Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1636–1649
Website:http://www.jaac-online.com
DOI:10.11948/20210510

INTERVAL-TYPE CRITERIA OF LIMIT-POINT CASE FOR CONFORMABLE FRACTIONAL STURM-LIOUVILLE OPERATORS*

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Abstract In this paper, the classification of limit-point case and limit-circle case for the 2α-order conformable fractional Sturm-Liouville operator

$$\ell_\alpha(y) = -T_\alpha\left(pT_\alpha y\right) + qy, x \in [a,\infty), a > 0$$

is considered. Two interval criteria of limit-point case in the frame of conformable fractional derivatives are obtained. Examples illustrating the main results are presented.

Keywords Conformable fractional calculus, Sturm-Liouville operator, limit-point case, interval-type criterion.

MSC(2010) 34B12, 47E0575.

1. Introduction

Differential operators play an increasingly important role in describing many phenomena and processes in various fields of science and engineering such as quantum mechanics, finance, medicine.

For the second-order singular symmetric differential operator (Sturm-Liouville operator):

$$\ell(y) = -\left(p(x)y'\right)' + q(x)y, x \in [a,\infty), \tag{1.1}$$

where p, q are real-valued functions with $p > 0$ and $p^{-1}, q \in L_{\text{loc}}[a,\infty)$, there are a lot of spectral analysis results for (1.1), we refer the readers to papers [2, 5, 10, 14, 16–19] and references cited therein.

Regarding the study of the spectral theory of differential operator (1.1), the classification problem of limit-point case(LPC) and limit-circle case(LCC) was first proposed by H. Weyl in 1910 [21]. After then, Levinson [16], Sears [20], Read [18] and other researchers gave a series of classical LPC criteria. By separating the potential function $q(x)$, Read gave a more general LPC criterion in 1970s.

Theorem 1.1 (Read criterion [18]). *Assume $w(x)$ be a non-negative local absolutely continuous function on $[a,\infty)$ and $q(x) = q_1(x) + q_2(x) + q_3(x)$. If*

(1) $q_1(x) \geq 0$, for a positive number δ, there exists constant K, such that $(1+\delta)p(w')^2 - q_1w^2 \leq K$;

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*This work was completed with the support of NSF of Shandong Province (No. ZR2019MA034).

(2) $-q_2 w^2 \leq K$*;*

(3) There exists a constant d *with* $0 \leq d \leq 1$ *such that* $w^d p^{-1/2}|Q| \leq K$*, where* $Q' = q_3 w^{1-d}$*;*

(4) $\displaystyle\int_0^\infty w^2 (q_1/p)^{1/2}\, \mathrm{d}x = \infty.$

Then (1.1) *is of the LPC at* ∞*.*

Kuffman, Read and Zettl in [12] replaced condition (4) by $\int_0^\infty wp^{-1/2}\mathrm{d}x = \infty$, conditions (1)-(3) being unchanged, then they deduced that (1.1) is also of the LPC at ∞. The Read criterion is an extension of the Levinson criterion. It generalizes Hartman-Wintner criterion, Levinson criterion and Sears criterion (see [12] for details).

The above LPC criteria are considering the properties of p and q on the entire positive half axis. As we all know, LPC for differential operator on $[a, \infty)$ is an interval-type property, it is necessary to consider the nature of p and q in a series of interval sequences tending to ∞. There are two typical methods to obtain the interval-type LPC criterion, one is restriction p and q on given sequence of interval by constants, this method appeared in the early 1960s. The first interval-type LPC criterion was proposed by the Soviet mathematician R. S. Ismagilov in 1963.

Theorem 1.2 ([10])**.** *Suppose* $\{I_k\}$ *are mutually disjoint intervals whose endpoints tend to* ∞*. The length of* I_k *is denoted by* h_k *and* $p(x) = 1$*. If there is a sequence of real numbers* q_k*, such that*

(1) $\displaystyle\sum_{k=1}^{\infty} \sqrt{q_k} h_k^3 = \infty;$

(2) $q(x) \geq q_k$ *for* $x \in I_k$*,*

then $\ell(y)$ *is of the LPC at* ∞*.*

In 1973, Knowles improved Ismagilov criterion to a more general form, he proved the following conclusions.

Theorem 1.3 (Knowles criterion [14])**.** *If there are mutually disjoint intervals* I_n *(n=1, 2,* $\cdots$*) with the endpoints tending to infinity as* $n \to \infty$*, and* p_n*,* q_n *are constants such that*

(1) $q(x) \geq q_n > 0, p(x) \geq p_n > 0, \quad (x \in I_n);$

(2) $\displaystyle\sum_{n=1}^{\infty} p_n^{3/2} q_n^{1/2} \left(\int_{I_n} \frac{dx}{p(x)} \right)^3 = \infty,$

then $\ell(y)$ *is of the LPC at* ∞*.*

Another method of interval-type LPC criterion is obtained by constructing suitable nonnegative function $w(x)$ in Therem 1.1. Followed this line, Eastham and Thompson gave the following result.

Theorem 1.4 (Eastham and Thompson interval-type criterion [8])**.** *If there are mutually disjoint intervals* $I_n = [a_n, b_n]$ *(n=1, 2,* $\cdots$*) with* $b_n \to \infty$ *as* $n \to \infty$*, and a sequence of positive numbers* $\{v_n\}$*, such that*

(1) $v_n P_n \geq K > 0$, *where* $P_n = \int_{a_n}^{b_n} \frac{dx}{\sqrt{p(x)}}$;

(2) $\sum_{n=1}^{\infty} \frac{1}{v_n} = \infty$;

(3) $\int_{a_n}^{b_n} q_-(x)dx \leq C v_n^2 P_n^3 \min_{I_n} \sqrt{p(x)}$.

