Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1672–1678

Website:http://www.jaac-online.com
DOI:10.11948/20220107

AN IMPROVED UPPER BOUND ON THE LINEAR 2-ARBORICITY OF TOROIDAL GRAPHS*

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Abstract The linear 2-arboricity $\mathrm{la}_2(G)$ of a graph G is the smallest integer k such that G can be partitioned into k edge-disjoint forests, whose components are paths of length at most 2. In this paper, we prove that every toroidal graph G has $\mathrm{la}_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 6$. Since K_7 is a toroidal graph with $\mathrm{la}_2(K_7) = 6 = \lceil \frac{\Delta(K_7)+1}{2} \rceil + 2$, our solution is within four from optimal.

Keywords Toroidal graph, linear 2-arboricity, maximum degree, edge-partition.

MSC(2010) 05C15.

1. Introduction

In this paper, we will consider only simple graphs. For a graph G, we use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, edge set, minimum degree, and maximum degree, respectively. A *linear forest* is one in which each connected component is a path. Given a graph G, we define its *linear arboricity*, denoted by $\mathrm{la}(G)$, to be the minimum number of edge-disjoint linear forests in G whose union is $E(G)$. A *linear k-forest* is a graph whose components are paths of length at most k. The *linear k-arboricity* $\mathrm{la}_k(G)$ of G is the smallest integer m for which G can be edge-partitioned into m linear k-forests.

The linear k-arboricity of a graph was first introduced by Habib and Péroche [5], who were based on the linear arboricity, see the most recent research [1,4,6]. Among other things, they made the conjecture that any graph G with n vertices has

$$\mathrm{la}_2(G) \leq \begin{cases} \lceil \dfrac{n\Delta(G)+1}{2\lfloor \frac{2}{3}n \rfloor} \rceil, & \text{if } \Delta(G) \neq n-1; \\ \lceil \dfrac{n\Delta(G)}{2\lfloor \frac{2}{3}n \rfloor} \rceil, & \text{if } \Delta(G) = n-1. \end{cases}$$

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*The authors were supported by National Science Foundation of China (11801512, 11971437, 11901525, 12071048) and Science and Technology Commission of Shanghai Municipality (18dz2271000).

The linear 2-arboricity of graphs has been extensively studied in the past decades. Let $f(\Delta(G)) = \lceil \frac{\Delta(G)+1}{2} \rceil$. In 2004, Lih, Tong and Wang [7] proved that every planar graph G has $\mathrm{la}_2(G) \le f(\Delta(G)) + 12$; and if G does not contain 3-cycles, then $\mathrm{la}_2(G) \le f(\Delta(G)) + 6$. In 2009, Ma, Wu and Hu [10] proved that if a planar graph G contains no 5-cycles or 6-cycles, then $\mathrm{la}_2(G) \le f(\Delta(G)) + 6$. Wang et al. [12,13] provided a significant improvement by showing that $\mathrm{la}_2(G) \le f(\Delta(G)) + 6$ for any planar graph G. Graphs are referred to as *1-planar* if each edge can be drawn in the plane and only one other edge is crossed by it. Liu et al. [8] proved that $\mathrm{la}_2(G) \le f(\Delta(G)) + 14$ for any 1-planar graph G. Recently, this result was improved to $\mathrm{la}_2(G) \le f(\Delta(G)) + 7$ in [9].

A graph is *toroidal* if it can be embedded in the torus such that any two edges cross only at their ends. Wang et al. [11] showed that every toroidal graph G has $\mathrm{la}_2(G) \le f(\Delta(G)) + 7$. The purpose of this paper is to improve this result by reducing the constant 7 to 6.

2. Results

Let G be a graph embedded in the torus. Denote by $F(G)$ the set of faces. A vertex of degree k (at most k or at least k) is called a *k-vertex* (k^--vertex or k^+-vertex). Similarly, we can define k-face, k^--face, and k^+-face. For a vertex $v \in V(G)$ and an integer $i \ge 1$, let $n_i(v)$ denote the number of i-vertices in G adjacent to v.

