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WEAK N-BEST POAFD FOR SOLVING PARABOLIC EQUATIONS IN REPRODUCING KERNEL HILBERT SPACE

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Abstract The analytical solutions and numerical ones of parabolic equations in one space variable and the time variable are constructed by weak N-best pre-orthogonal adaptive Fourier decomposition method (weak N-best POAFD) in reproducing kernel Hilbert space (RKHS). To apply weak N-best POAFD, we first choose a dictionary for weak N-best POAFD and implement pre-orthonormalization to all dictionary elements. Then select some parameters by weak N-best maximal selection principle and determine some normalized dictionary elements iteratively. Thus, the analytical solution can be expressed as a linear combination of these determined normalized dictionary elements with a fast convergence rate. Some numerical examples confirm the good accuracy and applicability of the weak N-best POAFD method in solving the partial differential equations.

Keywords Weak N-best POAFD, weak N-best maximum selection principle, parabolic equation, reproducing kernel Hilbert space.

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1. Introduction

Consider the following one-dimensional time-dependent partial differential equation defined on $\Omega = [0,1]^2$

$$u_t - a(t)u_{xx} = f(x,t), \quad (x,t) \in (0,1)^2, \tag{1.1}$$

with the conditions

$$\begin{aligned} &u(0,t) = g_0(t),\ u(1,t) = g_1(t), && 0 \le t \le 1, \\ &u(x,0) = v_0(x), && 0 \le x \le 1, \end{aligned} \tag{1.2}$$

where u is a dependent variable, t is an initial-value independent variable, x is a boundary-value independent variable, u is unknown, and the variables f, v_0, g_0, g_1 are known. $a > 0$, $a, f \in W^{(1,1)}(\Omega)$, $u \in W^{(3,2)}(\Omega)$, where $W^{(1,1)}(\Omega)$ and $W^{(3,2)}(\Omega)$ are reproducing kernel Hilbert spaces defined in Section 2. Suppose that f is given such that PDE (1.1)–(1.2) satisfies the existence and uniqueness of the solution.

First, we homogenize the initial and boundary conditions in (1.1)-(1.2) to obtain

$$\bar{u}_t - a(t)\bar{u}_{xx} = F(x,t), \qquad (x,t) \in (0,1)^2,$$

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$$\bar{u}(0,t)=0,\ \bar{u}(1,t)=0, \qquad 0\le t\le 1, \tag{1.3}$$
$$\bar{u}(x,0)=0, \qquad 0\le x\le 1,$$

where $\bar{u}=u-v_0(x)-x(g_1(t)-g_1(0))-(1-x)(g_0(t)-g_0(0))$, $F(x,t)=f(x,t)-x\dfrac{dg_1(t)}{dt}-(1-x)\dfrac{dg_0(t)}{dt}+a(t)\dfrac{d^2v_0(x)}{dx^2}$. For convenience and without ambiguity, $\bar{u}$ is still marked with u.

We introduce a linear operator $L:W^{(3,2)}(\Omega)\to W^{(1,1)}(\Omega)$,

$$(Lu)(x,t)=u_t-a(t)u_{xx}.$$

Then the problem (1.3) can be converted to the following problem with homogeneous conditions

$$\begin{cases} Lu(x,t)=F(x,t), & (x,t)\in(0,1)^2, \\ u(0,t)=0,\ u(1,t)=0, & 0\le t\le 1, \\ u(x,0)=0, & 0\le x\le 1. \end{cases} \tag{1.4}$$

PDEs have many applications in various fields, but the analytical solutions of most PDEs are hard to obtain. So numerical procedures are needed to find approximate solutions. Many computational methods have been developed and applied to approximate the solution of PDE. Such as finite difference methods (FDMs) [7]; the method of weighted residuals (MWR) [5], like the Galerkin methods and collocation methods [13]; Wavelet methods [12], and finite element method [14] etc. In RKHS, Cui et al. presented the reproducing kernel method (RKM), and many researchers devoted themselves to applying RKM to solve PDEs [1,2,4,6].

On the other hand, Qian and his co-workers have proposed adaptive Fourier decomposition (AFD) type methods, including AFD, pre-orthogonal adaptive Fourier decomposition (POAFD) and several other variations [8,10,11], as sparse non-basis methods for more general approximation problems in RKHS. Based on the maximal selection principle they adopted, this type of approximation representation converges fast in the first few steps.

Thanks to the previous work of Cui et al. on RKM, and inspired by the advantage of the AFD type method benefitting from its maximal selection principle, we aim to obtain the solution of PDE (1.4) by weak N-best POAFD. For the weak N-best POAFD method, we choose the system $\{L^*K_P\}_{P\in\Omega}$ used by Cui et al. [4] as a dictionary such that the solution of (1.4) has a simple expression. However, the boundary vanishing condition for POAFD does not hold under the dictionary we choose. Thus, we select parameters by the weak N-best maximal selection principle. Here, "weak" means a weak type optimal selection strategy that establishes the existence of parameters to make the residual as small as possible at each step. "N-best" means we choose more than one parameter minimizing the residual as much as possible iteratively. Such selection principle is more greedy than that of the existing orthogonal greedy algorithm, thus the weak N-best POAFD has better convergence property.

This paper is organized as follows. In Section 2, we introduce RKHSs related to (1.4). In Section 3, we construct the analytical solutions and numerical ones by the weak N-best POAFD method. And the convergence properties of the numerical solutions, as well as the stability of the weak N-best POAFD method, are discussed. Numerical experiments are carried out in Section 4. The paper ends with a brief conclusion in Section 5.

2. Several reproducing kernel Hilbert spaces

In this section, we introduce the following RKHSs $W^1[0,1]$, $W^2[0,1]$ and $W^3[0,1]$. For convenience, we use the notations $\|\cdot\|_{W^1}= \|\cdot\|_{W^1[0,1]}$, $\|\cdot\|_{W^2}= \|\cdot\|_{W^2[0,1]}$, and $\|\cdot\|_{W^3}= \|\cdot\|_{W^3[0,1]}$. The same is true for their inner products.

Definition 2.1 (Theorem 1.3.2, [4]). The Sobolev space $W^1[0,1]$ is defined as follows

$$W^1[0,1] = \{w|w \text{ is absolutely continuous on } [0,1],\ w' \in L^2[0,1].\}$$

under the inner product and the norm

$$\langle w,v\rangle_{W^1} = w(0)v(0) + \int_0^1 w'(\xi)v'(\xi)d\xi, \quad w,v\in W^1[0,1],$$
$$\|w\|_{W^1}= \sqrt{\langle w,w\rangle_{W^1}}, \quad w\in W^1[0,1].$$

The Hilbert space $W^1[0,1]$ is a RKHS admitting the reproducing kernel

$$K^1(s,t) = K^1_t(s) = \begin{cases} 1+s, & s\le t,\\ 1+t, & s>t. \end{cases} \tag{2.1}$$

Definition 2.2 (Theorem 1.3.5, [4]). The Sobolev space $W^2[0,1]$ is defined as

$$W^2[0,1] = \{w|w' \text{ is absolutely continuous on } [0,1], w'' \in L^2[0,1], w(0)=0\}$$

with the inner product and the norm respectively given by

$$\begin{aligned}&\langle w,v\rangle_{W^2} = w(1)v(1) + \int_0^1 w''(\xi)v''(\xi)d\xi, \quad w,v\in W^2[0,1],\\ &\|w\|_{W^2}= \sqrt{\langle w,w\rangle_{W^2}}, \quad w\in W^2[0,1].\end{aligned} \tag{2.2}$$

The space $W^2[0,1]$ is a RKHS with the reproducing kernel

$$K^2(s,t) = K^2_t(s) = \begin{cases} \dfrac{1}{6}s(s^2(t-1)+t(8-3t+t^2)), & s\le t,\\ \dfrac{1}{6}t(-3s^2+s^3-t^2+s(8+t^2)), & s>t. \end{cases} \tag{2.3}$$

Definition 2.3 (Theorem 1.3.5, [4]). The RKHS $W^3[0,1]$ is defined as

$$W^3[0,1] = \{w|w'' \text{ is absolutely continuous}, w^{(3)} \in L^2[0,1], w(0)=w(1)=0\}$$

with the inner product and the norm defined by

$$\begin{aligned}&\langle w,v\rangle_{W^3} = w'(0)v'(0) + \int_0^1 w^{(3)}(\xi)v^{(3)}(\xi)d\xi, \quad w,v\in W^3[0,1],\\ &\|w\|_{W^3}= \sqrt{\langle w,w\rangle_{W^3}}, \quad w\in W^3[0,1].\end{aligned} \tag{2.4}$$

The RKHS $W^3[0,1]$ has the reproducing kernel

$$K^3(s,t) = \begin{cases} -\dfrac{s(t-1)}{120}(120t-5s^3t+s^4(1+t)+st(t^3-4t^2+6t-120)), & s\le t,\\ -\dfrac{t(s-1)}{120}(6s^2t-4s^3t+s^4t+t^4+s(120-120t-5t^3+t^4)), & s>t. \end{cases}$$

Remark 2.1. For the reproducing kernels $K^1(s,t)$ of $W^1[0,1]$, $K^2(s,t)$ of $W^2[0,1]$ and $K^3(s,t)$ of $W^3[0,1]$, they satisfy the following:
(i) The reproducing properties: for any $s,t \in [0,1]$, any $w_i \in W^i[0,1]$, $i = 1,2,3$, then

$$\langle w_i, K_t^i \rangle_{W^i} = w_i(t). \tag{2.5}$$

(ii) The symmetric properties:

$$K^i(s,t) = K^i(t,s). \tag{2.6}$$

Let $\Omega = [0,1] \times [0,1]$. We can define the tensor products of RKHSs $W^{(3,2)}(\Omega) = W^3[0,1] \times W^2[0,1]$ and $W^{(1,1)}(\Omega) = W^1[0,1] \times W^1[0,1]$ as follows. For simplicity, we use the following notations $\|\cdot\| = \|\cdot\|_{W^{(3,2)}(\Omega)}$ and $\|\cdot\|_{W^{(1,1)}} = \|\cdot\|_{W^{(1,1)}(\Omega)}$. The same is true for their inner products.