Then $\ell(y)$ *is of the LPC at* ∞.

Read improved Knowles' result and he gave the following conclusion.

Theorem 1.5 (Read interval-type criterion [19]). *If there are mutually disjoint intervals* I_n *(n=1, 2,* $\cdots$*) with the endpoints tending to infinity as* $n \to \infty$*, such that*

(1) $q(x) \geq 0 \quad (x \in I_k)$;

(2) there exist constants $\gamma > 0$, $0 < c < 1$ *and interval* $I_n \subset J_n (n = 1, 2, \cdots)$*, such that* $\int_J p^{-\frac{1}{2}} dx \geq \gamma \int_{J_n} p^{-\frac{1}{2}} dx$ *on each connected component* J *of* $J_n \backslash I_n$*, and*

$$\sum_{n=1}^{\infty} \left\{ \exp\left[c \int_{I_n} \left(\frac{q}{p}\right)^{\frac{1}{2}} dx \right] - 1 \right\} \left(\int_{J_n} p^{-\frac{1}{2}} dx \right)^2 = \infty,$$

then $\ell(y)$ *is of the LPC at* ∞.

For spectral theory of fractional Sturm-Liouville operator, the research draw many researchers' attention in recent years (see [4, 6, 7, 11, 22] for detials). Meanwhile, there are a lot of attention being paid to finding the more suitable definitions of fractional derivatives, and many definitions in the existing literature, such as the Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald-Letnikov, Hadamard, are defined. In 2014, R. Khalil et al. in [13] introduced a new simple well-behaved definition of the fractional derivative called conformable fractional derivative. This new definition satisfies almost all the requirements of standard derivative, such as, the chain rule, integration by part, fractional power series expansions. For recent results from conformable fractional calculus we refer the readers to [3, 13].

In this paper, we consider the following 2α-order conformable fractional Sturm-Liouville operator

$$\ell_\alpha(y) = -T_\alpha\left(pT_\alpha y\right) + qy, x \in [a, \infty), a > 0, \tag{1.2}$$

where $p > 0, q$ are real-valued continuous functions and p is α-differentiable and T_α denotes the conformable fractional derivative of order $0 < \alpha \leq 1$, we note that when $\alpha = 1$, (1.2) reduce to (1.1).

In 2018, Dumitru Baleanu [4] promoted the Levinson criterion to 2α-order conformable fractional Sturm-Liouville operator.

Theorem 1.6 (Dumitru Baleanu [4]). *Suppose* $M(x)$ *is a positive,* $\alpha-$*nondecreasing function on* $[a, \infty), a > 0$. *If*

(i) for sufficiently large value of x, $q(x) > -KM(x)$*, where* K *is a constant;*

(ii) $\displaystyle\int_a^\infty \frac{1}{\sqrt{pM}} \mathrm{d}_\alpha x = \infty$;

(iii) $\displaystyle\limsup_{x\to\infty} \frac{T_\alpha M \cdot \sqrt{p}}{\sqrt{M^3}} < \infty$,

then the operator $\ell_\alpha(y)$ is of the LPC at ∞.

In 2020, Z. Zheng et.al. [22] promoted the Read criterion to 2α-order conformable fractional Sturm-Liouville operator.

Theorem 1.7. *Suppose that $w(x)$ is a non-negative local absolutely continuous function defined on $[a,\infty), a > 0$ and $q(x) = q_1(x) + q_2(x) + q_3(x)$, such that*

(1) $q_1(x) \geq 0$, for some $\delta > 0$, there exists a constant K, such that $(1+\delta)p(T_\alpha w)^2 - q_1 w^2 \leq K$;

(2) $-q_2 w^2 \leq K$;

(3) There exists a constant d with $0 \leq d \leq 1$ such that $w^d p^{-\frac{1}{2}}|Q| \leq K$, where $T_\alpha Q = q_3 w^{1-d}$;

(4) $\displaystyle\int_a^\infty w^2 \left(\frac{q_1}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x = \infty.$

Then the operator $\ell_\alpha(y)$ is of the LPC at ∞.

Theorem 1.8. *Suppose that $w(x)$ is a non-negative local absolutely continuous function on $[a,\infty), a > 0$, and $q(x) = q_1(x) + q_2(x) + q_3(x)$. If conditions (1)-(3) are fulfilled, and*

(4)' $\displaystyle\int_a^\infty w p^{-\frac{1}{2}} \mathrm{d}_\alpha x = \infty,$

then the operator $\ell_\alpha(y)$ is of the LPC at ∞.

Theorem 1.9. *For sufficiently large value of x, if $q(x) > -Kx^{2\alpha}$, $K > 0$, then the operator $\ell_\alpha(y)$ is of the LPC at ∞.*

Based on the above information, a natural problem is that whether the Eastham-Thompson-Read interval-type LPC criterion and Read interval-type criterion can be improved to the 2α-order conformable fractional Sturm-Liouville operator (1.2). In this paper, we give a confirm answer to these questions.

