{2/3}(h)$ with polynomial coefficients of $\hbar$:

$$\hbar J_m(h)=\alpha_{m,1}(\hbar)J_{-\frac{1}{3}}(h)+\beta_{m,1}(\hbar)J_0(h)+\gamma_{m,1}(\hbar)J_{\frac{1}{3}}(h)+\delta_{m,1}(\hbar)J_{\frac{2}{3}}(h),$$

where $5\le\deg(\alpha_{m,1}(\hbar))\le 3m-1$, $4\le\deg(\beta_{m,1}(\hbar))\le 3m-2$, $3\le\deg(\gamma_{m,1}(\hbar))\le 3m-3$, $2\le\deg(\delta_{m,1}(\hbar))\le 3m-1$, when $m\ge 2$; $\deg(\alpha_{m,1}(\hbar))=2$, $\beta_{m,1}(\hbar)=0$, $\gamma_{m,1}(\hbar)=0$, $\deg(\delta_{m,1}(\hbar))=2$, when $m=1$.

$\mathcal{B}$) When $m = 0$, then $\hbar J_m(h)$ can also be a linear combination of $J_{-1/3}(h)$, $J_0(h)$, $J_{1/3}(h)$ and $J_{2/3}(h)$ as

$$\hbar J_0(h) = \hbar J_0(h).$$

$\mathcal{C}$) When $m < 0$, then we rewrite (2.10) as

$$\hbar J_m(h) = \frac{2(6m+1)}{3m-4}\hbar J_{m+1}(h) + \frac{48(3m+2)}{3m-4} J_{m+\frac{4}{3}}(h),$$

which indicates that $\hbar J_{-1}(h)$ can be expressed in terms of $J_{-1/3}(h)$, $J_0(h)$, $J_{1/3}(h)$ and $J_{2/3}(h)$:

$$\hbar J_{-1}(h) = \frac{10}{7}\hbar J_0(h) + \frac{48}{7} J_{\frac{1}{3}}(h).$$

As a consequence, all $\hbar J_m(h)$ can be expressed in terms of $J_{-1/3}(h)$, $J_0(h)$, $J_{1/3}(h)$ and $J_{2/3}(h)$. From (2.9), substituting these formulas to $\hbar A(h)$, therefore

$$\hbar A(h) = J(h) = \alpha(\hbar) J_{-\frac{1}{3}}(h) + \beta(\hbar) J_0(h) + \gamma(\hbar) J_{\frac{1}{3}}(h) + \delta(\hbar) J_{\frac{2}{3}}(h),$$

where $2 \le \deg(\alpha(\hbar)) \le 3n-4$, $1 \le \deg(\beta(\hbar)) \le 3n-5$, $0 \le \deg(\gamma(\hbar)) \le 3n-6$, $2 \le \deg(\delta(\hbar)) \le 3n-4$, when $n \ge 2$; $\alpha(\hbar) = 0$, $\deg(\beta(\hbar)) = 1$, $\deg(\gamma(\hbar)) = 0$, $\delta(\hbar) = 0$, when $n = 1$.

(1) When $n \ge 2$, assume that $K(h) := h^{3n-4} J(h)$, that is $K(h) = h^{3n-5} A(h)$, we can get

$$K(h) = \alpha(h) J_{-\frac{1}{3}}(h) + \beta(h) J_0(h) + \gamma(h) J_{\frac{1}{3}}(h) + \delta(h) J_{\frac{2}{3}}(h),$$

where $0 \le \deg(\alpha(h)) \le 3n-6$, $1 \le \deg(\beta(h)) \le 3n-5$, $2 \le \deg(\gamma(h)) \le 3n-4$, $0 \le \deg(\delta(h)) \le 3n-6$.

(2) When $n = 1$, assume that $K(h) := hJ(h)$, that is $K(h) = A(h)$, we can get

$$A(h) = K(h) = \beta(h) J_0(h) + \gamma(h) J_{\frac{1}{3}}(h),$$

where $\deg(\beta(h)) = 0$, $\deg(\gamma(h)) = 1$.

(3) When $n = 0$, from (2.1), we can get

$$A(h) = -\frac{7}{3} a_{0,0} J_{-1}(h) = \zeta(h) J_{-1}(h),$$

where $\zeta(h) = -7a_{0,0}/3$, and $\deg(\zeta(h)) = 0$. $\square$

3. Picard-Fuchs Equation and Riccati Equation

In this section, we give a relation among functions $J_m(h)$ and their derivatives $J'_m(h)$ for $m = -1/3, 0, 1/3, 2/3$; a relation among functions $J'_m(h)$ and their derivatives $J''_m(h)$ for $m = -1/3, 0, 1/3$; a relation between $J'_{-1/3}(h)$ and $J'_{1/3}(h)$, obtaining two Picard-Fuchs equations and a Riccati equation.