Definition 2.4 (Theorem 1.5.2, [4])**.** The space $W^{(3,2)}(\Omega)$ is defined as

$$W^{(3,2)}(\Omega) = \left\{ u \;\middle|\; \begin{array}{l} u : \Omega \to \mathbb{R}^2, \dfrac{\partial^3 u}{\partial x^2 \partial t} \text{ exists and is completely continuous,} \\ \dfrac{\partial^5 u}{\partial x^3 \partial t^2} \in L^2(\Omega), u(0,t) = u(1,t) = u(x,0) = 0. \end{array} \right\}$$

The inner product and the norm in $W^{(3,2)}(\Omega)$ are given by

$$\begin{aligned} \langle u, v \rangle &= \left\langle \frac{du(0,t)}{dx}, \frac{dv(0,t)}{dx} \right\rangle_{W^2} + \int_0^1 \frac{d^3u(x,1)}{dx^3} \cdot \frac{d^3v(x,1)}{dx^3} dx \\ &\quad + \int_0^1 \int_0^1 \frac{\partial^5 u(x,t)}{\partial x^3 \partial t^2} \cdot \frac{\partial^5 v(x,t)}{\partial x^3 \partial t^2} dt dx, \\ \|u\| &= \sqrt{\langle u, u \rangle}, \quad u \in W^{(3,2)}(\Omega). \end{aligned} \tag{2.7}$$

$W^{(3,2)}(\Omega)$ is a RKHS with reproducing kernel [4]

$$K^{(3,2)}(x,\xi,t,\eta) = K^{(3,2)}_{(\xi,\eta)}(x,t) = K^3(x,\xi)K^2(t,\eta), \tag{2.8}$$

where K^3, K^2 are the reproducing kernels of $W^3[0,1]$ and $W^2[0,1]$, respectively.
Similarly, we can define the second tensor product of RKHS $W^{(1,1)}(\Omega)$ [4].

$$W^{(1,1)}(\Omega) = \left\{ u | u : \Omega \to \mathbb{R}^2, u \text{ is completely continuous on } \Omega, \ \frac{\partial^2 u}{\partial x \partial t} \in L^2(\Omega) \right\}$$

with the inner product and the norm as follows:

$$\begin{aligned} \langle f, g \rangle_{W^{(1,1)}} &= \langle f(0,t), g(0,t) \rangle_{W^1} + \int_0^1 \frac{df(x,0)}{dx} \cdot \frac{dg(x,0)}{dx} dx \\ &\quad + \int_0^1 \int_0^1 \frac{\partial^2 f(x,t)}{\partial x \partial t} \cdot \frac{\partial^2 g(x,t)}{dx dt} dt dx, \\ \|f\|_{W^{(1,1)}} &= \sqrt{\langle f, f \rangle_{W^{(1,1)}}}. \end{aligned} \tag{2.9}$$

The RKHS $W^{(1,1)}(\Omega)$ has reproducing kernel [4]

$$K^{(1,1)}(x,\xi,t,\eta) = K^{(1,1)}_{(\xi,\eta)}(x,t) = K^1(x,\xi)K^1(t,\eta), \tag{2.10}$$

where K^1 is the reproducing kernel of $W^1[0,1]$.

Remark 2.2. For any x, t, ξ, η in $[0,1]$, any $f \in W^{(1,1)}(\Omega)$ and any $u \in W^{(3,2)}(\Omega)$, the reproducing kernels $K^{(1,1)}$ of $W^{(1,1)}(\Omega)$ and $K^{(3,2)}$ of $W^{(3,2)}(\Omega)$ satisfy the following:
(i) Reproducing properties:

$$u(\xi,\eta) = \langle u(x,t), K^{(3,2)}(x,\xi,t,\eta)\rangle = \langle u(x,t), K^{(3,2)}_{(\xi,\eta)}(x,t)\rangle; \tag{2.11}$$

$$f(\xi,\eta) = \langle f(x,t), K^{(1,1)}(x,\xi,t,\eta)\rangle_{W^{(1,1)}} = \langle f(x,t), K^{(1,1)}_{(\xi,\eta)}(x,t)\rangle_{W^{(1,1)}}. \tag{2.12}$$

(ii) Symmetric properties: it follows from the symmetric property of K^1, K^2, K^3 (2.6) that

$$K^{(3,2)}_{(\xi,\eta)}(x,t) = K^{(3,2)}(x,\xi,t,\eta) = K^{(3,2)}(\xi,x,\eta,t) = K^{(3,2)}_{(x,t)}(\xi,\eta); \tag{2.13}$$

$$K^{(1,1)}_{(\xi,\eta)}(x,t) = K^{(1,1)}(x,\xi,t,\eta) = K^{(1,1)}(\xi,x,\eta,t) = K^{(1,1)}_{(x,t)}(\xi,\eta). \tag{2.14}$$

Lemma 2.1. *The operator $L : W^{(3,2)}(\Omega) \to W^{(1,1)}(\Omega)$ defined in* (1.4) *is a bounded linear operator.*

3. Weak N-best POAFD for solutions and error estimate

In the following discussion, we assume that L has bounded inverse L^{-1} mapping from $W^{(1,1)}(\Omega)$ to $W^{(3,2)}(\Omega)$. We construct the dictionary $\mathcal{D}$ in $W^{(3,2)}(\Omega)$ for weak N-best POAFD first.

Let $X = \{(x_i, t_i)\}_{j\geq 1} \subset \Omega$, $Y = \{(\xi_j, \eta_j)\}_{j\geq 1} \subset \Omega$, $Q_i = (x_i, t_i)$, $Q = (x,t)$, $Q' = (\xi, \eta)$. By the adjoint operator L^* of L and the reproducing property of $K^{(1,1)}_Q \in W^{(1,1)}(\Omega)$ in (2.12), we have the following important observation:

$$F(Q) = \langle F, K^{(1,1)}_Q\rangle_{W^{(1,1)}} = \langle Lu, K^{(1,1)}_Q\rangle_{W^{(1,1)}} = \langle u, L^* K^{(1,1)}_Q\rangle = \langle u, \mathbf{h}_{Q_i}\rangle \tag{3.1}$$

for any $f \in W^{(1,1)}(\Omega)$ and any $u \in W^{(3,2)}(\Omega)$.

3.1. The dictionary of weak N-best POAFD

Lemma 3.1. *Define functions $\phi : \Omega \to W^{(3,2)}(\Omega)$ and $\psi : \Omega \to \mathbb{R}$ as follows: for any $Q \in \Omega$,*

$$\phi(Q) = \mathbf{h}_Q = L^* K^{(1,1)}_Q, \quad \psi(Q) = \|\mathbf{h}_Q\|.$$

Then (i) ϕ and ψ are continuous on Ω; (ii) $\phi(Q) \neq 0$ and $\psi(Q) > 0$ for any $Q \in \Omega$.

Proof. (i) By definition in (2.10), $K(Q,Q')$ is a continuous function on Ω^2, so

$$\begin{aligned}
&\|K^{(1,1)}_Q - K^{(1,1)}_P\|^2_{W^{(1,1)}} \\
&= \langle K^{(1,1)}_Q - K^{(1,1)}_P, K^{(1,1)}_Q - K^{(1,1)}_P\rangle_{W^{(1,1)}} \\
&= \langle K^{(1,1)}_Q, K^{(1,1)}_Q\rangle_{W^{(1,1)}} + \langle K^{(1,1)}_P, K^{(1,1)}_P\rangle_{W^{(1,1)}} - 2\langle K^{(1,1)}_Q, K^{(1,1)}_P\rangle_{W^{(1,1)}} \\
&= K^{(1,1)}(Q,Q) + K^{(1,1)}(P,P) - 2K^{(1,1)}(P,Q) \to 0, \quad \text{as } Q \to P.
\end{aligned}$$

As L and L^* are continuous operators on $W^{(3,2)}(\Omega)$ and $W^{(1,1)}(\Omega)$ respectively, so

$$\lim_{Q\to P}\phi(Q)=\lim_{Q\to P}\mathbf{h}_Q=\lim_{Q\to P}L^*K_Q^{(1,1)}=L^*K_P^{(1,1)}=\mathbf{h}_P=\phi(P).$$

Thus, ϕ is continuous on Ω. The continuity of ψ follows from that of ϕ and $\psi(Q)=\|\phi(Q)\|$.

(ii) As L is invertible, so is L^* for any $Q\in\Omega$. It follows from $K_Q^{(1,1)}\neq 0$ that $\phi(Q)=\mathbf{h}_Q=L^*K_Q^{(1,1)}\neq 0$, and hence $\psi(Q)=\|\phi(Q)\|=\|\mathbf{h}_Q\|>0$. □

Theorem 3.1. *If $X=\{Q_i\}_{i\geq 1}$ is dense on Ω, then $\{\mathbf{h}_{Q_i}\}_{i\geq 1}$ is the complete function system of the space $W^{(3,2)}(\Omega)$. If X consists of distinct points, then $\{\mathbf{h}_{Q_i}\}_{i\geq 1}$ is linearly independent.*

Proof. First, consider the completeness of $\{\mathbf{h}_{Q_i}\}_{i\geq 1}$. For any V in the orthogonal complement of the subspace spanned by $\{\mathbf{h}_{Q_i}\}_{i\geq 1}$ in $W^{(3,2)}(\Omega)$, we have $\langle V,\mathbf{h}_{Q_i}\rangle=0, i=1,2,\cdots$, which means

$$\langle V,L^*K_{Q_i}^{(1,1)}\rangle=\langle LV,K_{Q_i}^{(1,1)}\rangle_{W^{(1,1)}}=(LV)(Q_i)=(LV)(x_i,t_i)=0.$$

As $X=\{(x_i,t_i)\}_{j\geq 1}$ is dense on Ω and LV is continuous on Ω, we have $(LV)(x,t)=0$ for all $(x,t)\in\Omega$. It follows from the existence of L^{-1} that $V\equiv 0$. Therefore, the span of $\{\mathbf{h}_{Q_i}(Q')\}_{i\geq 1}$ is a dense subspace of $W^{(3,2)}(\Omega)$.