The rest of the paper is organized as follows: In Section 2, we recall some definitions and results of conformable fractional calculus. In Section 3, we use an example to introduce the possibility of 2α-order conformable fractional interval-type criteria and obtain limit-point interval-type criteria for 2α-order conformable fractional Sturm-Liouville operator. Two examples illustrating the main results are presented in Section 4.

2. Conformable Fractional Calculus

In this section, we give the definitions and properties of conformable fractional derivative and integral, which are important to the proofs of the main results.

Definition 2.1 ([13]). For given a function $u : [0, \infty) \to \mathbb{R}$, the left conformable fractional derivative of u of order α is defined by

$$(\mathbf{T}_\alpha^{\mathbf{t_0}} u)(t) = u^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon(t - t_0)^{1-\alpha}) - u(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0, 1]$. When $\alpha = 1$, this derivative of $u(t)$ coincides with $u'(t)$. If $(\mathbf{T}_\alpha^{\mathbf{t_0}} u)(t)$ exists on (t_0, t_1) £¬ then

$$(\mathbf{T}_\alpha^{\mathbf{t_0}} u)(t_0) = \lim_{t \to t_0} u^{(\alpha)}(t).$$

By the definition of the left conformable fractional derivative, we see that $(\mathbf{T}_\alpha^{\mathbf{t_0}} u)(t) > 0$ on an interval implies the monotone increasing of u on this interval.

Definition 2.2. Let $\alpha \in (0, 1]$. The left conformable fractional integral of order α starting at t_0 is defined by

$$(\mathbf{I}_\alpha^{\mathbf{t_0}} u)(t_0) = \int_{t_0}^{t} (s - t_0)^{\alpha - 1} u(s)\, \mathrm{d}s := \int_{t_0}^{t} u(s)\, \mathrm{d}_{t_0}^{\alpha} s.$$

If the conformable fractional integral exists, we call u is α–integrable.

Lemma 2.1 (see [1]). *Let $\alpha \in (0, 1]$, and $u \in C^1([t_0, \infty), \mathbb{R})$. Then for all $t > t_0$ we have*

$$\mathbf{I}_\alpha^{\mathbf{t_0}} \mathbf{T}_\alpha^{\mathbf{t_0}} u(t) = u(t) - u(t_0),$$

and

$$\mathbf{T}_\alpha^{\mathbf{t_0}} \mathbf{I}_\alpha^{\mathbf{t_0}} u(t) = u(t).$$

Lemma 2.2 (see [13]).

(1) $\mathbf{T}_\alpha^{\mathbf{t_0}}(au + bv) = a\mathbf{T}_\alpha^{\mathbf{t_0}}(u) + b\mathbf{T}_\alpha^{\mathbf{t_0}}(v)$ *for all real constant a, b.*

(2) $\mathbf{T}_\alpha^{\mathbf{t_0}}(uv) = u\mathbf{T}_\alpha^{\mathbf{t_0}}(v) + v\mathbf{T}_\alpha^{\mathbf{t_0}}(u)$.

(3) $\mathbf{T}_\alpha^{\mathbf{t_0}}(t^p) = pt^{p-\alpha}$ *for all p.*

(4) $\mathbf{T}_\alpha^{\mathbf{t_0}}(\frac{u}{v}) = \dfrac{v\mathbf{T}_\alpha^{\mathbf{t_0}}(u) - u\mathbf{T}_\alpha^{\mathbf{t_0}}(v)}{v^2}$.

(5) $\mathbf{T}_\alpha^{\mathbf{t_0}}(c) = 0$, *where c is a constant.*

Lemma 2.3 (see [1]). *Let $u, v : [t_0, t_1] \to \mathbb{R}$ be two functions with u, v being differentiable. Then*

$$\int_{t_0}^{t_1} u(s) T_\alpha v(s) \mathrm{d}_{t_0}^{\alpha} s = u(s)v(s)|_{t_0}^{t_1} - \int_{t_0}^{t_1} v(s) T_\alpha u(s) \mathrm{d}_{t_0}^{\alpha} s.$$

Lemma 2.4 (Chain Rule, see [15]). *Let $u : \mathbb{R} \to \mathbb{R}$ be a differential function and $y(t) : \mathbb{R} \to \mathbb{R}$ be an $\alpha-$differentiable function. Then we get*

$$T_\alpha u(y(t)) = u'(y(t)) T_\alpha y(t). \tag{2.1}$$

Similar to the second order Sturm-Liouvile equation, we introduce the second order conformable fractional Sturm-Liouville equation as

$$\ell_\alpha(y) = -T_\alpha(pT_\alpha y) + qy = \lambda y, x > 0, \tag{2.2}$$

where $p > 0$, q are real-valued continuous functions and p is α-differentiable on given interval. By a solution of (2.2), we mean a function $y(t)$ which satisfies (2.2) a.e. on $[0, \infty)$. If the coefficients in (2.2) are smooth sufficiently, then the equation (2.2) turns into

$$-x^{1-\alpha}\left(px^{1-\alpha}y'\right)' + qy = \lambda y, x > 0. \tag{2.3}$$

3. Limit-point interval-type criteria

In this section, we give an example to illustrate the possibility of interval-type criteria of LPC for 2α-order conformable fractional differential operator firstly. Then we give interval-type theorems of LPC for 2α-order conformable fractional Sturm-Liouville operators.