The following lemma gives a relation among functions $J_m(h)$ and their derivatives $J'_m(h)$ for $m = -1/3, 0, 1/3, 2/3$.

Lemma 3.1. *Functions $J_m(h)$ for $m = -1/3, 0, 1/3, 2/3$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_{-\frac{1}{3}}(h) \\ J_0(h) \\ J_{\frac{1}{3}}(h) \\ J_{\frac{2}{3}}(h) \end{pmatrix} = \begin{pmatrix} \frac{4}{5}h & 0 & \frac{1}{20} & 0 \\ \frac{1}{16} & h & 0 & 0 \\ 0 & \frac{1}{12} & \frac{4}{3}h & 0 \\ 0 & 0 & \frac{1}{8} & 2h \end{pmatrix} \begin{pmatrix} J'_{-\frac{1}{3}}(h) \\ J'_0(h) \\ J'_{\frac{1}{3}}(h) \\ J'_{\frac{2}{3}}(h) \end{pmatrix}. \tag{3.1}$$

Proof. By (1.5), we have $y^2 = 2hx^{4/3} + x/6 - 1/24$, $\partial y/\partial h = x^{4/3}/y$, and $ydy = (4hx^{1/3}/3 + 1/12)dx$. Since $J_i(h) = \oint_{\Gamma_h} x^{i-7/3} ydx$, $J'_i(h) = \oint_{\Gamma_h} x^{i-1}/ydx$. Thus

$$\begin{aligned} \left(i - \frac{4}{3}\right) J_i(h) &= \oint_{\Gamma_h} \left(i - \frac{4}{3}\right) x^{i-\frac{7}{3}} ydx = \oint_{\Gamma_h} ydx^{i-\frac{4}{3}} \\ &= -\oint_{\Gamma_h} \frac{x^{i-\frac{4}{3}} \left(\frac{4}{3}hx^{\frac{1}{3}} + \frac{1}{12}\right)}{y} dx = -\frac{4}{3}hJ'_i(h) - \frac{1}{12}J'_{i-\frac{1}{3}}(h). \end{aligned} \tag{3.2}$$

By (3.2), let $i = -1/3, 0, 1/3, 2/3$ respectively, we obtain

$$\begin{cases} J_{-\frac{1}{3}}(h) = \frac{4}{5}hJ'_{-\frac{1}{3}}(h) + \frac{1}{20}J'_{-\frac{2}{3}}(h), \\ J_0(h) = \frac{1}{16}J'_{-\frac{1}{3}}(h) + hJ'_0(h), \\ J_{\frac{1}{3}}(h) = \frac{1}{12}J'_0(h) + \frac{4}{3}hJ'_{\frac{1}{3}}(h), \\ J_{\frac{2}{3}}(h) = \frac{1}{8}J'_{\frac{1}{3}}(h) + 2hJ'_{\frac{2}{3}}(h). \end{cases} \tag{3.3}$$

From equalities (3.3) and (2.7), we obtain (3.1). □

The following lemma gives a relation among functions $J'_m(h)$ and their derivatives $J''_m(h)$ for $m = -1/3, 0, 1/3$.

Lemma 3.2. *Functions $J'_m(h)$ for $m = -1/3, 0, 1/3$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J''_{-\frac{1}{3}}(h) \\ J''_0(h) \\ J''_{\frac{1}{3}}(h) \end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix} -64h^2 & -4h \\ 4h & \frac{1}{4} \\ -\frac{1}{4} & 64h^2 \end{pmatrix} \begin{pmatrix} J'_{-\frac{1}{3}}(h) \\ J'_{\frac{1}{3}}(h) \end{pmatrix}, \tag{3.4}$$

where $B(h) = -256(h^3 + 1/16^3) = -256(h + 1/16)(h^2 - h/16 + 1/256)$.