For the linear independence, suppose that for any fixed natural number n and $\sum\limits_{j=1}^{n}c_j\mathbf{h}_{Q'_j}=0$ for some $c_1,c_2,\cdots,c_n\in\mathbb{R}$. Then, we have

$$0=\sum_{j=1}^{n}c_j\mathbf{h}_{Q'_j}=\sum_{j=1}^{n}c_jL^*K_{Q'_j}^{(1,1)}=L^*\left[\sum_{j=1}^{n}c_jK_{Q'_j}^{(1,1)}\right]. \tag{3.2}$$

It follows from the boundedness of L^{-1} that L^* also has a bounded inverse. Therefore, (3.2) implies that

$$\sum_{j=1}^{n}c_jK_{Q'_j}^{(1,1)}=0.$$

It follows from (2.1), (2.10) and (2.14) that

$$\begin{aligned}\det(K^{(1,1)}(x_i,\xi_j,t_i,\eta_j))_{1\leq i,j\leq n}&=\det(K_{(\xi_j,\eta_j)}^{(1,1)}(x_i,t_i))_{1\leq i,j\leq n}\\&=\det(K_{Q'_j}^{(1,1)}(Q_i))_{1\leq i,j\leq n}\\&=\det(Diag(K^1(x_i,\xi_j)K^1(t_i,\eta_j)))\\&=\prod_{j=1}^{n}\prod_{i=1}^{n}K(x_i,\xi_j)K(t_i,\eta_j)>0.\end{aligned} \tag{3.3}$$

Then from (3.3) and Cramer's rule that

$$\sum_{j=1}^{n}c_jK_{Q'_j}^{(1,1)}(Q_i)=\sum_{j=1}^{n}c_jK_{(\xi_j,\eta_j)}^{(1,1)}(x_i,t_i)=\sum_{j=1}^{n}c_jK^{(1,1)}(x_i,\xi_j,t_i,\eta_j)=0$$

has only trivial solution. Hence $\{\mathbf{h}_{Q'_j}\}_{j\geq 1}$ is linearly independent. The proof is complete. □

Theorem 3.1 shows that $\{\mathbf{h}_Q\}_{Q\in\Omega}$ can be a dictionary $\mathcal{D}$ of $W^{(3,2)}(\Omega)$. Let $L^{(1)}$ and $L^{(2)}$ be operators on the space of functions defined on Ω^2, in which L acts the function $K^{(3,2)}(Q,Q')$ on the first and second variable respectively. By symmetric property of $K^{(3,2)}$ in $W^{(3,2)}(\Omega)$ (2.11), we have

$$L^{(1)}K_Q^{(3,2)}(Q') = L^{(2)}K_{Q'}^{(3,2)}(Q).$$

Then we can express $\{\mathbf{h}_Q\}_{Q\in\Omega}$ as follows:

Lemma 3.2. *For any* $Q\in\Omega$ *define* $\mathbf{h}_Q = L^*K_Q^{(1,1)} \in W^{(3,2)}(\Omega)$. *Then*

$$\mathbf{h}_{Q'}(Q) = (LK_Q^{(3,2)})(Q'), \qquad \text{for any } \; Q,Q'\in\Omega, \tag{3.4}$$

$$\langle \mathbf{h}_P, \mathbf{h}_{P'}\rangle = (L^{(1)}L^{(2)}K^{(3,2)})(P,P'), \qquad \text{for any } \; P,P'\in\Omega. \tag{3.5}$$

Proof. Due to reproducing property of $K^{(1,1)}(x,y)$, we have

$$\begin{aligned}\mathbf{h}_{Q'}(Q) &= L^*K_{Q'}^{(1,1)}(Q) = \langle L^*K_{Q'}^{(1,1)}, K_Q^{(3,2)}\rangle = \langle K_{Q'}^{(1,1)}, LK_Q^{(3,2)}\rangle_{W^{(1,1)}}\\ &= LK_Q^{(3,2)}(Q') = L^{(2)}K_{Q'}^{(3,2)}(Q),\\ L\mathbf{h}_{P'}(P) &= (L^{(1)})(L^{(2)}K_{P'}^{(3,2)}(P)) = (L^{(1)}L^{(2)})K_{P'}^{(3,2)}(P).\end{aligned}$$

And from (3.4) we have

$$\begin{aligned}\langle \mathbf{h}_P, \mathbf{h}_{P'}\rangle &= \langle L^*K_P^{(1,1)}, \mathbf{h}_{P'}\rangle = \langle K_P^{(1,1)}, L\mathbf{h}_{P'}\rangle_{W^{(1,1)}} = L\mathbf{h}_{P'}(P)\\ &= L^{(1)}L^{(2)}K_{P'}^{(3,2)}(P).\end{aligned}$$

This completes the proof. □

We now explain how we can solve PDE (1.4) by weak N-best POAFD. The key strategy of weak N-best POAFD is to choose a sequence $\{Q_i\}_{i\geq 1}$ of distinct points successively by the weak N-best maximum selection principle below on minimizing the norm of the residual u_i as much as possible step by step, then the solution u of equation PDE (1.4) is a linear combination of $\mathbf{h}_{Q_i}$'s from the dictionary $\mathcal{D}$.

3.2. The weak 1-best POAFD

For $N = 1$, suppose that we have selected $n-1$ distinct parameters $\{Q_i = (x_i,t_i)\}_{i=1}^{n-1}$ in Ω by weak maximal selection principle (3.8) in the first $n-1$ steps. The related normalized dictionary elements are $\{B_i = B_{Q_i}\}_{i=1}^{n-1}$ in $W^{(3,2)}(\Omega)$ obtained by applying Gram-Schmidt (G-S) orthonormalization to $\{\mathbf{h}_{Q_i}\}_{i=1}^{n-1}$, where

$$D_i(Q_i) = \mathbf{h}_{Q_i} - \sum_{j=1}^{i-1}\langle \mathbf{h}_{Q_i}, B_j\rangle B_j, \quad 1\leq i\leq n-1, \tag{3.6}$$

$$B_i = \frac{D_i(Q_i)}{\|D_i(Q_i)\|} = \frac{\mathbf{h}_{Q_i} - \sum\limits_{j=1}^{i-1}\langle \mathbf{h}_{Q_i}, B_j\rangle B_j}{\|\mathbf{h}_{Q_i} - \sum\limits_{j=1}^{i-1}\langle \mathbf{h}_{Q_i}, B_j\rangle B_j\|}. \tag{3.7}$$

And $u_n = u - \sum\limits_{i=1}^{n-1}\langle u, B_i\rangle B_i$ is the $(n-1)$–th residual of u.

To select the n–th parameter Q_n at n–th step, we apply G-S orthonormalization to the sequence $(B_1, \cdots, B_{n-1}, \mathbf{h}_Q)$ for any $Q \in \Omega \setminus \{Q_i\}_{i=1}^{n-1}$ to obtain a new normalized sequence $(B_1, \cdots, B_{n-1}, B_n^Q)$, where

$$B_n^Q = \frac{\mathbf{h}_Q - \sum\limits_{i=1}^{n-1} \langle \mathbf{h}_Q, B_i \rangle B_i}{\|\mathbf{h}_Q - \sum\limits_{i=1}^{n-1} \langle \mathbf{h}_Q, B_i \rangle B_i\|}.$$

Then the n–th point $Q_n \in \Omega \setminus \{Q_i\}_{i=1}^{n-1}$ can be chosen by the following weak maximal selection principle:

$$|\langle u_n, B_{Q_n} \rangle| \geq \rho \sup \left\{ |\langle u_n, B_n^Q \rangle| \;\middle|\; Q \in \Omega \setminus \{Q_i\}_{i=1}^{n-1} \right\}, \tag{3.8}$$

where $\rho \in (0, 1)$ is any fixed constant.

With the selected parameter $\{Q_i\}_{i \geq 1}$ and the normalized vector $\{B_i = B_{Q_i}\}_{i \geq 1}$, we can express the solution $u \in W^{(3,2)}(\Omega)$ of PDE (1.4) as follows:

$$u = \lim_{n \to \infty} S_n = \sum_{n \geq 1} \langle u, B_n \rangle B_n, \tag{3.9}$$

where n–th partial sum S_n is defined by

$$S_n = \sum_{i=1}^{n} \langle u, B_i \rangle B_i. \tag{3.10}$$

Such representation method is also called weak pre-orthogonal adaptive Fourier decomposition (W-POAFD). The convergence of the series (3.9) can be checked from [3, 9].

3.3. The weak 2-best POAFD

For $N = 2$, after successive $n - 1$ steps with $n \geq 2$, suppose that we have chosen

(i) the sequence $(Q_1, P_1, Q_2, P_2, \cdots, Q_{n-1}, P_{n-1})$ of $2(n-1)$ distinct points in Ω.

(ii) $2(n-1)$ orthonormal vectors

$$(B_{Q_1}, B_{P_1}, \cdots, B_{Q_{n-1}}, B_{P_{n-1}}) \in W^{(3,2)}(\Omega)$$

obtained by applying G-S orthonormalization to $(\mathbf{h}_{Q_1}, \mathbf{h}_{P_1}, \cdots, \mathbf{h}_{Q_{n-1}}, \mathbf{h}_{P_{n-1}})$.

Define

$$O_i(Q_i) = \mathbf{h}_{Q_i} - \sum_{k=1}^{i-1} (\langle \mathbf{h}_{Q_i}, B_{Q_k} \rangle B_{Q_k} + \langle \mathbf{h}_{Q_i}, B_{P_k} \rangle B_{P_k}), \tag{3.11}$$

then

$$B_{Q_i} = \frac{O_i(Q_i)}{\|O_i(Q_i)\|} = \frac{\mathbf{h}_{Q_i} - \sum\limits_{k=1}^{i-1} (\langle \mathbf{h}_{Q_i}, B_{Q_k} \rangle B_{Q_k} + \langle \mathbf{h}_{Q_i}, B_{P_k} \rangle B_{P_k})}{\|\mathbf{h}_{Q_i} - \sum\limits_{k=1}^{i-1} (\langle \mathbf{h}_{Q_i}, B_{Q_k} \rangle B_{Q_k} + \langle \mathbf{h}_{Q_i}, B_{P_k} \rangle B_{P_k})\|}$$

$$= \sum_{k=1}^{i-1} (\beta_{ik}^{(Q_i,Q_k)} \mathbf{h}_{Q_k} + \beta_{ik}^{(Q_i,P_k)} \mathbf{h}_{P_k}) + \beta_{ii}^{(Q_i,Q_i)} \mathbf{h}_{Q_i}, \tag{3.12}$$

and

$$E_i(P_i) = \mathbf{h}_{P_i} - \sum_{k=1}^{i-1} (\langle \mathbf{h}_{P_i}, B_{Q_k} \rangle B_{Q_k} + \langle \mathbf{h}_{P_i}, B_{P_k} \rangle B_{P_k}) - \langle \mathbf{h}_{P_i}, B_{Q_i} \rangle B_{Q_i}, \tag{3.13}$$

$$\begin{aligned} B_{P_i} &= \frac{E_i(P_i)}{\|E_i(P_i)\|} \\ &= \frac{\mathbf{h}_{P_i} - \sum_{k=1}^{i-1} (\langle \mathbf{h}_{P_i}, B_{Q_k} \rangle B_{Q_k} + \langle \mathbf{h}_{P_i}, B_{P_k} \rangle B_{P_k}) - \langle \mathbf{h}_{P_i}, B_{Q_i} \rangle B_{Q_i}}{\|\mathbf{h}_{P_i} - \sum_{k=1}^{i-1} (\langle \mathbf{h}_{P_i}, B_{Q_k} \rangle B_{Q_k} + \langle \mathbf{h}_{P_i}, B_{P_k} \rangle B_{P_k}) - \langle \mathbf{h}_{P_i}, B_{Q_i} \rangle B_{Q_i}\|} \\ &= \sum_{k=1}^{i} (\beta_{ik}^{(P_i,Q_k)} \mathbf{h}_{Q_k} + \beta_{ik}^{(P_i,P_k)} \mathbf{h}_{P_k}) \end{aligned} \tag{3.14}$$

for $1 \le i \le n-1$. Thus the sequence $\{B_{Q_1}, B_{P_1}, \cdots, B_{Q_{n-1}}, B_{P_{n-1}}\}$ is orthonormal set.