Example 3.1. Consider the 2α-order conformable fractional differential operator

$$M_\alpha(y) \equiv -T_\alpha(T_\alpha y) - (x^n + \frac{1}{\alpha}x^\alpha e^{\frac{1}{\alpha}x^\alpha}\cos e^{\frac{1}{\alpha}x^\alpha})y \quad \text{on} \quad [a,\infty)$$

with $a>0, \alpha\in(0,1]$. We obtain that $M_\alpha(y)$ is of the LPC at ∞ when $n\le 2\alpha$.

Proof. By $M_\alpha(y)=0$, we see that $p(x)=1$. Taking

$$w = x^{-\alpha}, q_1 = 0, q_2 = -x^n, q_3 = -\frac{1}{\alpha}x^\alpha e^{\frac{1}{\alpha}x^\alpha}\cos e^{\frac{1}{\alpha}x^\alpha},$$

we verify all the conditions of Theorem 1.8. According to the conformable fractional calculus, we get $(1+\delta)p(T_\alpha x^m)^2 - q_1 w^2 = (1+\delta)\alpha^2 x^{-4\alpha} \le K$, thus condition (1) holds; we get $-q_2 w^2 = x^n x^{-2\alpha} = x^{n-2\alpha}$, and since $n\le 2\alpha$, we get $x^{n-2\alpha}$ is bounded, hence condition (2) is fulfilled. Moreover, we see that $\int_a^\infty wp^{-\frac{1}{2}}\mathrm{d}_\alpha x = \int_a^\infty x^{-\alpha}\mathrm{d}_\alpha x = \frac{1}{\alpha}\ln(x^\alpha)\big|_a^\infty = \infty$ implies (4)' holds. Now, let $d=1$, $q_3 = -\frac{1}{\alpha}x^\alpha e^{\frac{1}{\alpha}x^\alpha}\cos e^{\frac{1}{\alpha}x^\alpha}$, $Q = \int_1^x -\frac{1}{\alpha}t^\alpha e^{\frac{1}{\alpha}t^\alpha}\cos e^{\frac{1}{\alpha}t^\alpha}\mathrm{d}_\alpha t$, we obtain that $w^d p^{-\frac{1}{2}}|Q| = \frac{1}{x^\alpha}\left|\int_1^x -\frac{1}{\alpha}t^\alpha e^{\frac{1}{\alpha}t^\alpha}\cos e^{\frac{1}{\alpha}t^\alpha}\mathrm{d}_\alpha t\right| \le K$. According to Lemma 2.3 and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{x^\alpha}\left|\int_1^x -\frac{1}{\alpha}t^\alpha e^{\frac{1}{\alpha}t^\alpha}\cos e^{\frac{1}{\alpha}t^\alpha}\mathrm{d}_\alpha t\right| \\
=&\frac{1}{x^\alpha}\left|-\frac{1}{\alpha}t^\alpha\cdot\sin e^{\frac{1}{\alpha}t^\alpha}\Big|_1^x + \int_1^x \sin e^{\frac{1}{\alpha}t^\alpha}\mathrm{d}t\right| \\
\le&\frac{1}{x^\alpha}\left(\left|-\frac{1}{\alpha}x^\alpha\cdot\sin e^{\frac{1}{\alpha}x^\alpha}\right| + |\frac{1}{\alpha}\sin e^{\frac{1}{\alpha}}| + \frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}\right) \\
\le&\frac{2}{\alpha}\le K.
\end{aligned}$$

Hence all conditions of Theorem 1.8 are fulfilled, so the 2α-order conformable fractional differential operator $M_\alpha(y)$ is of the LPC at ∞ when $n\le 2\alpha$. □

This example shows that although the oscillation of $q(x)$ is strong, it even break through the speed of $q(x)\to\infty$ in Theorem 1.8, but it still maintains the LPC for conformable fractional Sturm-Liouville operators. Therefore, in order to ensure that M is of the LPC, there is no requirement for the whole property of $q(x)$ in the interval $[a,\infty)$, $q(x)$ needs only to have better properties in a part of the range. This leads to the following interval-type criteria of LPC.

Theorem 3.1. *If there exist mutually disjoint intervals $I_n=[a_n,b_n]$ (n=1, 2, $\cdots$) with $b_n\to\infty$ as $n\to\infty$, and a sequence of positive numbers $\{v_n\}$, such that*

(1) $v_n P_n^2 \ge K > 0$, where $P_n = \int_{a_n}^{b_n}\frac{1}{\sqrt{p(x)}}\mathrm{d}_\alpha x$;

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{v_n}=\infty$;

(3) $\displaystyle\int_{a_n}^{b_n} q_-(x)\mathrm{d}_\alpha x \le C v_n^2 P_n^3 \min_{I_n}\sqrt{p(x)}$, *where* $q_-(x)=\min\{q(x),0\}$.

Then $\ell_\alpha(y)$ *is of the LPC at* ∞.

Proof. We give the proof by three steps.

Step One. Defining function $w(x)$.

We define $\left[v_nP_n^2\right]$ as the integer part of $v_nP_n^2$, then we get $1\le\left[v_nP_n^2\right]\le v_nP_n^2<\left[v_nP_n^2\right]+1$. We divide I_n into $\left[v_nP_n^2\right]+1$ intervals J_{n_i} (i=1, 2, $\cdots$, $\left[v_nP_n^2\right]+1$), we see that

$$\frac{1}{2v_nP_n}\le\int_{J_{n_i}} p^{-\frac{1}{2}}(x)\mathrm{d}_\alpha x\le\frac{1}{v_nP_n} \tag{3.1}$$

at each of such intervals J_{n_i}.