Proof. From the first three equations of (3.3), differentiated both sides of equations with respect to h, we obtain

$$\begin{pmatrix} J'_{-\frac{1}{3}}(h) \\ 0 \cdot J'_0(h) \\ J'_{\frac{1}{3}}(h) \end{pmatrix} = \begin{pmatrix} 4h & 0 & \frac{1}{4} \\ 1 & 16h & 0 \\ 0 & -\frac{1}{4} & -4h \end{pmatrix} \begin{pmatrix} J''_{-\frac{1}{3}}(h) \\ J''_0(h) \\ J''_{\frac{1}{3}}(h) \end{pmatrix}. \tag{3.5}$$

From (3.5), we can get

$$\begin{pmatrix} J''_{-\frac{1}{3}}(h) \\ J''_0(h) \\ J''_{\frac{1}{3}}(h) \end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix} -64h^2 & -\frac{1}{16} & -4h \\ 4h & -16h^2 & \frac{1}{4} \\ -\frac{1}{4} & h & 64h^2 \end{pmatrix} \begin{pmatrix} J'_{-\frac{1}{3}}(h) \\ 0 \cdot J'_0(h) \\ J'_{\frac{1}{3}}(h) \end{pmatrix}, \tag{3.6}$$

where $B(h) = -256(h + 1/16)(h^2 - h/16 + 1/256)$.

From equation (3.6), we get (3.4). □

Lemma 3.3. $J_{-1}(h) < 0$, $J'_m(h) > 0$ $(m = -1/3, 0, 1/3, 2/3)$ *when* $h \in (-1/16, 0)$; $J_m(-1/16) = 0$ $(m = -1/3, 0, 1/3, 2/3)$.

Since $J_m(h) = \oint_{\Gamma_h} x^{m-7/3} y dx$, $J'_m(h) = \oint_{\Gamma_h} x^{m-1}/y dx$. The proof only requires some simple calculations, so it is omitted.

For the relation between $J'_{-1/3}(h)$,$J'_{1/3}(h)$, assume that $U(h) := J'_{-1/3}(h)/J'_{1/3}(h)$, we obtain the following corollary.

Corollary 3.1. *Function* $U(h)$ *satisfies the following Riccati equation*

$$B(h)U'(h) = \frac{1}{4}U^2(h) - 128h^2 U(h) - 4h, \tag{3.7}$$

where $B(h) = -256(h + 1/16)(h^2 - h/16 + 1/256)$.

Proof. Using Lemma 3.2, and differentiated both sides of $U(h)$ with respect to h, we obtain (3.7). □

4. The Number of Zeros for Abelian Integrals $A(h)$

In this section, we obtain three variable coefficient first order linear ordinary differential equations and a Riccati equation. Finally, we prove Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation.

(1) If $n \geq 2$, by (2.2) and (3.1), we have

$$K(h) = \alpha_1(h) J'_{-\frac{1}{3}}(h) + \beta_1(h) J'_0(h) + \gamma_1(h) J'_{\frac{1}{3}}(h) + \delta_1(h) J'_{\frac{2}{3}}(h), \tag{4.1}$$

where $\alpha_1(h) = 4h\alpha(h)/5 + \beta(h)/16$, $\beta_1(h) = h\beta(h) + \gamma(h)/12$, $\gamma_1(h) = \alpha(h)/20 + 4h\gamma(h)/3 + \delta(h)/8$, $\delta_1(h) = 2h\delta(h)$. Thus $1 \leq \deg(\alpha_1(h)) \leq 3n-5$, $2 \leq \deg(\beta_1(h)) \leq 3n-4$, $0 \leq \deg(\gamma_1(h)) \leq 3n-3$, and $1 \leq \deg(\delta_1(h)) \leq 3n-5$.

Differentiating both sides of (2.2) with respect to h and using (3.1), we obtain

$$K'(h) = \alpha_2(h) J'_{-\frac{1}{3}}(h) + \beta_2(h) J'_0(h) + \gamma_2(h) J'_{\frac{1}{3}}(h) + \delta_2(h) J'_{\frac{2}{3}}(h), \tag{4.2}$$

where $\alpha_2(h) = \alpha(h) + 4h\alpha'(h)/5 + \beta'(h)/16$, $\beta_2(h) = \beta(h) + h\beta'(h) + \gamma'(h)/12$, $\gamma_2(h) = \gamma(h) + 4h\gamma'(h)/3 + \alpha'(h)/20 + \delta'(h)/8$, $\delta_2(h) = \delta(h) + 2h\delta'(h)$. So $0 \leq \deg(\alpha_2(h)) \leq 3n-6$, $1 \leq \deg(\beta_2(h)) \leq 3n-5$, $2 \leq \deg(\gamma_2(h)) \leq 3n-4$, $0 \leq \deg(\delta_2(h)) \leq 3n-6$.