(iii) For all $k = 1, \cdots, n-1$, $u_1 = u$, we have

$$u = \sum_{i=1}^{k} (\langle u_i, B_{Q_i} \rangle B_{Q_i} + \langle u_i, B_{P_i} \rangle B_{P_i}) + u_{k+1}, \tag{3.15}$$

$$u_{k+1} = u - \sum_{i=1}^{k} (\langle u_i, B_{Q_i} \rangle B_{Q_i} + \langle u_i, B_{P_i} \rangle B_{P_i}), \tag{3.16}$$

where u_{k+1} is the k–th residual of u at the k–th step.

For all $1 \le k \le n-1$, from (3.16) and (ii), we have

$$\begin{aligned} \langle u_k, B_{Q_k} \rangle &= \langle u - \sum_{i=1}^{k-1} (\langle u_i, B_{Q_i} \rangle B_{Q_i} + \langle u_i, B_{P_i} \rangle B_{P_i}), B_{Q_k} \rangle = \langle u, B_{Q_k} \rangle, \\ \langle u_k, B_{P_k} \rangle &= \langle u, B_{P_k} \rangle. \end{aligned} \tag{3.17}$$

And for $1 \le k \le j-1$,

$$\begin{aligned} \langle u_j, B_{Q_k} \rangle &= \langle u - \sum_{i=1}^{j-1} (\langle u_i, B_{Q_i} \rangle B_{Q_i} + \langle u_i, B_{P_i} \rangle B_{P_i}), B_{Q_k} \rangle \\ &= \langle u, B_{Q_k} \rangle - \langle u_k, B_{Q_k} \rangle = 0, \\ \langle u_j, B_{P_k} \rangle &= 0. \end{aligned} \tag{3.18}$$

Define $C_i(Q_i, P_i) = |\langle u_i, B_{Q_i} \rangle|^2 + |\langle u_i, B_{P_i} \rangle|^2$. Then

$$\langle u_i, B_{Q_i} \rangle B_{Q_i} + \langle u_i, B_{P_i} \rangle B_{P_i} = \langle u, B_{Q_i} \rangle B_{Q_i} + \langle u, B_{P_i} \rangle B_{P_i}, \tag{3.19}$$

$$C_i(Q_i, P_i) = |\langle u, B_{Q_i} \rangle|^2 + |\langle u, B_{P_i} \rangle|^2. \tag{3.20}$$

If $u_n = 0$, stop the decomposition; otherwise, we continue to work out the n–th decomposition of u. That is, we shall choose 2 distinct points $Q_n, P_n \in \Omega \setminus \{Q_i, P_i\}_{i=1}^{n-1}$ by the following strategy.

Denote by $A_n := \Omega \setminus \{Q_i, P_i\}_{i=1}^{n-1}$ the punctured region. For any fixed 2 distinct points $P,\ Q \in A_n$, we get a new point sequence $(Q_1, P_1, \cdots, Q_{n-1}, P_{n-1}, Q, P)$ of distinct points. Then from Theorem 3.1, the new vector

$$(\mathbf{h}_{Q_1}, \mathbf{h}_{P_1}, \cdots, \mathbf{h}_{Q_{n-1}}, \mathbf{h}_{P_{n-1}}, \mathbf{h}_Q, \mathbf{h}_P)$$

is linearly independent. We can obtain the orthonormal set

$$\{B_{Q_1}, B_{P_1}, \cdots, B_{Q_{n-1}}, B_{P_{n-1}}, B_n^Q, B_n^P\}$$

by applying G-S orthonormalization to $(\mathbf{h}_{Q_1}, \mathbf{h}_{P_1}, \cdots, \mathbf{h}_{Q_{n-1}}, \mathbf{h}_{P_{n-1}}, \mathbf{h}_Q, \mathbf{h}_P)$.

Denote the pre-orthogonalization at n–th step.

$$O_n(Q) = \mathbf{h}_Q - \sum_{j=1}^{n-1}(\langle \mathbf{h}_Q, B_{Q_j}\rangle B_{Q_j} + \langle \mathbf{h}_Q, B_{P_j}\rangle B_{P_j}) \neq 0, \tag{3.21}$$

$$B_n^Q = \frac{O_n(Q)}{\|O_n(Q)\|}, \tag{3.22}$$

$$E_n(P) = \mathbf{h}_P - \sum_{j=1}^{n-1}(\langle \mathbf{h}_P, B_{Q_j}\rangle B_{Q_j} + \langle \mathbf{h}_P, B_{P_j}\rangle B_{P_j}) - \langle \mathbf{h}_P, B_n^Q\rangle B_n^Q \neq 0, \tag{3.23}$$

$$B_n^P = \frac{E_n(P)}{\|E_n(P)\|}. \tag{3.24}$$

Then from (3.21)– (3.24) and (3.18), we have

$$\begin{aligned}
C_n(Q,P) =& |\langle u_n, B_n^Q\rangle|^2 + |\langle u_n, B_n^P\rangle|^2 = \frac{|\langle u_n, O_n(Q)\rangle|^2}{\|O_n(Q)\|^2} + \frac{|\langle u_n, E_n(P)\rangle|^2}{\|E_n(P)\|^2} \\
=& \frac{\left|\left\langle u_n, \mathbf{h}_Q - \sum\limits_{j=1}^{n-1}(\langle \mathbf{h}_Q, B_{Q_j}\rangle B_{Q_j} + \langle \mathbf{h}_Q, B_{P_j}\rangle B_{P_j})\right\rangle\right|^2}{\|O_n(Q)\|^2} \\
&+ \frac{\left|\left\langle u_n, \mathbf{h}_P - \sum\limits_{j=1}^{n-1}(\langle \mathbf{h}_P, B_{Q_j}\rangle B_{Q_j} + \langle \mathbf{h}_P, B_{P_j}\rangle B_{P_j}) - \langle \mathbf{h}_P, B_n^Q\rangle B_n^Q\right\rangle\right|^2}{\|E_n(P)\|^2} \\
=& \frac{|\langle u_n, \mathbf{h}_Q\rangle|^2}{\|O_n(Q)\|^2} + \frac{|\langle u_n, \mathbf{h}_P - \langle \mathbf{h}_P, B_n^Q\rangle B_n^Q\rangle|^2}{\|E_n(P)\|^2} \\
=& \frac{|Lu_n(Q)|^2}{\|O_n(Q)\|^2} + \frac{|u_n(P) - \langle \mathbf{h}_P, B_n^Q\rangle\langle u_n, B_n^Q\rangle|^2}{\|E_n(P)\|^2} \geq \frac{|Lu_n(Q)|^2}{\|O_n(Q)\|^2} > 0.
\end{aligned} \tag{3.25}$$

The existence of the n–th 2 distinct points $Q_n, P_n \in A_n$ follows the following Weak 2-Best Maximal Selection Principle to establish.

Lemma 3.3 (Weak 2-Best Maximal Selection Principle)**.** *For any $n \geq 2$, let $A_n := \Omega \setminus \{Q_i, P_i\}_{i=1}^{n-1}$ be the punctured region. Suppose that $u_n \neq 0$, then for any distinct two points $Q, P \in A_n$*

(i) $\sup\{C_n(Q,P) = |\langle u, B_n^Q\rangle|^2 + |\langle u, B_n^P\rangle|^2 : Q, P \in A_n\ \}$ *is finite.*

(ii) For any fixed $0 < \rho < 1$, there exists 2 distinct points $Q_n, P_n \in A_n$ such that

$$\begin{aligned} C_n(Q_n, P_n) &= |\langle u, B_{Q_n}\rangle|^2 + |\langle u, B_{P_n}\rangle|^2 \\ &\geq \rho \sup\{ \ |\langle u, B_n^Q\rangle|^2 + |\langle u, B_n^P\rangle|^2 \ \big| \ Q, P \in A_n \ \} > 0. \end{aligned} \tag{3.26}$$

Proof. (i) First, from definition of $C_n(Q, P)$ in (3.25) and the Cauchy-Schwarz inequality, we have

$$0 \leq C_n(Q, P) = |\langle u, B_n^Q\rangle|^2 + |\langle u, B_n^P\rangle|^2 \leq \|u\|^2 \, \|B_n^Q\|^2 + \|u_n\|^2 \, \|B_n^P\|^2 \leq 2\|u\|^2,$$

which shows that C_n is bounded on A_n.

To ensure the finiteness of supremum of $C_n(Q, P)$ on A_n, it needs to show that C_n is continuous on A_n. First, we notice that $O_n(Q) \neq 0$ and $E_n(P) \neq 0$ for any $Q, P \in A_n$ from Theorem 3.1. On one hand, by definition of $O_n(Q)$ in (3.21), the triangle inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\|O_n(Q) - O_n(Q_0)\|^2 \\ =&\|(\mathbf{h}_Q - \mathbf{h}_{Q_0}) - \sum_{j=1}^{n-1} (\langle(\mathbf{h}_Q - \mathbf{h}_{Q_0}), B_{Q_j}\rangle B_{Q_j} + \langle(\mathbf{h}_Q - \mathbf{h}_{Q_0}), B_{P_j}\rangle B_{P_j})\|^2 \\ =&\|(\mathbf{h}_Q - \mathbf{h}_{Q_0})\|^2 - \sum_{j=1}^{n-1} \left(|\langle(\mathbf{h}_Q - \mathbf{h}_{Q_0}), B_{Q_j}\rangle|^2 + |\langle(\mathbf{h}_Q - \mathbf{h}_{Q_0}), B_{P_j}\rangle|^2\right) \\ \leq&\|\mathbf{h}_Q - \mathbf{h}_{Q_0}\|^2. \end{aligned} \tag{3.27}$$

It follows from Lemma 3.1 (ii) and (3.27) $\lim\limits_{Q\to Q_0} O_n(Q) = O_n(Q_0)$, and $\lim\limits_{Q\to Q_0} \|O_n(Q)\|^2 = \|O_n(Q_0)\|^2 > 0$ as well as $E_n(P)$. Thus $O_n(Q)$ and $E_n(P)$ are non-zero and continuous on A_n.