A *linear-k-coloring* of graph G is a mapping $\phi : E(G) \to \{1, 2, \dots, k\}$ such that each color class induces a subgraph whose components are paths of length at most 2. It is clear that a graph G has linear 2-arboricity at most k if and only if G is linear-k-colorable.

Lemma 2.1 ([2]). *For any graph G,* $\mathrm{la}_2(G) \le \Delta(G)$.

Theorem 2.1. *If G is a toroidal graph with $\Delta(G) \le 10$, then* $\mathrm{la}_2(G) \le 9$.

Proof. It suffices to prove that G has a linear-9-coloring. If $\Delta(G) \le 9$, then the result holds automatically by Lemma 2.1, so assume that $\Delta(G) = 10$. The proof is given by contradiction. Let G be such a counterexample to the theorem that $|V(G)| + |E(G)|$ is the smallest possible. So G is connected and $\delta(G) \ge 1$. For any proper subgraph H of G, H has a linear-9-coloring ϕ. In what follows, we denote by $C = \{1, 2, \dots, 9\}$ a set of nine colors used in the proof. For a vertex $v \in V(H)$, we denote by $C(v)$ the set of colors used in edges incident to v in H, and by $S(v)$ the sequence of colors assigned to the edges incident to v in H. For example, $S(v) = (1, 1, 2, \dots, 9)$ means that color 1 occurs twice, and each of colors $2, 3, \dots, 9$ occurs exactly once in $C(v)$. This means that v is a 10-vertex of H. Usually, we write $S(v) = (1, 1, 2, \dots, 9)$ for short as $S(v) = (1, 1, 2-9)$. For an edge $xy \in E(G) \setminus E(H)$, we define a list assignment $L(xy) = C \setminus (C(x) \cup C(y))$, whose colors can be applied to color xy.

Choose H as the largest component of the graph obtained from G by removing all 1-vertices and 2-vertices. Then H is a connected toroidal graph with $\Delta(H) \le 10$.

Claim 1 below can be similarly established as in [12].

Claim 1. ([12])

(1) Every edge $xy \in E(G)$ has $d_G(x) + d_G(y) \ge 11$.

(2) $\delta(H) \ge 3$.

(3) If $v \in V(H)$ with $d_H(v) \leq 6$, then $d_H(v) = d_G(v)$.

(4) Let $v \in V(G)$ be a k-vertex with $5 \leq k \leq 10$ and $v_1, v_2, \ldots, v_k$ be its neighbors with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$. Then

(4.1) $n_{11-k}(v) \leq 1$;

(4.2) If $n_{11-k}(v) = 1$, then $n_{12-k}(v) \leq 1$; moreover, if $n_{12-k}(v) = 1$, then $n_{13-k}(v) \leq 1$;

(4.3) $n_{11-k}(v) + n_{12-k}(v) \leq 3$; moreover, if $n_{11-k}(v) + n_{12-k}(v) = 3$, then $n_{13-k}(v) = 0$;

(4.4) If $k = 10$, then $n_1(v)+n_2(v)+n_3(v) \leq 5$; moreover, if $n_1(v)+n_2(v) \geq 3$, then $n_3(v) = 0$.

For a vertex $v \in V(H)$ and an integer $i \geq 3$, let $n'_i(v)$ denote the number of i-vertices adjacent to v in H, and let $m'_3(v)$ denote the number of 3-faces incident to v in H. According to Claim 1(3), $n'_i(v) = n_i(v)$ for $i = 3, 4, 5$.

The proof of Claim 2 below can be similarly given as in [13].

Claim 2. ([13])

(1) If $d_H(v) = 7$, then $n'_3(v) = 0$, $n'_4(v) \leq 1$; and if $n'_4(v) = 1$, then $n'_5(v) \leq 1$.

(2) If $d_H(v) = 8$, then $n'_3(v) \leq 1$; and if $n'_3(v) = 1$, then $n'_4(v) \leq 1$.

(3) If $d_H(v) = 9$, then $n'_3(v) \leq 3$.