If more than half of the intervals satisfy

$$\int_{J_{n_i}} q_-(x)\mathrm{d}_\alpha x>2Cv_nP_n\min_{I_n}\sqrt{p(x)},$$

then we get

$$\frac{\left[v_nP_n^2\right]+1}{2}\int_J q_-(x)\mathrm{d}_\alpha x>Cv_nP_n\left(\left[v_nP_n^2\right]+1\right)\min_{I_n}\sqrt{p(x)}>Cv_n^2P_n^3\min_{I_n}\sqrt{p(x)}.$$

So the remaining intervals J_{n_j} must be satisfied

$$\int_{J_{n_j}} q_-(x)\mathrm{d}_\alpha x\le 2Cv_nP_n\min_{I_n}\sqrt{p(x)}. \tag{3.2}$$

On each such an interval $J_{n_j}=[c,d]$, we take $e\in[c,d]$, such that

$$\int_c^e p^{-\frac{1}{2}}(x)\mathrm{d}_\alpha x=\frac{1}{2}\int_c^d p^{-\frac{1}{2}}(x)\mathrm{d}_\alpha x, \tag{3.3}$$

we define

$$w(x)=\begin{cases}\displaystyle\int_c^x p^{-\frac{1}{2}}(t)\mathrm{d}_\alpha t, & c\le x\le e,\\ \displaystyle w(e)-\int_e^x p^{-\frac{1}{2}}(t)\mathrm{d}_\alpha t, & e<x\le d.\end{cases}$$

On the other parts of I_n and the complementary set of $\cup_n I_n$, we define $w(x)=0$. Hence $w(x)$ is a non-negative local absolutely continuous function, and

$$\max_J w(x)=w(e)=\frac{1}{2}\int_J p^{-\frac{1}{2}}(x)\mathrm{d}_\alpha x\le\frac{1}{2v_nP_n}. \tag{3.4}$$

Moreover, on the intervals $J_{n_j}=[c,d]$ satisfying (3.2),

$$T_\alpha(w(x))=\begin{cases}\displaystyle T_\alpha\int_c^x p^{-\frac{1}{2}}(t)dt=p^{-\frac{1}{2}}(x), & c\le x\le e,\\ \displaystyle T_\alpha\left[w(e)-\int_e^x p^{-\frac{1}{2}}(t)dt\right]=-p^{-\frac{1}{2}}(x), & e<x\le d.\end{cases}$$

On the other parts of I_n and the complementary set of $\cup_n I_n$, $T_\alpha(w(x)) = 0$ a.e., so $p(T_\alpha w)^2 \le K$.

Step Two. Dividing potential function $p(x)$.
On the $\left[v_n P_n^2\right] + 1$ subintervals $J_{n_j} = [c, d]$ of each interval I_n, we define

$$q_0(x) = \frac{\alpha}{d^\alpha - c^\alpha} \int_c^d q_-(x) \mathrm{d}_\alpha x.$$

Between I_n and I_{n+1}, i.e., $[b_n, a_{n+1}]$, we define

$$q_0(x) = \frac{\alpha}{a_{n+1}^\alpha - b_n^\alpha} \int_{b_n}^{a_{n+1}} q_-(x) \mathrm{d}_\alpha x.$$

Then $q_0(x)$ is the step function on $[a, \infty)$. Obviously, $[a, \infty)$ contains some subintervals I, which satisfies

$$\int_I \left[q_-(x) - q_0(x)\right] \mathrm{d}_\alpha x = 0.$$

Let $q_1(x) = 0, q_2(x) = q_+(x) - q_0(x), q_3(x) = -q_-(x) + q_0(x)$. Then $q(x) = q_1(x) + q_2(x) + q_3(x)$.

Step Three. Verifying the conditions of Theorem 1.8.
Firstly, it is easy to see that

$$(1 + \delta)p(T_{\alpha w})^2 - q_1 w^2 = (1 + \delta)p(T_{\alpha w})^2 \le (1 + \delta)K. \tag{3.5}$$

Secondly, on the intervals where the inequality (3.2) is not satisfied, $w(x) = 0$, then $q_0 w^2 = 0$. And on the intervals satisfying the inequality (3.2),

$$q_0 w^2 = \left(\frac{\alpha}{d^\alpha - c^\alpha} \int_c^d q_-(x) \mathrm{d}_\alpha x\right) w^2 \le \frac{\alpha}{d^\alpha - c^\alpha} 2C v_n P_n \min_{I_n} \sqrt{p(x)} \frac{1}{4v_n^2 P_n^2}.$$

Because of

$$\int_c^d p^{-\frac{1}{2}}(x) \mathrm{d}_\alpha x \le \max\left(\frac{1}{\sqrt{p(x)}}\right) \int_c^d \mathrm{d}_\alpha x = \frac{d^\alpha - c^\alpha}{\alpha \min_{I_n} \sqrt{p(x)}},$$

we have

$$\frac{\alpha \min_{I_n} \sqrt{p(x)}}{d^\alpha - c^\alpha} \le \frac{1}{\int_c^d p^{-\frac{1}{2}}(x) \mathrm{d}_\alpha x} \le 2v_n P_n,$$

then we get

$$q_0 w^2 \le C.$$

So for the interval $\cup_n I_n$,

$$-q_2 w^2 = (q_0 - q_+)\, w^2 \le C - q_+ w^2 \le C. \tag{3.6}$$

Thirdly, we take $d = 1$ and notice that $wp^{-\frac{1}{2}} \left|\int_a^x q_3(t) \mathrm{d}_\alpha t\right| \ne 0$ only in the interval J_{n_j} that satisfies inequality (3.2). When $x \in J_{n_j} = [c, d]$,

$$\int_a^x q_3(t) \mathrm{d}_\alpha t = \int_a^c q_3(t) \mathrm{d}_\alpha t + \int_c^x q_3(t) \mathrm{d}_\alpha t = \int_c^x q_3(t) \mathrm{d}_\alpha t.$$