In addition, by the Cauchy-Schwarz inequality, we have

$$|\langle u_n, O_n(Q) - O_n(Q_0)\rangle| \leq \|u_n\| \, \|O_n(Q) - O_n(Q_0)\|,$$

which implies $\lim\limits_{Q\to Q_0} \langle u_n, O_n(Q)\rangle = \langle u_n, O_n(Q_0)\rangle$. Thus $\langle u, O_n(Q)\rangle$ and $\langle u, E_n(P)\rangle$ are continuous on A_n. Noticing that $C_n(Q, P) = \frac{|\langle u, O_n(Q)\rangle|^2}{\|O_n(Q)\|^2} + \frac{|\langle u, E_n(P)\rangle|^2}{\|E_n(P)\|^2}$, consequently, $\lim\limits_{(Q,P)\to(Q_0,P_0)} C_n(Q, P) = C_n(Q_0, P_0)$. As C_n is continuous on A_n, and A_n is a finite union of disjoint sub-regions of Ω. From intermediate value theorem, we see that the range of C_n on A_n is also a finite union of disjoint bounded sub-regions, and that

$$\sup\{ \ |\langle u, B_n^Q\rangle|^2 + |\langle u, B_n^P\rangle|^2 \ \big| \ Q, P \in A_n\}$$

is finite.

(ii) Since $u_n \neq 0$ and L is bounded, then $L(u_n) \neq 0$, and hence $|L(u_n)(Q)| > 0$. From (3.25), we see that

$$C_n(Q, P) \geq \frac{|Lu_n(Q)|^2}{\|O_n(Q)\|^2} > 0.$$

Recall that $0 < \rho < 1$, so

$$0 < \rho \sup\{ \ C_n(Q, P) = |\langle u, B_n^Q\rangle|^2 + |\langle u, B_n^P\rangle|^2 \mid Q, P \in A_n \ \}$$

$$< \sup\{\ C_n(Q,P) = |\langle u, B_n^Q\rangle|^2 + |\langle u, B_n^P\rangle|^2 \mid Q,P \in A_n\ \}.$$

It follows from definition of supremum, there exists 2 distinct points $Q_n, P_n \in A_n$ such that

$$\begin{aligned} C_n(Q_n,P_n) &= |\langle u, B_{Q_n}\rangle|^2 + |\langle u, B_{P_n}\rangle|^2 \\ &\geq \rho \sup\left\{\ |\langle u, B_n^Q\rangle|^2 + |\langle u, B_n^P\rangle|^2 \ \middle|\, Q,P \in A_n\right\} > 0, \end{aligned}$$

which is called the Weak 2-best Maximal Selection Principle (3.26). □

Thus we have the n–th orthonormal decomposition of u in terms of

$$\{B_{Q_1}, B_{P_1}, \cdots, B_{Q_{n-1}}, B_{P_{n-1}}, B_{Q_n}, B_{P_n}\}$$

and the n–th residual $u_{n+1} = u_n - \langle u, B_{Q_n}\rangle B_{Q_n} - \langle u, B_{P_n}\rangle B_{P_n}$:

$$u = \sum_{i=1}^{n}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}) + u_{n+1}; \tag{3.28}$$

$$\|u_{n+1}\|^2 = \|u\|^2 - \sum_{i=1}^{n}(|\langle u, B_{Q_i}\rangle|^2 + |\langle u, B_{P_i}\rangle|^2). \tag{3.29}$$

Theorem 3.2. *Suppose that L^{-1} exists and is bounded. For any 2 distinct points $Q,P \in A_n\,(n \geq 1)$, we orthonormalize $(B_{Q_1}, B_{P_1}\cdots, B_{Q_{n-1}}, B_{P_{n-1}}, \mathbf{h}_Q, \mathbf{h}_P)$ to obtain a new normalized set $(B_{Q_1}, B_{P_1}\cdots, B_{Q_{n-1}}, B_{P_{n-1}}, B_n^Q, B_n^P)$. One can find the n–th pair of distinct points $Q_n, P_n \in A_n$ by the weak 2-best maximal selection principle* (3.26)*, then the solution $u \in W^{(3,2)}(\Omega)$ of PDE* (1.4) *has the following series form:*

$$u = \lim_{n\to\infty} S_n = \sum_{i=1}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}), \tag{3.30}$$

where n–th partial sum S_n is defined by

$$S_n = \sum_{i=1}^{n}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}). \tag{3.31}$$

Proof. If L^{-1} exists, then the solution $u = L^{-1}f \in W^{(3,2)}(\Omega)$. Let $\{Q_i, P_i\}_{i\geq 1}$ be a sequence of distinct points in Ω for the solution u satisfying the weak 2-best maximal selection principle (3.26). It remains to prove that the solution of PDE (1.4) can be represented by the series $u = \sum_{i\geq 1}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i})$. The proof is rather long, we divide it into 3 parts.

(I) We first prove the series $\sum_{i\geq 1}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}) \in W^{(3,2)}(\Omega)$. As the corresponding sequence $(B_{Q_1}, B_{P_1}, \cdots, B_{Q_i}, B_{P_i}, \cdots)$ is an orthonormal set in $W^{(3,2)}(\Omega)$, so Bessel's inequality implies that

$$\sum_{i\geq 1} C_i(Q_i,P_i) = \sum_{i\geq 1}(|\langle u, B_{Q_i}\rangle|^2 + |\langle u, B_{P_i}\rangle|^2) \leq \|u\|^2, \tag{3.32}$$

and hence the series $\sum\limits_{i\geq 1}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}) \in W^{(3,2)}(\Omega)$.

(II) We are going to prove that this series converges to the solution u of PDE (1.4). Suppose to the contrary that the residual

$$g = u - \sum_{i\geq 1}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}) \in W^{(3,2)}(\Omega) \tag{3.33}$$

is non-zero. Then the weak 2-best maximal selection principle in Lemma 3.3 implies that the sequence $\{Q_i, P_i\}_{i\geq 1}$ is infinite.

Define $\Omega' = \Omega\backslash\{Q_1, P_1, \cdots, Q_{N-1}, P_{N-1}\}$. For any distinct $S, T \in \Omega'$, we define two unit vectors

$$e_S = \frac{\mathbf{h}_S}{\|\mathbf{h}_S\|}, \tag{3.34}$$

$$e_T = \frac{\mathbf{h}_T - \langle \mathbf{h}_T, B_N^S\rangle B_N^S}{\|\mathbf{h}_T - \langle \mathbf{h}_T, B_N^S\rangle B_N^S\|}, \tag{3.35}$$

where $B_N^S = \frac{\mathbf{h}_S - \sum\limits_{j=1}^{N-1}(\langle \mathbf{h}_S, B_{Q_j}\rangle B_{Q_j} + \langle \mathbf{h}_S, B_{P_j}\rangle B_{P_j})}{\|\mathbf{h}_S - \sum\limits_{j=1}^{N-1}(\langle \mathbf{h}_S, B_{Q_j}\rangle B_{Q_j} + \langle \mathbf{h}_S, B_{P_j}\rangle B_{P_j})\|}$ defined in (3.22).

It follows from Lemma 3.1 (ii) that $\mathbf{h}_S$, $\|\mathbf{h}_S\|$ are continuous on Ω'. And $\|\mathbf{h}_S\| \neq 0$ follows from Lemma 3.1 (iii). Thus e_S (3.34) is continuous on Ω'.

On the other hand, it follows from Theorem 3.1 that $\|\mathbf{h}_T - \langle \mathbf{h}_T, B_N^S\rangle B_N^S\| \neq 0$. As $\mathbf{h}_T$ is continuous on Ω', then $\mathbf{h}_T - \langle \mathbf{h}_T, B_N^S\rangle B_N^S$ is continuous in Ω'. Since

$$\begin{aligned}\|\mathbf{h}_T - \langle \mathbf{h}_T, B_N^S\rangle B_N^S\|^2 &= \langle \mathbf{h}_T, \mathbf{h}_T\rangle + |\langle \mathbf{h}_T, B_N^S\rangle|^2 - 2|\langle \mathbf{h}_T, B_N^S\rangle|^2 \\ &= \|\mathbf{h}_T\|^2 - |\langle \mathbf{h}_T, B_N^S\rangle|^2 \leq \|\mathbf{h}_T\|^2,\end{aligned}$$

from continuity of $\|\mathbf{h}_T\|$, we see that $\|\mathbf{h}_T - \langle \mathbf{h}_T, B_N^S\rangle B_N^S\|$ is continuous in Ω'. Thus e_T in (3.35) is continuous in Ω'. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle g, e_S\rangle| &\leq \|g\|\|e_S\|, \\ |\langle g, e_T\rangle| &\leq \|g\|\|e_T\|,\end{aligned}$$

which imply that $|\langle g, e_S\rangle|$ and $|\langle g, e_T\rangle|$ are continuous in Ω'. As Ω' is a region removing finite points from a bounded closed region Ω, from intermediate value theorem, we see that the range of $|\langle g, e_S\rangle|$ and $|\langle g, e_T\rangle|$ in Ω' are bounded, and $\sup\{|\langle g, e_S\rangle|^2 + |\langle g, e_T\rangle|^2 \big| S, T \in \Omega'\}$ is finite.

Hence there exists a closed region $\bar{\Omega} \subset \Omega'$ such that

$$\frac{3}{2}C_0 = \inf_{S,T\in\bar{\Omega}}\{|\langle g, e_S\rangle|^2 + |\langle g, e_T\rangle|^2\} > 0. \tag{3.36}$$

Recall that $u_N = u - \sum\limits_{i=1}^{N-1}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i})$ is the $(N-1)$–th residual of u. We are going to estimate $|\langle u_N, e_Q\rangle|^2 + |\langle u_N, e_P\rangle|^2$ for any $Q, P \in \bar{\Omega}$ in two different ways.