(4) No 9-vertex v of H satisfies $n'_3(v) = 3$ and $m'_3(v) = 9$.

(5) No 3-face $[uvw]$ of H satisfies $d_H(u) = 5$ and $d_H(v) = 6$.

(6) No 10-vertex v of H with $n'_3(v) \geq 4$ such that every adjacent 3-vertex is incident to a 3-face that is incident to v.

Claim 3. *No 10-vertex $v \in V(H)$ satisfies $n'_3(v) \geq 3$ and $m_3(v) = 10$.*

Proof. Suppose H contains such a 10-vertex v. Let $v_0, v_1, \ldots, v_9$ be the neighbors of v in clockwise order. For $0 \leq i \leq 9$, let f_i denote the incident face of v with vv_i and vv_{i+1} as two of the boundary edges, where subscripts are taken modulo 10. Without loss of generality, we assume that v_1, v_3, v_5 are 3-vertices; for other cases we have a similar proof. By Claim 1(3), $d_G(v_i) = d_H(v_i)$ for $i = 1, 3, 5$. Let $G' = G - vv_1$. Then G' is a torodial graph with $|V(G')|+|E(G')| < |V(G)|+|E(G)|$ and $\Delta(G') \leq \Delta(G) \leq 10$. By the minimality of G, G' has a linear-9-coloring ϕ using the color set C. First, we remove the colors of vv_3 and vv_5. To color vv_1, vv_3 and vv_5, we consider four cases as follows by symmetry.

Case 1. $|L(vv_i)| \geq 1$ for $i = 1, 3, 5$.

If $|L(vv_i)| \geq 2$ for some $i \in \{1, 3, 5\}$ or there are $i, j \in \{1, 3, 5\}$ such that $L(vv_i) \neq L(vv_j)$, then we can color vv_1, vv_3, vv_5. Otherwise, we assume that $L(vv_1) = L(vv_3) = L(vv_5) = \{9\}$. The proof is split into two subcases.

Case 1.1. There is a color, say 1, that occurs twice in $S(v)$.

We can now assume that $S(v) = (1, 1, 2-6)$ and $C(v_i) = \{7, 8\}$ for $i = 1, 3, 5$. Let us also assume that $\phi(vv_6) = 4$, $\phi(v_4v_5) = 7$ and $\phi(v_5v_6) = 8$. We first assign color 9 to vv_1 and vv_3. If 4 or 8 occurs exactly once in $S(v_6)$, then we color vv_5 with 4 or 8. If there is $a \in \{1, 2, 3, 5, 6, 7\} \setminus C(v_6)$, then we recolor v_5v_6 with a and color vv_5 with 8. Otherwise, it follows that $S(v_6) = (1-3, 4, 4, 5-7, 8, 8)$. This implies that $9 \notin C(v_6)$. We can recolor v_5v_6 with 9 and vv_5 with 8.

Case 1.2. $S(v) = (1-7)$.

Then $8 \in C(v_i)$ for $i = 1, 3, 5$, assume that $\phi(v_4v_5) = 8$. Let $\phi(vv_0) = 1$, $\phi(vv_2) = 2$, $\phi(vv_4) = 3$, and $\phi(vv_6) = 4$. First color vv_1 and vv_3 with 9.

Suppose that $\phi(v_5v_6) \neq 3$. If $3 \notin C(v_1)$, then we exchange the colors of vv_4 and v_4v_5, recolor vv_1 with 3, and color vv_5 with 9. Otherwise, $3 \in C(v_1)$. A similar discussion can show that $3 \in C(v_3)$. It follows that $S(v_1) = S(v_3) = (3, 8, 9)$, say $\phi(v_0v_1) = 3$ and $\phi(v_1v_2) = 8$. If there is $a \in \{1, 4-7, 9\} \setminus C(v_2)$, then we recolor v_1v_2 with a and vv_1 with 8, and color vv_5 with 9. If 2 or 8 appears exactly once in $S(v_2)$, then we recolor vv_1 with 2 or 8 and color vv_5 with 9. Otherwise, $S(v_2) = \{1, 2, 2, 4-7, 8, 8, 9\}$. This implies that $3 \notin C(v_2)$. We exchange the colors of vv_4 and v_4v_5, recolor vv_1 with 2 and vv_2 with 3, and color vv_5 with 9.