Hence we get

$$\begin{aligned} wp^{-\frac{1}{2}}\left|\int_a^x q_3(t)\mathrm{d}_\alpha t\right| &\le \frac{1}{2v_nP_n\sqrt{p}}\left|\int_c^x q_3(t)\mathrm{d}_\alpha t\right| \\ &= \frac{1}{2v_nP_n\sqrt{p}}\left|-\int_c^x q_-(t)\mathrm{d}_\alpha t+\int_c^x q_0(t)\mathrm{d}_\alpha t\right| \\ &= \frac{1}{2v_nP_n\sqrt{p}}\left|-\int_c^x q_-(t)\mathrm{d}_\alpha t+\frac{\frac{1}{\alpha}(x^\alpha-c^\alpha)\alpha}{d^\alpha-c^\alpha}\int_c^x q_-(t)\mathrm{d}_\alpha t\right| \\ &\le \frac{1}{v_nP_n\sqrt{p}}\int_c^d q_-(t)\mathrm{d}_\alpha t \le \frac{2Cv_nP_n\min\sqrt{p}}{v_nP_n\sqrt{p}} \le 2C. \end{aligned} \tag{3.7}$$

Finally, since

$$\begin{aligned} &\int_J wp^{-\frac{1}{2}}\mathrm{d}_\alpha x \\ &= \int_c^e \frac{1}{\sqrt{p}}\left(\int_c^x p^{-\frac{1}{2}}\mathrm{d}_\alpha t\right)\mathrm{d}_\alpha x+\int_e^d \frac{w(e)}{\sqrt{p}}\mathrm{d}_\alpha x-\int_e^d \frac{1}{\sqrt{p}}\left(\int_c^x p^{-\frac{1}{2}}\mathrm{d}_\alpha t\right)\mathrm{d}_\alpha x \\ &= \frac{1}{2}\left(\int_c^e p^{-\frac{1}{2}}\mathrm{d}_\alpha x\right)^2+\frac{1}{4}\left(\int_J p^{-\frac{1}{2}}\mathrm{d}_\alpha x\right)^2-\frac{1}{2}\left(\int_e^d p^{-\frac{1}{2}}\mathrm{d}_\alpha x\right)^2 \\ &= \frac{1}{4}\left(\int_J p^{-\frac{1}{2}}\mathrm{d}_\alpha x\right)^2 \ge \left(\frac{1}{4v_nP_n}\right)^2, \end{aligned}$$

so we get

$$\begin{aligned} \int_a^\infty wp^{-\frac{1}{2}}\mathrm{d}_\alpha x \ge \sum_n \int_{I_n} wp^{-\frac{1}{2}}\mathrm{d}_\alpha x &\ge \sum_n \frac{[v_nP_n^2]+1}{2}\left(\frac{1}{4v_nP_n}\right)^2 \\ &\ge \frac{1}{32}\sum_n \frac{1}{v_n} = \infty. \end{aligned} \tag{3.8}$$

By (3.5)-(3.8), the conditions (1)-(4) of Theorem 1.8 are fulfilled, so $\ell_\alpha(y)$ is of the LPC at ∞. ☐

Remark 3.1. For the special case of $\alpha = 1$, Theorem 3.1 reduces to Theorem 1.4 in [8].

Theorem 3.2. *If there exist mutually disjoint intervals J_n (n=1, 2, $\cdots$), such that*

(1) $q(x) \ge 0$ for $x \in J_n$.

(2) There exist constants $\gamma > 0$, $0 < c < 1$ and subinterval $I_n \subset J_n (n = 1, 2, \cdots)$, such that $\int_J p^{-\frac{1}{2}}\mathrm{d}_\alpha x \ge \gamma \int_{I_n} p^{-\frac{1}{2}}\mathrm{d}_\alpha x$ on each connected component J of $J_n \backslash I_n$, and

$$\sum_{n=1}^\infty \left\{\exp\left[c\int_{I_n}\left(\frac{q}{p}\right)^{\frac{1}{2}}\mathrm{d}_\alpha x\right]-1\right\}\left(\int_{J_n} p^{-\frac{1}{2}}\mathrm{d}_\alpha x\right)^2 = \infty.$$

Then $\ell_\alpha(y)$ is of the LPC at ∞.

Proof. We give the proof by three steps.