Firstly, it follows from (3.32) and $\rho, C_0 > 0$ that there exists a positive integer N_0 such that for all $N \geq N_0$, one has

$$|\langle u_N, B_{Q_N}\rangle|^2 + |\langle u_N, B_{P_N}\rangle|^2 \leq \sum_{i=N}^{\infty}(|\langle u, B_{Q_i}\rangle|^2 + |\langle u, B_{P_i}\rangle|^2) < \frac{\rho C_0}{2}. \tag{3.37}$$

Secondly, for any fixed $N \geq N_0$, we select distinct points $P, Q \in \bar{\Omega}$. Considering another sequence $\{Q_1, P_1, \ldots, Q_{N-1}, P_{N-1}, Q, P\} \in \Omega$ of distinct points, then $(\mathbf{h}_{Q_1}, \mathbf{h}_{P_1}, \cdots, \mathbf{h}_{Q_{N-1}}, \mathbf{h}_{P_{N-1}}, \mathbf{h}_Q, \mathbf{h}_P)$ is linearly independent. Let the sequence

$(B_{Q_1}, B_{P_1}, \cdots, B_{Q_{N-1}}, B_{P_{N-1}}, B_N^Q, B_N^P)$ in (3.22) be the G-S orthonormalization of $(B_{Q_1}, B_{P_1}, \cdots, B_{Q_{N-1}}, B_{P_{N-1}}, \mathbf{h}_Q, \mathbf{h}_P)$, where

$$\begin{aligned} B_N^Q &= \frac{\mathbf{h}_Q - \sum_{i=1}^{N-1} (\langle \mathbf{h}_Q, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_Q, B_{P_i}\rangle B_{P_i})}{\|\mathbf{h}_Q - \sum_{i=1}^{N-1} (\langle \mathbf{h}_Q, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_Q, B_{P_i}\rangle B_{P_i})\|}, \\ B_N^P &= \frac{\mathbf{h}_P - \sum_{i=1}^{N-1} (\langle \mathbf{h}_P, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_P, B_{P_i}\rangle B_{P_i}) - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q}{\|\mathbf{h}_P - \sum_{i=1}^{N-1} (\langle \mathbf{h}_P, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_P, B_{P_i}\rangle B_{P_i}) - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q\|}. \end{aligned} \tag{3.38}$$

Since Q_n, P_n are selected according to the weak 2-best maximal selection principle (3.26) in Lemma 3.3, we have

$$\begin{aligned} |\langle u_N, B_{Q_N}\rangle|^2 + |\langle u_N, B_{P_N}\rangle|^2 &\geq \rho \sup\left\{ |\langle u_N, B_N^Q\rangle|^2 + |\langle u_N, B_N^P\rangle|^2 \;\middle|\; Q, P \in \bar{\Omega} \right\} \\ &\geq \rho\big[|\langle u_N, B_N^Q\rangle|^2 + |\langle u_N, B_N^P\rangle|^2\big]. \end{aligned} \tag{3.39}$$

In order to arrive at a contradiction, we can consider 2 unit vectors

$$e_Q = \frac{\mathbf{h}_Q}{\|\mathbf{h}_Q\|}, \quad e_P = \frac{\mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q}{\|\mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q\|} \tag{3.40}$$

in $W^{(3,2)}(\Omega)$, where B_N^Q defined in (3.38). And note that

$$\begin{aligned} &\|\mathbf{h}_Q - \sum_{i=1}^{N-1} (\langle \mathbf{h}_Q, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_Q, B_{P_i}\rangle B_{P_i})\|^2 \\ =&\|\mathbf{h}_Q\|^2 - \sum_{i=1}^{N-1} (|\langle \mathbf{h}_Q, B_{Q_i}\rangle|^2 + |\langle \mathbf{h}_Q, B_{P_i}\rangle|^2) \leq \|\mathbf{h}_Q\|^2, \\ &\|\mathbf{h}_P - \sum_{i=1}^{N-1} (\langle \mathbf{h}_P, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_P, B_{P_i}\rangle B_{P_i}) - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q\|^2 \\ \leq&\|\mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q\|^2. \end{aligned} \tag{3.41}$$

It follows from (3.18) that for $1 \leq i \leq N-1$, we have

$$\langle u_N, B_{Q_i}\rangle = \langle u_N, B_{P_i}\rangle = 0. \tag{3.42}$$

Then, (3.38) and (3.40)–(3.42) imply that

$$\begin{aligned} |\langle u_N, e_Q\rangle| = |\langle u_N, \frac{\mathbf{h}_Q}{\|\mathbf{h}_Q\|}\rangle| &= \frac{\left|\left\langle u_N, \mathbf{h}_Q - \sum_{i=1}^{N-1} (\langle \mathbf{h}_Q, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_Q, B_{P_i}\rangle B_{P_i})\right\rangle\right|}{\|\mathbf{h}_Q\|} \\ &\leq \frac{\left|\left\langle u_N, \mathbf{h}_Q - \sum_{i=1}^{N-1} (\langle \mathbf{h}_Q, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_Q, B_{P_i}\rangle B_{P_i})\right\rangle\right|}{\left\|\mathbf{h}_Q - \sum_{i=1}^{N-1} (\langle \mathbf{h}_Q, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_Q, B_{P_i}\rangle B_{P_i})\right\|} = |\langle u_N, B_N^Q\rangle|, \end{aligned} \tag{3.43}$$

$$
\begin{aligned}
|\langle u_N, e_P\rangle| &= \frac{|\langle u_N, \mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q\rangle|}{\|\mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q\|} \\
&= \frac{\left|\left\langle u_N, \mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q - \sum\limits_{i=1}^{N-1}(\langle \mathbf{h}_P, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_P, B_{P_i}\rangle B_{P_i})\right\rangle\right|}{\|\mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q\|} \\
&\le \frac{\left|\left\langle u_N, \mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q - \sum\limits_{i=1}^{N-1}(\langle \mathbf{h}_P, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_P, B_{P_i}\rangle B_{P_i})\right\rangle\right|}{\left\|\mathbf{h}_P - \langle \mathbf{h}_P, B_N^Q\rangle B_N^Q - \sum\limits_{i=1}^{N-1}(\langle \mathbf{h}_P, B_{Q_i}\rangle B_{Q_i} + \langle \mathbf{h}_P, B_{P_i}\rangle B_{P_i})\right\|} \\
&= |\langle u_N, B_N^P\rangle|.
\end{aligned} \tag{3.44}
$$

Thus, it follows from (3.43), (3.44), (3.39) and (3.37) that

$$
\begin{aligned}
|\langle u_N, e_Q\rangle|^2 + |\langle u_N, e_P\rangle|^2 &\le |\langle u_N, B_N^Q\rangle|^2 + |\langle u_N, B_N^P\rangle|^2 \\
&\le \frac{1}{\rho}[|\langle u_N, B_{Q_N}\rangle|^2 + \langle u_N, B_{P_N}\rangle|^2] \\
&< \frac{1}{\rho}\frac{\rho C_0}{2} = \frac{C_0}{2}.
\end{aligned} \tag{3.45}
$$

On the other hand, it follows from (3.33) that one has

$$
u_N = u - \sum_{i=1}^{N-1}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}) = g + \sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}). \tag{3.46}
$$

Recall that $0 < \rho < 1$, it follows from the triangle inequality, the Cauchy-Schwarz inequality, (3.36), (3.37) and (3.46) that

$$
\begin{aligned}
&|\langle u_N, e_Q\rangle|^2 + |\langle u_N, e_P\rangle|^2 \\
=&\left|\left\langle g + \sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}), e_Q\right\rangle\right|^2 \\
&+ \left|\left\langle g + \sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}), e_P\right\rangle\right|^2 \\
\ge&|\langle g, e_Q\rangle|^2 - \left|\left\langle \sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}), e_Q\right\rangle\right|^2 \\
&+ |\langle g, e_P\rangle|^2 - \left|\left\langle \sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i}), e_P\right\rangle\right|^2 \\
\ge&|\langle g, e_Q\rangle|^2 - \left\|\sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i})\right\|^2 \|e_Q\||^2 \\
&+ |\langle g, e_P\rangle|^2 - \left\|\sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i} + \langle u, B_{P_i}\rangle B_{P_i})\right\|^2 \|e_P\||^2
\end{aligned}
$$

$$
\begin{aligned}
&=|\langle g, e_Q\rangle|^2+|\langle g, e_P\rangle|^2-2\left\|\sum_{i=N}^{\infty}(\langle u, B_{Q_i}\rangle B_{Q_i}+\langle u, B_{P_i}\rangle B_{P_i})\right\|^2\\
&\geq \inf_{Q,P\in\bar{\Omega}}\{|\langle g, e_Q\rangle|^2+|\langle g, e_P\rangle|^2\}-2\sum_{i=N}^{\infty}(|\langle u, B_{Q_i}\rangle|^2+|\langle u, B_{P_i}\rangle|^2)\\
&>\frac{3C_0}{2}-2\frac{\rho C_0}{2}\geq\frac{3C_0}{2}-2\frac{C_0}{2}=\frac{C_0}{2},
\end{aligned}
$$

which contradicts to (3.45). Consequently, the residual $g=0$. And this completes the proof of (II).

(III) We are going to derive another analytic formula (3.30) of the solution u of PDE $Lu=f$. We may assume that $u_i\neq 0$ for all i; otherwise the series in (3.30) is a finite sum. For this, we prove that the sequence $S_n=\sum_{i=1}^{n}(\langle u, B_{Q_i}\rangle B_{Q_i}+\langle u, B_{P_i}\rangle B_{P_i})$ satisfies $\lim_{n\to\infty}L(S_n)=f$. The result of part (II) implies that $\lim_{n\to\infty}\|u-S_n\|=0$. As L is bounded and

$$
\|f-LS_n\|=\|Lu-LS_n\|=\|L(u-S_n)\|\leq\|L\|\,\|u-S_n\|,
$$

we know that the series $LS_n=\sum_{i=1}^{n}(\langle u, B_{Q_i}\rangle LB_{Q_i}+\langle u, B_{P_i}\rangle LB_{P_i})$ converges to f in $W^{(1,1)}(\Omega)$. This completes the proof. □

Remark 3.1. As it is well-known that in any RKHS the norm-convergence implies pointwise convergence, we can even prove the uniform convergence of the series solution in RKHS $W^{(3,2)}(\Omega)$.