Suppose that $\phi(v_5v_6) = 3$. If vv_5 cannot be colored, we may derive that $S(v_4) = (1-7, 8, 8, 9)$. If there is $b \in \{1, 2, 5, 6, 7, 9\} \setminus C(v_6)$, then we recolor v_5v_6 with b and color vv_5 with 3. If $8 \notin C(v_6)$, then we color or recolor $\{v_5v_6, vv_5\}$ with 8, and v_4v_5 with 3. If 4 occurs exactly once in $S(v_6)$, we color vv_5 with 4. This shows that $S(v_6) = (1-3, 4, 4, 5-9)$. If $3 \notin C(v_1)$, then we recolor vv_1 with 3 and color vv_5 with 9. Otherwise, $S(v_1) = (3, 8, 9)$. Recolor or color $\{v_5v_6, vv_1\}$ with 4, vv_6 with 3, and vv_5 with 9.

Since, in each of the following remaining cases, there is at least one index $i \in \{1, 3, 5\}$ such that $|L(vv_i)| = 0$, we always assume that $S(v) = (1-7)$, $\phi(vv_0) = 1, \phi(vv_2) = 2, \phi(vv_4) = 3$, and $\phi(vv_6) = 4$.

Case 2. $|L(vv_1)|, |L(vv_3)| \geq 1$ and $|L(vv_5)| = 0$.

Note that $S(v_5) = (8, 9)$, we assume that $9 \in C(v_1)$ and $a \in C(v_3)$, where $a \in \{8, 9\}$. We first color vv_1 with 9 and vv_3 with a. Assume that $\phi(v_4v_5) = 8$ and $\phi(v_5v_6) = 9$. If there is a color $c \in \{1, 2, 4, 5, 6, 7\} \setminus C(v_4)$, then we recolor v_4v_5 with c and color vv_5 with 8. If 3 appears once in $S(v_4)$, then we color vv_5 with 3.

Assume that $a = 8$. If $9 \notin C(v_4)$, we recolor vv_4 with 9 and color vv_5 with 3. Otherwise, $S(v_4) = (1, 2, 3, 3, 4-9)$ and $S(v_6) = (1-3, 4, 4, 5-9)$. If $4 \notin C(v_1)$, then we exchange the colors of vv_6 and v_5v_6, recolor vv_1 with 4, and color vv_5 with 9. If $8 \notin C(v_1)$, then we recolor vv_1 with 8 and color vv_5 with 9. Otherwise, $S(v_1) = (4, 8, 9)$. Similarly, $S(v_3) = (3, 8, 9)$. Recolor or color $\{vv_1, v_4v_5\}$ with 3, vv_4 with 8, and vv_5 with 9.

Suppose $a = 9$. Similarly, we can deduce that $S(v_4) = (1, 2, 3, 3, 4-7, 8, 8)$. This implies that $9 \notin C(v_4)$. If $d \in \{1-3, 5-8\} \setminus C(v_6)$, then we recolor v_5v_6 with d, v_4v_5 with 9, and color vv_5 with 8. If 4 occurs only once in $S(v_6)$, we color vv_5 with 4. Otherwise, $S(v_6) = (1-3, 4, 4, 5-9)$. It sufficient to recolor v_4v_5 with 9 and color vv_5 with 8.

Case 3. $|L(vv_1)| \geq 1, |L(vv_3)| = |L(vv_5)| = 0$.