By (4.1) and (4.2), we have

$$\delta_1(h)K'(h) = \delta_2(h)K(h) + W(h), \tag{4.3}$$

$$W(h) = \alpha_3(h)J'_{-\frac{1}{3}}(h) + \beta_3(h)J'_0(h) + \gamma_3(h)J'_{\frac{1}{3}}(h), \tag{4.4}$$

where $\alpha_3(h) = \alpha_1(h)\delta_2(h) - \delta_1(h)\alpha_2(h)$, $\beta_3(h) = \beta_1(h)\delta_2(h) - \delta_1(h)\beta_2(h)$, $\gamma_3(h) = \gamma_1(h)\delta_2(h) - \delta_1(h)\gamma_2(h)$. Thus $1 \leq \deg(\alpha_3(h)) \leq 6n-11$, $2 \leq \deg(\beta_3(h)) \leq 6n-10$, $0 \leq \deg(\gamma_3(h)) \leq 6n-9$.

By (3.4) and (4.4), we have

$$B(h)W'(h) = \alpha_4(h)J'_{-\frac{1}{3}}(h) + B(h)\beta'_3(h)J'_0(h) + \gamma_4(h)J'_{\frac{1}{3}}(h), \tag{4.5}$$

where $\alpha_4(h) = B(h)\alpha'_3(h) - 64h^2\alpha_3(h) + 4h\beta_3(h) - \gamma_3(h)/4$, $\gamma_4(h) = B(h)\gamma'_3(h) + 64h^2\gamma_3(h) - 4h\alpha_3(h) + \beta_3(h)/4$. Thus $0 \leq \deg(\alpha_4(h)) \leq 6n-9$, $2 \leq \deg(\gamma_4(h)) \leq 6n-7$.

By (4.4) and (4.5), we have

$$B(h)\beta_3(h)W'(h) = B(h)\beta'_3(h)W(h) + V(h), \tag{4.6}$$

$$V(h) = \alpha_5(h)J'_{-\frac{1}{3}}(h) + \gamma_5(h)J'_{\frac{1}{3}}(h), \tag{4.7}$$

where $\alpha_5(h) = \beta_3(h)\alpha_4(h) - B(h)\beta'_3(h)\alpha_3(h)$, $\gamma_5(h) = \beta_3(h)\gamma_4(h) - B(h)\beta'_3(h)\gamma_3(h)$. Thus $2 \leq \deg(\alpha_5(h)) \leq 12n-19$, $1 \leq \deg(\gamma_5(h)) \leq 12n-17$.

(2) If $n = 1$, by (2.3) and (3.1), we have

$$A(h) = \alpha_1(h)J'_{-\frac{1}{3}}(h) + \beta_1(h)J'_0(h) + \gamma_1(h)J'_{\frac{1}{3}}(h), \tag{4.8}$$

where $\alpha_1(h) = \beta(h)/16$, $\beta_1(h) = h\beta(h) + \gamma(h)/12$, $\gamma_1(h) = 4h\gamma(h)/3$. Thus $\deg(\alpha_1(h)) = 0$, $\deg(\beta_1(h)) = 1$, $\deg(\gamma_1(h)) = 2$.

Differentiating both sides of (2.3) with respect to h and using (3.1), we obtain

$$A'(h) = \alpha_2(h)J'_{-\frac{1}{3}}(h) + \beta_2(h)J'_0(h) + \gamma_2(h)J'_{\frac{1}{3}}(h), \tag{4.9}$$

where $\alpha_2(h) = \beta'(h)/16$, $\beta_2(h) = \beta(h) + h\beta'(h) + \gamma'(h)/12$, $\gamma_2(h) = \gamma(h) + 4h\gamma'(h)/3$. Thus $\alpha_2(h) = 0$, $\deg(\beta_2(h)) = 0$, $\deg(\gamma_2(h)) = 1$.

By (4.8) and (4.9), we have

$$\beta_1(h)A'(h) = \beta_2(h)A(h) + V(h), \tag{4.10}$$

$$V(h) = \alpha_5(h)J'_{-\frac{1}{3}}(h) + \gamma_5(h)J'_{\frac{1}{3}}(h), \tag{4.11}$$

where $\alpha_5(h) = \beta_1(h)\alpha_2(h) - \beta_2(h)\alpha_1(h)$, $\gamma_5(h) = \beta_1(h)\gamma_2(h) - \beta_2(h)\gamma_1(h)$. Thus $\deg(\alpha_5(h)) = 0$, $\deg(\gamma_5(h)) = 2$.

For the relation between $V(h)$ and $J'_{1/3}(h)$, assume that $E(h) := V(h)/J'_{1/3}(h)$, we obtain the following lemma.