Corollary 3.1. *The numerical solutions* $\{S_n\}_{n\geq 1}$ *in* (3.10) *and* (3.31) *constructed by weak 1-best POAFD method and weak 2-best POAFD method converge uniformly to the solution* u *of PDE* (1.4) *on* Ω.

To discuss the convergence rate of the weak 2-best POAFD, we define the subclass $W_M^{(3,2)}(\Omega)$ of $W^{(3,2)}(\Omega)$

$$
W_M^{(3,2)}(\Omega)=\left\{u\;\middle|\;\begin{array}{l} u\in W^{(3,2)}(\Omega),\ \text{there exists } \{c_i\}_{i\geq1},\{d_i\}_{i\geq1}\\ \text{and } \{\mathbf{h}_{Q_i}\}_{i\geq1},\{\mathbf{h}_{P_i}\}_{i\geq1}\text{in } W^{(3,2)}(\Omega)\text{ such that}\\ u=\sum_{i\geq1}(c_i\mathbf{h}_{Q_i}+d_i\mathbf{h}_{P_i}),\sum_{i\geq1}(|c_i|+|d_i|)\leq M.\end{array}\right\}
$$

One can follow the proof in the paper of Qian et al. [3, 9] to obtain the following result:

Theorem 3.3 (Convergence rate, [3, 9])**.** *For* $N=2$, *let* $u=\sum_{i\geq1}(\langle u, B_{Q_i}\rangle B_{Q_i}+\langle u, B_{P_i}\rangle B_{P_i})=\sum_{i\geq1}(c_i\mathbf{h}_{Q_i}+d_i\mathbf{h}_{P_i})\in W_M^{(3,2)}(\Omega)$ *be the series solution of PDE* (1.4) *obtained by weak 2-best POAFD method. Denote by* $u_n=u-\sum_{i=1}^{n-1}(\langle u, B_{Q_i}\rangle B_{Q_i}+\langle u, B_{P_i}\rangle B_{P_i})$ *the* $(n-1)$*–th residual of* u. *We have*

$$
\|u_n\|\leq\frac{MR_n}{\rho\sqrt{2n}},
$$

where $R_n=\max_{1\leq i\leq k\leq n-1}\{\|O_k(Q_i)\|,\|E_k(P_i)\|\}$, *and* $O_k(Q_i), E_k(P_i)$ *defined in* (3.11) *and* (3.13) *depend on the first* $2(k-1)$ *functions* $\mathbf{h}_{Q_j},\mathbf{h}_{P_j}$ $(1\leq j\leq n-1)$.

To discuss the stability of weak 2-best POAFD for the solution of PDE (1.4), denote by $F_\epsilon = F + \delta_\epsilon$ the ϵ–perturbation of the non-homogeneous term F. We shall prove that the variations of both the numerical solutions and the solution u using weak 2-best POAFD are controlled by ϵ.

Theorem 3.4. *Suppose that $L : W^{(3,2)}(\Omega) \to W^{(1,1)}(\Omega)$ has bounded inverse, then the solution u and numerical solutions S_n obtained by weak 2-best method are stable.*

Proof. For any $\epsilon > 0$, and $\delta_\epsilon \in W^{(1,1)}(\Omega)$ with $\|\delta_\epsilon\|_{W^{(1,1)}} < \epsilon$. Let u and u_ϵ be the solutions of $Lu = F$ and $Lu_\epsilon = F + \delta_\epsilon$, respectively. Let S_n and S_n^ϵ be the n–th partial sum in (3.31) of weak 2-best POAFD solutions of (1.4) with inhomogeneous term F and $F + \delta_\epsilon$ respectively. Then $u = \lim_{n\to\infty} S_n$ and $u_\epsilon = \lim_{n\to\infty} S_n^\epsilon$.

Clearly, $L(u_\epsilon - u) = \delta_\epsilon$. As L^{-1} is bounded, let $C = \|L^{-1}\| > 0$. Then

$$\|u - u_\epsilon\| = \|L^{-1}(F - F_\epsilon)\| \leq \|L^{-1}\delta_\epsilon\| = \|L^{-1}\| \, \|\delta_\epsilon\|_{W^{(1,1)}} \leq C\epsilon. \tag{3.47}$$

For any $\epsilon > 0$, there exist $N \in \mathbb{N}$, if $n > N$, then

$$\|S_n - u\| \leq \frac{\epsilon}{2}, \quad \text{and} \quad \|u_\epsilon - S_n^\epsilon\| \leq \frac{\epsilon}{2}. \tag{3.48}$$

It follows from

$$\begin{aligned} \|S_n - S_n^\epsilon\| &= \|S_n - u + u - u_\epsilon + u_\epsilon - S_n^\epsilon\| \\ &\leq \|S_n - u\| + \|u - u_\epsilon\| + \|u_\epsilon - S_n^\epsilon\|, \end{aligned}$$

(3.47) and (3.48) that we have

$$\|S_n - S_n^\epsilon\| \leq (C+1)\epsilon, \tag{3.49}$$

for all $n > N$. Meanwhile, by reproducing property, we have

$$|S_n(Q) - S_n^\epsilon(Q)| = \left|\langle S_n - S_n^\epsilon, K_Q^{(3,2)}\rangle\right| \leq \|S_n - S_n^\epsilon\| \|K_Q^{(3,2)}\| \leq M_0^2(C+1)\epsilon. \tag{3.50}$$

Inequalities (3.49) and (3.50) mean numerical solutions obtained by weak 2-best POAFD method depend continuously on the inhomogeneous term of the equation (1.4) with respect to norm topology in $W^{(3,2)}(\Omega)$. □

3.4. The weak N-best POAFD

For any sequence $\{P_1, P_2, \cdots, P_N\}$ of distinct points in Ω, there relates a sequence of dictionary elements $\{\mathbf{h}_{P_1}, \mathbf{h}_{P_2}, \cdots, \mathbf{h}_{P_N}\}$. Applying G-S orthonormalization to $\{\mathbf{h}_{P_1}, \mathbf{h}_{P_2}, \cdots, \mathbf{h}_{P_N}\}$, we obtain the orthonormal set $\{B_{P_1}, B_{P_2}, \cdots, B_{P_N}\}$, where

$$B_{P_i} = \frac{\mathbf{h}_{P_i} - \sum_{j=1}^{i-1} \langle \mathbf{h}_{P_i}, B_{P_j}\rangle B_{P_j}}{\|\mathbf{h}_{P_i} - \sum_{j=1}^{i-1} \langle \mathbf{h}_{P_i}, B_{P_j}\rangle B_{P_j}\|}, \quad 1 \leq i \leq N.$$

For any fixed $0 < \rho < 1$, we determine a sequence $\{Q_1, Q_2, \cdots, Q_N\}$ of distinct points by the following weak N–best maximal selection principle:

$$\sum_{i=1}^{N} \left|\langle u, B_{Q_i}\rangle\right|^2 \geq \rho \sup\{ \sum_{i=1}^{N} \left|\langle u, B_{P_i}\rangle\right|^2 \left|\{P_i\}_{i=1}^{N} \in \Omega\}. \tag{3.51}$$

Denote $B_i = B_{Q_i}$, $1 \leq i \leq N$, then we can approximate the solution $u \in W^{(3,2)}(\Omega)$ of PDE (1.4) by weak N–best POAFD as follows:

$$u = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, B_i\rangle B_i. \tag{3.52}$$

The existence of these selected parameters (3.51), the convergence properties, and the stability of the series (3.52) can be similarly discussed as shown in 2-best case.

4. Numerical experiments

In this section, we implement two numerical experiments of PDE (1.1)–(1.2) by weak N-best POAFD method (N=1, N=2) and the reproducing kernel method discussed in [4]. Meanwhile, we consider the robustness of the weak N-best POAFD method. All these numerical experiments are carried out by MATLAB.

Denote by S_n the n–th partial sum constructed by weak N-best POAFD method and the RKM in [4]. Let m be the number of discrete points $x_i = \frac{i}{m}, i = 1, 2, \cdots, m$ in $[0, 1]$. Suppose $n \leq m^2$. Then for square region Ω, we obtain $m \times m$ grid points $(x_i, t_j) = (\dfrac{i}{m}, \dfrac{j}{m})$, $i, j = 1, 2, \cdots, m$. For the weak N-best POAFD method, we select n paremeters $\{(x_{n_i}, t_{n_j})\}_{i,j=1}^{n}$ from $\{(x_i, t_j)\}_{i,j=1}^{m^2}$ by the weak N-best maximal selection principle to construct the numerical solutions S_n. For the RKM in [4], we take $\hat{Q}(x_i, t_j), x_i = \frac{i}{\sqrt{n}}, t_j = \frac{j}{\sqrt{n}}, i, j = 1, 2, \cdots, \sqrt{n}$ to construct the numeircal solutions S_n. To show the stability of weak N-best POAFD, we choose $\epsilon = 0.0001 * \text{rand}(1, n)$ as disturbance of the inhomogeneous term f of (1.1), where $\text{rand}(1, m)$ represents $1 \times n$ matrix containing pseudo-random values drawn from a uniform distribution on the unit interval. Then we compare the norms of the remainders $\|u - S_n^{\epsilon}\|$ obtained by weak N-best POAFD. In the following, we denote by $|\cdot|^R$, $\|\cdot\|^R$ the errors and norms obtained by RKM, $|\cdot|_b^1$, $\|\cdot\|_b^1$ the errors and norms obtained by weak 1-best POAFD, $|\cdot|_b^2$, $\|\cdot\|_b^2$ the errors and norms obtained by weak 2-best POAFD, respectively.

Example 4.1. Consider the following equation

$$\begin{cases} u_t - 10u_{xx} = x - x^2 + 20(1+t), & (x,t) \in (0,1)^2, \\ u(0,t) = u(1,t) = 0, & 0 \leq t \leq 1, \\ u(x,0) = x - x^2, & 0 \leq x \leq 1. \end{cases}$$

The exact solution is given by $u(x, t) = (x - x^2)(t + 1)$.

Table 1 lists the maximum absolute errors, the maximum relative errors, and the norms of remainders of u obtained by RKM [4], weak 1-best POAFD, and weak 2-best POAFD for Example 4.1. Fig. 1 gives the graphs of numerical solutions S_n obtained by these three methods with $n = 4, 16, 36$. Figure 3 compares the errors obtained by weak 1-best POAFD, and weak 2-best POAFD.