Note that $S(v_3) = S(v_5) = (8, 9)$ and $9 \in L(vv_1)$. We color vv_1 with 9 and assume that $\phi(v_4v_5) = 8$ and $\phi(v_5v_6) = 9$. If v_4v_5 may be recolored by one of the colors $1, 2, \ldots, 7$, we define a new list assignment L' for the edges vv_1, vv_3, vv_5 such that $|L'(vv_1)| \geq 1$, $|L'(vv_5)| \geq 1$ and $|L'(vv_3)| \geq 0$. The proof is then reduced to either Case 1 or Case 2. If 8 appears just once in $S(v_4)$, we exchange the colors of vv_4 and v_4v_5, and color vv_3 with 3 and vv_5 with 8. Otherwise, $S(v_4) = (1, 2, 3, 3, 4-7, 8, 8)$, implying that $9 \notin C(v_4)$. We recolor vv_4 with 9 and color $\{vv_3, vv_5\}$ with 3.

Case 4. $|L(vv_i)| = 0$ for $i = 1, 3, 5$.

For $i = 1, 3, 5$, $S(v_i) = (8, 9)$. Assume that $\phi(v_0v_1) = 8$ and $\phi(v_1v_2) = 9$.

If 8 appears only once in $S(v_0)$, then we interchange the colors of vv_0 and v_0v_1, and color vv_1 with 8 and $\{vv_3, vv_5\}$ with 1. If $9 \notin C(v_0)$, we recolor vv_0 with 9 and v_0v_1 with 1, and color vv_1 with 8, $\{vv_3, vv_5\}$ with 1. Otherwise, v_0v_1 can be recolored with one of the colors $1, 2, \ldots, 7$, we can define a new list assignment L' such that $|L'(vv_1)| \geq 1$ and the proof is reduced to the preceding cases.

By employing Euler's formula $|V(H)|-|E(H)|+|F(H)| = 0$ and the handshaking theorem

$$\sum_{v\in V(H)} d_H(v) = \sum_{f\in F(H)} d_H(f) = 2|E(H)|,$$

the following identity can be deduced:

$$\sum_{v\in V(H)} (d_H(v) - 4) + \sum_{f\in F(H)} (d_H(f) - 4) = 0. \tag{2.1}$$

Consider the weight function defined on H by $w(x) = d_H(x) - 4$ for any $x \in V(H) \cup F(H)$. We are going to redistribute the weight between vertices and faces in H while keeping the sum of all weights fixed so that the resultant weight $w'(x) \geq 0$ for all $x \in V(H) \cup F(H)$. In addition, there exists some $x^* \in V(H) \cup F(H)$ such that $w'(x^*) > 0$. This yields the following obvious contradiction

$$0 < \sum_{x\in V(H)\cup F(H)} w'(x) = \sum_{x\in V(H)\cup F(H)} w(x) = 0 \tag{2.2}$$

and hence the proof is complete.

Let $f = [v_1v_2v_3]$ be a 3-face with $d_H(v_1) \leq d_H(v_2) \leq d_H(v_3)$. If v_i sends the weight a_i to the face f, then we simply write $(d_H(v_1), d_H(v_2), d_H(v_3)) \to (a_1, a_2, a_3)$. Our discharging rules are as follows.

(R1) Every 8^+-vertex $v \in V(H)$ sends $\frac{1}{3}$ to each adjacent 3-vertex.

(R2) If $f = [v_1v_2v_3]$ is a 3-face of H, then

- **(R2.1)** $(3, 8^+, 8^+) \to (0, \frac{1}{2}, \frac{1}{2})$;
- **(R2.2)** $(4, 7^+, 7^+) \to (0, \frac{1}{2}, \frac{1}{2})$;
- **(R2.3)** $(5, 6^+, 6^+) \to (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$;
- **(R2.4)** $(6^+, 6^+, 6^+) \to (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Let w' denote the resultant weight function after (R1)-(R2) are carried out on H. Now we need to compute $w'(x)$ for all $x \in V(H) \cup F(H)$.

