Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1679–1701
Website:http://www.jaac-online.com
DOI:10.11948/20220161

A NEW PROOF OF MOSER'S THEOREM

Chang Liu[1] and Jiamin Xing[1,†]

Abstract In this paper, we consider the persistence of invariant tori for mappings under perturbations. Mainly, we give a new proof of Moser type theorem about invariant tori for twist mappings with intersection property.

Keywords High-dimensional Moser's theorem, twist mapping, intersection property.

MSC(2010) 37E40, 37J40.

1. Introduction

In 1954, Kolmogorov [7] first claimed the persistence of quasi-periodic invariant tori under small perturbations with non-degenerate conditions and Arnold [1] proved it soon after. In 1962, Moser [11] proved the existence of the invariant curves for twist mappings with one action and one angular variable. These works [1, 7, 11] established the celebrated KAM theory. Later, Zehnder [22, 23] took use of the generalized implicit function theorem to solve some divisor problems. Rüssmann [16] exhibited the optimal result that the invariant tori of twist mappings persist when the perturbations belong to class C^l, $l > 3$. Due to the efforts of Cheng and Sun [2], the persistence of invariant tori for three-dimensional measure-preserving mappings under small perturbations is proved. And the case of volume-preserving mappings on n-dimensional space was considered by Xia [20]. For Moser's invariant curve theorem, one can find some further developments in Svanidze [17], Herman [6], Levi [9], and Moser [12–14].

Cong et al. [3] proved the persistence of invariant tori of analytic nearly twist mappings with different numbers of action and angular variables when having intersection property. The so-called intersection property is that any n-dimensional torus close to the invariant torus of the unperturbed system intersects its image under the mapping. Li and Yi [10] introduced a parameter in their consideration of generalized Hamiltonian systems. It inspired us to consider the corresponding circumstances of twist mappings with a parameter. Recently, Yang and Li [21] investigated the existence of periodically invariant tori of twist mappings on resonant surfaces. There are more related works on KAM theory in recent years, for example, see Chierchia et al. [4], Calleja et al. [5], Koudjinan [8], Sevryuk [18], and Trujillo [19]. In this paper, we will show detailed proof of the persistence of invariant tori for twist mappings with intersection property.

[†]The corresponding author. Email: xingjiamin1028@126.com (J. Xing)
[1]School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, China

Consider the analytic mapping $\mathscr{F} : T^n \times D \times G \to T^n \times \mathbb{R}^{m+p}$ defined by

$$\mathscr{F} : \begin{cases} \theta^1 = \theta + \omega(r,\xi) + f(\theta, r, \xi), \\ r^1 = r + g(\theta, r, \xi), \end{cases} \tag{1.1}$$

where $\theta \in T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, $r \in D \subset \mathbb{R}^m$, $\xi \in G \subset \mathbb{R}^p$, D and G are bounded and closed domains. Assume the following conditions hold:

$(A1)$ f and g are analytic on $T^n \times D \times G$;

$(A2)$ there exists a positive integer K such that

$$\text{rank}\{\frac{\partial^\alpha \omega}{\partial(r,\xi)^\alpha} : \ 0 \le |\alpha| \le K-1\} = n;$$

$(A3)$ the frequency ω satisfies the Diophantine condition

$$|\langle k, \omega\rangle - k_0| \ge \frac{\delta}{|k|^\tau}, \qquad \forall\ k \in \mathbb{Z}^n \backslash \{0\}, \qquad \forall\ k_0 \in \mathbb{Z},$$

where $\tau > n+1$ is fixed.

The unperturbed mapping of (1.1) is

$$\begin{cases} \theta^1 = \theta + \omega(r,\xi), \\ r^1 = r, \end{cases} \tag{1.2}$$

which is called twist mapping.

Hereafter consider the mapping $\mathscr{F}$ in a complex neighborhood $\mathbb{T}^n \times \mathbb{D} \times \mathbb{G}$ of $T^n \times D \times G$, where $\mathbb{T}^n = \mathbb{C}^n/2\pi\mathbb{Z}^n$, $\mathbb{D} \subset \mathbb{C}^m$, and $\mathbb{G} \subset \mathbb{C}^p$. For given $0 < \rho, s, \sigma \le 1$, let

$$\mathcal{D}(\rho, s, \sigma) = \{(\theta, r, \xi) \in \mathbb{T}^n \times \mathbb{D} \times \mathbb{G} : \ |\text{Im } \theta| \le \rho, \ |\text{Im } r| \le s, \ |\text{Im } \xi| \le \sigma\},$$

and $\| \cdot \|_{\mathcal{D}(\rho,s,\sigma)}$ represents the sup-norm of functions in $\mathcal{D}(\rho, s, \sigma)$.