Lemma 4.1. *For $n \geq 1$, function $E(h)$ satisfies the following Riccati equation*

$$B(h)\alpha_5(h)E'(h) = \frac{1}{4}E^2(h) + D(h)E(h) + G(h), \tag{4.12}$$

where $D(h) = B(h)\alpha'_5(h) - 128h^2\alpha_5(h) - \gamma_5(h)/2$, $G(h) = B(h)\alpha_5(h)\gamma'_5(h) - B(h)\alpha'_5(h)\gamma_5(h) + 128h^2\alpha_5(h)\gamma_5(h) - 4h\alpha_5^2(h) + \gamma_5^2(h)/4$. Thus $1 \leq \deg(D(h)) \leq 12n-17$, $2 \leq \deg(G(h)) \leq 24n-34$ when $n \geq 2$; $\deg(D(h)) = 2$, $1 \leq \deg(G(h)) \leq 4$ when $n = 1$.

Proof. Using equalities (4.7), (4.11) and Corollary 3.1, differentiated both sides of $E(h)$ with respect to h, we obtain (4.12). □

We use $\sharp A(h)$ to denote the number of zeros of Abelian integrals $A(h)$ in Δ, and we need the following lemma.

Lemma 4.2 ([17]). *The smooth functions $S(h)$, $\phi(h)$, $\psi(h)$, $\xi(h)$, and $\eta(h)$ satisfy the following Riccati equation*

$$\eta(h)S'(h) = \phi(h)S^2(h) + \psi(h)S(h) + \xi(h),$$

then

$$\sharp S(h) \le \sharp\eta(h) + \sharp\xi(h) + 1.$$

Lemma 4.2 is Lemma 5.3 in [17], and the proof can be found in [17], so it is omitted.

Finally, we complete the proof of Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation.

Proof of Theorem 1.1. (1) When $n \ge 2$, using equalities (2.2), (4.3), (4.6), Lemma 4.1 and Lemma 4.2, therefore

$$\sharp A(h) = \sharp K(h) \le \sharp\delta_1(h) + \sharp W(h) + 1,$$

and

$$\sharp W(h) \le \sharp B(h) + \sharp\beta_3(h) + \sharp V(h) + 1,$$
$$\sharp V(h) = \sharp E(h) \le \sharp B(h) + \sharp\alpha_5(h) + \sharp G(h) + 1.$$

So,

$$\sharp A(h) \le 2\sharp B(h) + \sharp\delta_1(h) + \sharp\beta_3(h) + \sharp\alpha_5(h) + \sharp G(h) + 3.$$

Since $1 \le \deg(\delta_1(h)) \le 3n-5$, $2 \le \deg(\beta_3(h)) \le 6n-10$, $2 \le \deg(\alpha_5(h)) \le 12n-19$, $2 \le \deg(G(h)) \le 24n-34$, noticing that $B(h) = -256(h+1/16)(h^2 - h/16 + 1/256)$ and there is no zero in $(-1/16, 0)$, we obtain

$$\sharp A(h) \le (3n-6) + (6n-12) + (12n-21) + (24n-36) + 3 = 45n - 72.$$

(2) When $n = 1$, using equality (4.10), Lemma 4.1 and Lemma 4.2, therefore

$$\sharp A(h) \le \sharp\beta_1(h) + \sharp V(h) + 1,$$

and

$$\sharp V(h) = \sharp E(h) \le \sharp B(h) + \sharp\alpha_5(h) + \sharp G(h) + 1.$$

So,

$$\sharp A(h) \le \sharp B(h) + \sharp\beta_1(h) + \sharp\alpha_5(h) + \sharp G(h) + 2.$$

Since $\deg(\beta_1(h)) = 1$, $\deg(\alpha_5(h)) = 0$, $1 \le \deg(G(h)) \le 4$, we can get

$$\sharp A(h) \le 0 + 0 + 0 + 3 + 2 = 5.$$

(3) When $n = 0$, using equality (2.4) and Lemma 3.3, since $A(h) = \zeta(h)J_{-1}(h)$, where $\deg(\zeta(h)) = 0$, $J_{-1}(h) < 0$, therefore

$$\sharp A(h) = 0.$$

□

5. Conclusion

In this paper, we study the linear estimation to the number of zeros for Abelian integrals in the quadratic reversible system ($r7$) under arbitrary polynomial perturbations of degree n, according to the method of Picard-Fuchs equation and Riccati equation. We obtain that the upper bound of the number is $45n - 72$ when $n \geq 2$, 5 when $n = 1$, and 0 when $n = 0$. This result shows that the upper bound depends linearly on n.

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