Assume that $f \in F(H)$. If $d_H(f) \geq 4$, then $w'(f) = w(f) = d_H(f) - 4 = 0$. Consider that $f = [xyz]$ is a 3-face with $d_H(x) \leq d_H(y) \leq d_H(z)$. There are a number of subcases to be discussed in view of Claims 1 and 2 and Rule (R2). If $d_H(x) = 3$, then $d_H(y), d_H(z) \geq 8$, and thus $w'(f) = -1 + 2 \cdot \frac{1}{2} = 0$ by (R2.1). Assuming $d_H(x) = 4$, $d_H(y), d_H(z) \geq 7$, and so $w'(f) = -1 + 2 \cdot \frac{1}{2} = 0$ by (R2.2). If $d_H(x) = 5$, then $d_H(y), d_H(z) \geq 6$. By (R2.3), $w'(f) = -1 + 2 \cdot \frac{2}{5} + \frac{1}{5} = 0$. If $d_H(x) \geq 6$, in this case, $w'(f) = -1 + 3 \cdot \frac{1}{3} = 0$ by (R2.4).

Assume that $v \in V(H)$ is a k-vertex, where $k \geq 3$ by Claim 1(2). Let $v_0, v_1, \ldots, v_{k-1}$ be the clockwise neighbors of v. For $i = 0, 1, \ldots, k-1$, let f_i denote the incident face of v with vv_i, vv_{i+1} as two boundary edges, where indices are taken modulo k.

If $k = 3$, then each of the neighbors of v is of degree at least 8 in H by Claims 1 and 2. According to (R1), $w'(v) \geq -1 + 3 \cdot \frac{1}{3} = 0$. If $k = 4$, then $w'(v) = w(v) = d_H(v) - 4 = 0$. Since each incident face in (R2.3) receives a maximum weight of $\frac{1}{5}$ if $k = 5$, $w'(v) \geq 1 - 5 \cdot \frac{1}{5} = 0$.

Assume that $k = 6$, $w(v) = 2$. If v is adjacent to a 5-vertex, say v_1, then neither f_0 nor f_1 is a 3-face according to Claim 2(5). We get $w'(v) \geq 2 - 4 \cdot \frac{1}{3} = \frac{2}{3}$. Otherwise, only 6^+ vertices are adjacent to v, and therefore $w'(v) \geq 2 - 6 \cdot \frac{1}{3} = 0$ by (R2.4).

Assume that $k = 7$. By Claim 2(1), $n_3'(v) = 0$, $n_4'(v) \leq 1$; moreover, if $n_4'(v) = 1$, then $n_5'(v) \leq 1$. If $n_4'(v) = 1$, then $w'(v) \geq 3 - 2 \cdot \frac{1}{2} - 2 \cdot \frac{2}{5} - 3 \cdot \frac{1}{3} = \frac{1}{5}$ by (R2). Assuming $n_4'(v) = 0$, (R2) and Claim 1(1) assert that $w'(v) \geq 3 - 6 \cdot \frac{2}{5} - \frac{1}{3} = \frac{4}{15}$.

Assume that $k = 8$. As stated in Claim 2(2), $n_3'(v) \leq 1$, and if $n_3'(v) = 1$, then $n_4'(v) \leq 1$. If $n_3'(v) = 0$, then $w'(v) \geq 4 - 8 \cdot \frac{1}{2} = 0$ by (R2). Assuming that $n_3'(v) = 1$, then $w'(v) \geq 4 - 4 \cdot \frac{1}{2} - 4 \cdot \frac{2}{5} - \frac{1}{3} = \frac{1}{15}$ by (R1) and (R2).

Assume that $k = 9$. According to Claim 2(3), $n_3'(v) \leq 3$. If $n_3'(v) \leq 1$, then $w'(v) \geq 5 - 9 \cdot \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ by (R1) and (R2). Assume that $n_3'(v) = 2$. If $m_3'(v) \leq 8$, then $w'(v) \geq 5 - 8 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} = \frac{1}{3}$ by (R1) and (R2). Otherwise, $m_3'(v) = 9$. In this case, Claim 1(1) asserts that v is adjacent to at most four 5^--vertices, which means that $w'(v) \geq 5 - 8 \cdot \frac{1}{2} - \frac{1}{3} - 2 \cdot \frac{1}{3} = 0$ by (R1) and (R2). Next suppose that $n_3'(v) = 3$. If $m_3'(v) \leq 8$, then $w'(v) \geq 5 - 8 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0$ by (R1) and (R2). Otherwise, $m_3'(v) = 9$, which contradicts Claim 2(4).