Example 4.2. Consider the following equation

$$\begin{cases} u_t - 120t^2 u_{xx} = f(x,t), & (x,t) \in (0,1)^2, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le 1, \\ u(x,0) = 0, & 0 \le x \le 1, \end{cases}$$

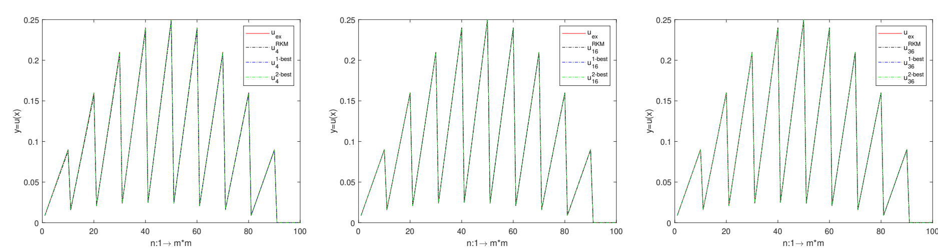
where $f(x,t) = -120(\mathrm{e}^{-1+x} - \mathrm{e}^{-1+x-t})x^2 - \mathrm{e}^{-t}\big(\frac{1}{\mathrm{e}} - \mathrm{e}^{-1+x} + \big(1 - \frac{x}{\mathrm{e}}\big)\big)$. The exact solution is $u(x,t) = -\frac{1}{\mathrm{e}} + \mathrm{e}^{-1+x} - \big(1 - \frac{1}{\mathrm{e}}\big)x + \mathrm{e}^{-t}\big(\frac{1}{\mathrm{e}} - \mathrm{e}^{-1+x} + \big(1 - \frac{x}{\mathrm{e}}\big)\big)$.

One can check the numerical results of Example 4.2 obtained by RKM, weak 1-best POAFD, and weak 2-best POAFD from Table 2, Figure 2, and Figure 4.

The comparison of these numerical results shows that the weak N-best POAFD converges fast in the first few steps and has higher accuracy than RKM. Since the implementation of weak 2-best POAFD is limited by hardware, we set $\Delta x = \frac{1}{10}$, and thus the weak 2-best POAFD has a slight advantage over weak 1-best POAFD. As Δx smaller, the weak 2-best POAFD greatly improves the accuracy than the weak 1-best POAFD in our previous work on solving integral equations. Of course, the N-best case will be better.

Table 1. Comparison of the maximum absolute errors, the maximum relative errors, and the norms of remainders obtained by RKM in [4] and the present method with $N = 1, N = 2$ for Example 4.1.

Errors	$n = 4$	$n = 8$	$n = 10$	$n = 16$	$n = 20$	$n = 36$
$\lvert u - S_n\rvert^R$	3.9527e-3	Null	Null	5.1110e-4	Null	1.8261e-4
$\lvert u - S_n\rvert_b^1$	5.7175e-4	8.5216e-5	5.9191e-5	5.7459e-5	5.7426e-5	5.7383e-5
$\lvert u - S_n\rvert_b^2$	2.6316e-4	7.0894e-5	5.7750e-5	5.7433e-5	5.7414e-5	5.7382e-5
$\left\lvert \frac{u-S_n}{u}\right\rvert^R$	2.4820e-2	Null	Null	3.9715e-3	Null	1.3068e-3
$\left\lvert \frac{u-S_n}{u}\right\rvert_b^1$	3.5320e-3	4.0847e-4	3.4549e-4	3.4146e-4	3.4134e-4	3.4119e-4
$\left\lvert \frac{u-S_n}{u}\right\rvert_b^2$	1.3334e-3	6.3662e-4	3.4230e-4	3.4137e-4	3.4130e-4	3.4118e-4
$\lVert u - S_n\rVert^R$	1.3857e-1	Null	Null	2.2914e-2	Null	2.9595e-3
$\lVert u - S_n\rVert_b^1$	2.0376e-2	3.3598e-3	9.7029e-4	7.0062e-4	7.3057e-4	7.2460e-4
$\lVert u - S_n\rVert_b^2$	7.8132e-3	1.9293e-3	5.8186e-4	5.5332e-4	5.5279e-4	7.2311e-4
$\lVert u - S_n\rVert_\epsilon^1$	2.0376e-2	3.3619e-3	9.6628e-3	1.0932e-3	2.0340e-3	1.6453e-3
$\lVert u - S_n\rVert_\epsilon^2$	1.0205e-2	2.5304e-3	5.9410e-3	2.8096e-3	2.5600e-3	2.6686e-3

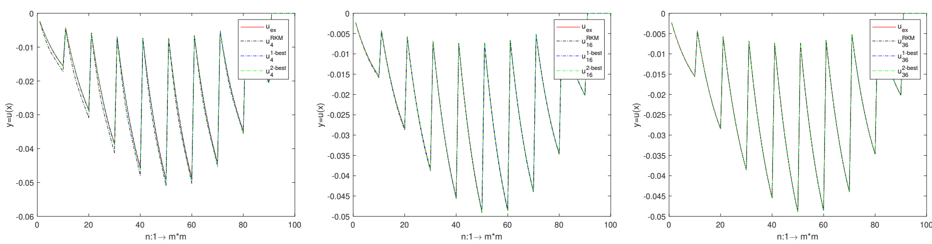


(a) $S_4(x,t)$ (b) $S_{16}(x,t)$ (c) $S_{36}(x,t)$

Figure 1. The numerical solutions $S_4(x,t)$, $S_{16}(x,t)$, S_{36} obtained by RKM in [4], weak 1-best POAFD and weak 2-best POAFD for Example 4.1, respectively.

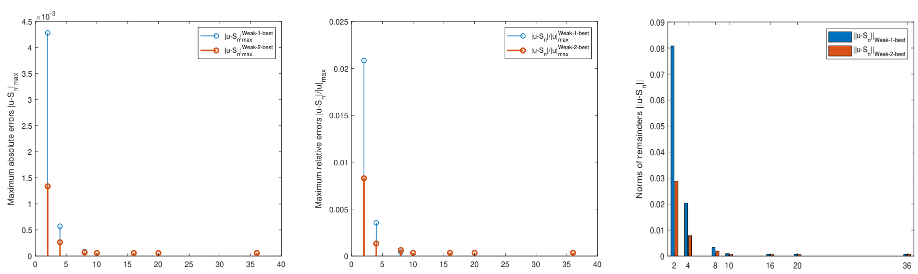
Table 2. Comparison of the maximum absolute errors, the maximum relative errors, and the norms of remainders obtained by RKM in [4] and the present method with $N = 1, N = 2$ for Example 4.2.

Errors	$n = 4$	$n = 8$	$n = 10$	$n = 16$	$n = 20$	$n = 36$
$\lvert u - S_n\rvert^R$	3.9957e-3	Null	Null	5.7104e-4	Null	1.6190e-4
$\lvert u - S_n\rvert_b^1$	8.6468e-4	1.9437e-4	2.6171e-4	1.4654e-4	5.1844e-5	3.4166e-5
$\lvert u - S_n\rvert_b^2$	1.2791e-3	2.3351e-4	2.9238e-4	5.1456e-5	5.3856e-5	3.3537e-5
$\left\lvert \frac{u-S_n}{u}\right\rvert^R$	1.5074e-1	Null	Null	3.1149e-2	Null	1.0665e-2
$\left\lvert \frac{u-S_n}{u}\right\rvert_b^1$	1.7804e-2	1.5095e-2	1.2125e-2	5.0212e-3	6.7118e-3	5.7902e-3
$\left\lvert \frac{u-S_n}{u}\right\rvert_b^2$	3.3596e-2	1.0368e-2	1.1537e-2	4.5756e-3	6.2193e-3	5.0876e-3
$\lVert u - S_n\rVert^R$	2.1028e-1	Null	Null	6.8686e-2	Null	3.1795e-2
$\lVert u - S_n\rVert_b^1$	8.8449e-2	3.3402e-2	3.0950e-2	1.9564e-2	1.4148e-2	5.9997e-3
$\lVert u - S_n\rVert_b^2$	8.8618e-2	3.1736e-2	2.7259e-2	1.6621e-2	1.3288e-2	5.8484e-3
$\lVert u - S_n\rVert_\epsilon^1$	8.8567e-2	3.3206e-2	3.0935e-2	1.8300e-2	1.5392e-2	7.8187e-3
$\lVert u - S_n\rVert_\epsilon^2$	8.8542e-2	3.0944e-2	2.7152e-2	1.8077e-2	1.5867e-2	8.5171e-3



(a) $S_4(x,t)$ (b) $S_{16}(x,t)$ (c) $S_{36}(x,t)$

Figure 2. The numerical solutions $S_4(x,t)$, $S_{16}(x,t)$, S_{36} obtained by RKM in [4], weak 1-best POAFD and weak 2-best POAFD for Example 4.2, respectively.

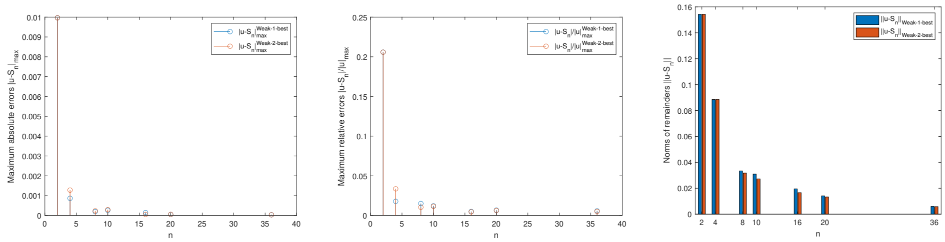


(a) The maximum absolute errors (b) The maximum relative errors (c) The norms of remainders

Figure 3. Comparison of numerical results for Example 4.1.

5. Conclusion

In this paper, we applied weak N-best POAFD method to solve the parabolic equation (1.4) in the RKHSs $W^{(3,2)}(\Omega)$ and $W^{(1,1)}(\Omega)$. And obtained the solutions u in a series form under some additional assumptions. This is the first trial in apply-



(a) The maximum absolute errors (b) The maximum relative errors (c) The norms of remainders

Figure 4. Comparison of numerical results for Example 4.2.

ing weak N-best POAFD to solve PDEs. Based on weak N-best maximal selection principle (3.8), (3.26), and (3.51), the numerical solutions S_n is a finite linear combination of $\{L^*K_{Q_i}\}_{Q_i\in\Omega}$ with a higher accuracy in $W^{(3,2)}(\Omega)$ than that of RKM method. Moreover, theoretical and numerical results show that the weak N-best POAFD method is robust and converges fast.

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