Step One. Defining function $w(x)$.
For given $J_n = [a_n, b_n]$, we divide it appropriately into three subintervals $[c_{0n}, c_{1n}]$, $I_n = [c_{1n}, c_{3n}]$ and $[c_{3n}, c_{4n}]$ with $c_{0n} = a_n, c_{4n} = b_n$ such that

$$\int_{c_{0n}}^{c_{1n}} p^{-\frac{1}{2}} \mathrm{d}_\alpha x = \int_{c_{3n}}^{c_{4n}} p^{-\frac{1}{2}} \mathrm{d}_\alpha x = \gamma \int_{J_n} p^{-\frac{1}{2}} \mathrm{d}_\alpha x, \tag{3.9}$$

then we take $c_{2n} \in (c_{1n}, c_{3n})$, such that

$$\int_{c_{1n}}^{c_{2n}} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x = \int_{c_{2n}}^{c_{3n}} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x. \tag{3.10}$$

Outside the interval $\cup_n J_n$, we take $w(x) = 0$, and define $w(x)$ on J_n as follows:

$$w(x) = \begin{cases} \displaystyle\int_{c_{0n}}^{x} p^{-\frac{1}{2}} \mathrm{d}_\alpha t, & c_{0n} \le x \le c_{1n}, \\ w\left(c_{1n}\right) \exp\left[c \displaystyle\int_{c_{1n}}^{x} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha t\right], & c_{1n} \le x \le c_{2n}, \\ w\left(c_{1n}\right) \exp\left[c \displaystyle\int_{x}^{c_{3n}} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha t\right], & c_{2n} \le x \le c_{3n}, \\ \displaystyle\int_{x}^{c_{4n}} p^{-\frac{1}{2}} \mathrm{d}_\alpha t, & c_{3n} \le x \le c_{4n}. \end{cases} \tag{3.11}$$

By (3.11), we get when $c_{0n} < x < c_{1n}$ and $c_{3n} < x < c_{4n}$, there is $p(T_\alpha w)^2 = 1$, and when $c_{1n} < x < c_{2n}$, we get

$$\begin{aligned} llT_\alpha(w(x)) &= w\left(c_{1n}\right) \exp\left[c \int_{c_{1n}}^{x} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha t\right] \cdot T_\alpha \left[c \int_{c_{1n}}^{x} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha t\right] \\ &= w(x) \cdot c \left(\frac{q}{p}\right)^{\frac{1}{2}}, \end{aligned}$$

and using the same method, when $c_{2n} < x < c_{3n}$, we have

$$T_\alpha(w(x)) = -w(x) \cdot c \left(\frac{q}{p}\right)^{\frac{1}{2}}.$$

Step Two. Dividing potential function $p(x)$.
We take $q_1(x) = 0$, $q_2(x) = p(x)$, $q_3(x) = 0$ outside the interval $\cup_n J_n$, and take $q_1(x) = q(x)$, $q_2(x) = 0$, $q_3(x) = 0$ on the interval $\cup_n J_n$.

Step Three. Verifying the conditions of Theorem 1.7.
Obviously, $q_1(x) \ge 0$, we take $\delta = \frac{1}{c^2} > 0$, $K \ge \frac{1}{c^2}$ then by (3.5),

$$(1 + \delta) p(T_\alpha w)^2 - q_1 w^2 \le K, \tag{3.12}$$

hence (1) holds. (2) holds obviously since

$$-q_2 w^2 = 0 \le K. \tag{3.13}$$

Now, $q_3 = 0$, then $Q = C$. So

$$w^d p^{-\frac{1}{2}}|Q| = w^d p^{-\frac{1}{2}}|C| \le K \tag{3.14}$$

implies (3) is true. Finally, since

$$\begin{aligned}
\int_{J_n} w^2 \left(\frac{q_1}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x &= \int_{J_n} w^2 \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x \ge \int_{c_1}^{c_2} w^2 \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x \\
&= \int_{c_1}^{c_2} w^2(c_1) \exp\left[2c\int_{c_1}^{x} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha t\right] \mathrm{d}_\alpha \left(\int_{c_1}^{x} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha t\right) \\
&= \frac{w^2(c_1)}{2c} \exp\left[2c\int_{c_1}^{x} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha dt\right]\Bigg|_{c_1}^{c_2} \\
&= \frac{w^2(c_1)}{2c} \left\{\exp\left[2c\int_{c_1}^{c_2} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x\right] - 1\right\} \\
&= \frac{\gamma^2}{2c} \left\{\exp\left[c\int_{I_n} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x\right] - 1\right\} \left(\int_{J_n} p^{-\frac{1}{2}} \mathrm{d}_\alpha x\right)^2,
\end{aligned} \tag{3.15}$$

so

$$\int_a^\infty w^2 \left(\frac{q_1}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x \ge \frac{\gamma^2}{2c} \sum_{n=1}^{\infty} \left\{\exp\left[c\int_{I_n} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x\right] - 1\right\} \left(\int_{J_n} p^{-\frac{1}{2}} \mathrm{d}_\alpha x\right)^2 = \infty. \tag{3.16}$$

i.e., (4) holds. By (3.5)-(3.8), the conditions (1)-(4) of Theorem 1.7 are fulfilled, then $\ell_\alpha(y)$ is of the LPC at ∞. □

Remark 3.2. For the special case of $\alpha = 1$, Theorem 3.2 reduces to Theorem 1.5 introduced in [19].

4. Examples

In this section, we give two examples to verify our main results.

Example 4.1. Consider the 2α-order conformable fractional differential operator

$$M_\alpha(y) \equiv -T_\alpha(T_\alpha y) - x^\delta \sin(x^\beta) y \quad \text{on} \quad [a, \infty)$$

with $a > 0, \alpha \in (0, 1]$. We obtain that $M_\alpha(y)$ is of the LPC at ∞ when $0 \le \beta \le 2\alpha$.