Assume that $k = 10$. Claim 1(4) states that $n_3'(v) \leq 5$. We have $w'(v) \geq 6 - 10 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} = \frac{1}{3}$ when $n_3'(v) \leq 2$ by (R1) and (R2). Suppose that $n_3'(v) \geq 3$. By Claim 3, $m'(v) \leq 9$. If $n_3'(v) = 3$, then $w'(v) \geq 6 - 9 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = \frac{1}{2}$. If $n_3'(v) = 4$, then $w'(v) \geq 6 - 9 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6}$ by (R1) and (R2). Now assume that $n_3'(v) = 5$. By Claim 2(6), $m_3'(v) \leq 8$, which means that $w'(v) \geq 6 - 8 \cdot \frac{1}{2} - 5 \cdot \frac{1}{3} = \frac{1}{3}$ by (R1) and (R2).

We have now shown that $w'(x) \geq 0$ for all $x \in V(H) \cup F(H)$. Note that a 10-vertex, say x^*, can always be found in G due to the fact that $\Delta(H) = 10$, and the above proof implies that $w'(x^*) > 0$. This completes proof of the theorem. □

Lemma 2.2 ([11]). *Every toroidal graph G with $\Delta(G) \geq 12$ can be edge-partitioned into two forests F_1, F_2 and a subgraph H such that $\Delta(H) \leq 10$ and $\Delta(F_i) \leq \lceil \frac{\Delta(G)-9}{2} \rceil$ for $i = 1, 2$.*

Lemma 2.3 ([3]). *For a forest F, we have $\mathrm{la}_2(F) \leq \lceil \frac{\Delta(F)+1}{2} \rceil$.*

The following conclusion is obvious:

Lemma 2.4. *If $G = G_1 \cup G_2$, then $\mathrm{la}_2(G) \leq \mathrm{la}_2(G_1) + \mathrm{la}_2(G_2)$.*

Theorem 2.2. *If G is a toroidal graph, then $\mathrm{la}_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 6$.*

Proof. Lemma 2.1 makes it simple to see $\mathrm{la}_2(G) \leq \Delta(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 6$ if $\Delta(G) \leq 11$, so assume that $\Delta(G) \geq 12$. According to Lemma 2.2, G has an edge-partition into two forests F_1, F_2 and a subgraph H with $\Delta(F_1) \leq \lceil \frac{\Delta(G)-9}{2} \rceil$, $\Delta(F_2) \leq \lceil \frac{\Delta(G)-9}{2} \rceil$, and $\Delta(H) \leq 10$. We establish the following sequence of inequalities by combining Lemmas 2.3 and 2.4 with Theorem 2.1:

$$\begin{aligned}\mathrm{la}_2(G) &\leq \mathrm{la}_2(F_1) + \mathrm{la}_2(F_2) + \mathrm{la}_2(H) \\ &\leq \left\lceil \frac{\Delta(F_1)+1}{2} \right\rceil + \left\lceil \frac{\Delta(F_2)+1}{2} \right\rceil + 9\end{aligned}$$

$$\leq 2\left\lceil \frac{1}{2}\left\lceil \frac{\Delta(G)-9)}{2}\right\rceil + \frac{1}{2}\right\rceil + 9$$
$$\leq \left\lceil \frac{\Delta(G)+1}{2}\right\rceil + 6.$$

□

Note that K_7 is a toroidal graph with $\Delta(K_7) = 6$ and it is simple to compute that $\mathrm{la}_2(K_7) = 6 = \lceil \frac{\Delta(K_7)+1}{2} \rceil + 2$. This means that Theorem 2.2 is within four of optimality. As a natural observation, we provide the following problem:

Problem. *Determine the smallest constant C such that every toroidal graph G has $\mathrm{la}_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + C$.*

Based on Theorem 2.2 and the linear 2-arboricity of K_7, we can conclude that $2 \leq C \leq 6$.

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