Proof. We take $I_n = \left[((2n-1)\pi)^{\frac{1}{\beta}}, (2n\pi)^{\frac{1}{\beta}}\right]$ $(n = 1, 2, \cdots)$, obviously $q(x) \ge 0$ on I_n, so $q_-(x) = 0$ and by the definition of P_n in Theorem 3.1,

$$P_n = (2n\pi)^{\frac{\alpha}{\beta}} - ((2n-1)\pi)^{\frac{\alpha}{\beta}} = (2n\pi)^{\frac{\alpha}{\beta}} \left[1 - \left(1 - \frac{1}{2n}\right)^{\frac{\alpha}{\beta}}\right] = O\left(n^{\frac{\alpha}{\beta}-1}\right). \tag{4.1}$$

Then we take $v_n = n^{-2(\frac{\alpha}{\beta}-1)}$ and verify the conditions of Theorem 3.1. According to the conformable fractional calculus, we get $v_n P_n^2 = n^{-2(\frac{\alpha}{\beta}-1)} \cdot O\left(n^{2(\frac{\alpha}{\beta}-1)}\right) \ge$

$K > 0$; and $\sum_{n=1}^{\infty} \frac{1}{v_n} = \sum_{n=1}^{\infty} n^{2(\frac{\alpha}{\beta}-1)}$, hence when $2(\frac{\alpha}{\beta}-1) \geq -1$, i.e., $0 < \beta \leq 2\alpha$,

$$\sum_{n=1}^{\infty} \frac{1}{v_n} = \sum_{n=1}^{\infty} n^{2(\frac{\alpha}{\beta}-1)} = \infty;$$

since $q_-(x) = 0$, it is obviously that $\int_{a_n}^{b_n} q_-(x)\mathrm{d}_\alpha x \leq Cv_n^2 P_n^3 \min_{I_n} \sqrt{p(x)}$ holds. Hence by Theorem 3.1, the 2α-order conformable fractional differential operator $M_\alpha(y)$ is of the LPC at ∞ when $0 \leq \beta \leq 2\alpha$. $\square$

Example 4.2. Consider the 2α-order conformable fractional differential operator

$$M_\alpha(y) \equiv -T_\alpha(T_\alpha y) - x^\delta \sin(\pi x^\beta) y \quad \text{on} \quad [a, \infty)$$

with $a > 0$, $\alpha \in (0,1]$, $\delta, \beta > 0$. We obtain that $M_\alpha(y)$ is of the LPC at ∞ when $\beta - \frac{\delta}{2} < \alpha$.

Proof. By the definition of $M_\alpha(y)$, it is clear that the potential function $q(x) \to \infty$ on some intervals. Then we take $J_n = \left[(2n)^{\frac{1}{\beta}}, (2n+1)^{\frac{1}{\beta}}\right]$, $I_n = \left[\left(2n+\frac{1}{6}\right)^{\frac{1}{\beta}}, \left(2n+\frac{5}{6}\right)^{\frac{1}{\beta}}\right]$, and verify the conditions of Theorem 3.2. We get:

(1) It is clear that $q(x) \geq 0$, when $x \in J_n$;

(2) According to the conformable fractional calculus, we get

$$\begin{aligned}
\int_{I_n} \left(\frac{q}{p}\right)^{\frac{1}{2}} dx &= \int_{I_n} x^{\frac{\delta}{2}} \sqrt{\sin \pi x^\beta} dx \geq \frac{1}{\sqrt{2}} \int_{I_n} x^{\frac{\delta}{2}} dx \\
&= \frac{1}{\sqrt{2}\left(\frac{\delta}{2}+\alpha\right)} \left[\left(2n+\frac{5}{6}\right)^{\frac{1}{\beta}\left(\frac{\delta}{2}+\alpha\right)} - \left(2n+\frac{1}{6}\right)^{\frac{1}{\beta}\left(\frac{\delta}{2}+\alpha\right)}\right] \\
&= \frac{1}{\sqrt{2}\left(\frac{\delta}{2}+1\right)} \left(2n+\frac{5}{6}\right)^{\frac{1}{\beta}\left(\frac{\delta}{2}+\alpha\right)} \left[1 - \left(1 - \frac{\frac{4}{6}}{2n+\frac{5}{6}}\right)^{\frac{1}{\beta}\left(\frac{\delta}{2}+\alpha\right)}\right] \\
&= O\left(n^{\frac{1}{\beta}\left(\frac{\delta}{2}+\alpha\right)-1}\right).
\end{aligned} \tag{4.2}$$

Since there is an exponential function in (2), we only need to make $\frac{1}{\beta}\left(\frac{\delta}{2}+\alpha\right)-1 > 0$, i.e. $\beta - \frac{\delta}{2} < \alpha$. Then

$$\sum_{n=1}^{\infty} \left\{\exp\left[c\int_{I_n} \left(\frac{q}{p}\right)^{\frac{1}{2}} \mathrm{d}_\alpha x\right] - 1\right\} \left(\int_{J_n} p^{-\frac{1}{2}} \mathrm{d}_\alpha x\right)^2 = \infty.$$

Hence, the 2α-order conformable fractional differential operator $M_\alpha(y)$ is of the LPC at ∞ when $\beta - \frac{\delta}{2} < \alpha$. $\square$

Acknowledgements

We would like to thank the referees for their valuable comments to improve our paper.

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