Now we are in a position to state the main results of this paper.

Theorem 1.1. *Consider the twist mapping* (1.1) *with intersection property on* $T^n \times D \times G$*. Assume* (A1)-(A3) *hold. Then there exists a nonempty Cantor set* $D_\delta \subset D \times G$ *and a sufficiently small constant* $\epsilon_* > 0$ *such that for all* $\epsilon \in (0, \epsilon_*)$*, if*

$$\| f(\theta, r, \xi) \|_{\mathcal{D}(\rho,s,\sigma)} + \| g(\theta, r, \xi) \|_{\mathcal{D}(\rho,s,\sigma)} \le \epsilon, \tag{1.3}$$

the mapping $\mathscr{F}$ *has a Whitney smooth family of invariant tori for* $(r, \xi) \in D_\delta \subset D \times G$*. And the measure estimate holds:*

$$\text{meas}((D \times G)\backslash D_\delta) = \mathcal{O}(\delta^{\frac{1}{K}}).$$

The case $m = 0$ in Theorem 1.1 can be regarded as a parameterized Herman's theorem under Rüssmann type nondegenerate condition. See the corollary as follows.

Corollary 1.1. *Consider the analytic mapping* $\mathscr{F}'$ *defined by*

$$\theta^1 = \theta + \omega(\xi) + f(\theta, \xi), \tag{1.4}$$

where $\theta \in T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, $\xi \in G \subset \mathbb{R}^p$ *is a parameter and* G *is a bounded and closed domain. Assume* $(A1)$, $(A3)$ *and* $(A2)'$ *hold, where*
$(A2)'$ *there exists a positive integer* K *such that*

$$\text{rank}\{\frac{\partial^\alpha \omega}{\partial \xi^\alpha}:\ 0 \le |\alpha| \le K-1\} = n.$$

Then there exist a complex domain $\mathcal{D}(\rho,\sigma) = \{(\theta,\xi) \in \mathbb{T}^n \times \mathbb{G}:\ |\text{Im}\ \theta| \le \rho,\ |\text{Im}\ \xi| \le \sigma\}$*, a nonempty Cantor set* $G_\delta \subset G$*, and a sufficiently small constant* $\epsilon_* > 0$ *such that for all* $\epsilon \in (0,\epsilon_*)$*, if*

$$\| f(\theta,\xi) \|_{\mathcal{D}(\rho,\sigma)} \le \epsilon,$$

then mapping $\mathcal{F}'$ *has a Whitney smooth family of invariant tori. And the measure estimate holds:*

$$\text{meas}(G \backslash G_\delta) = \mathcal{O}(\delta^{\frac{1}{K}}).$$

The rest of this paper is organized as follows. Section 2 gives general analytic KAM steps for twist mappings with intersection property; section 3 draws up the proof of Theorem 1.1. In Appendix, we introduce the method of measure estimates mentioned in [15].

2. KAM Steps

In this section, mainly, we consider the KAM steps for mapping (1.1). The symbol $'\cdot'$ represents the abbreviation of constants appearing in estimates for convenience that are independent of the iterative process.

2.1. Description of the 0-th KAM step

We first define the following parameters that appear in the 0-th KAM step.

$$s_0 = \epsilon_0^{\frac{1}{2K}}, \qquad \delta_0 = \epsilon_0^{\frac{a_0}{4K}}, \qquad \mu_0 = \epsilon_0^{\frac{1}{2}-\frac{a_0}{4}},$$
$$\epsilon_0 = \epsilon = s_0^K \delta_0^K \mu_0, \qquad \rho_0 = \rho, \qquad \sigma_0 = \epsilon_0^{\frac{1}{2K}},$$
$$\mathcal{D}(\rho_0, s_0, \sigma_0) = \{(\theta_0, r_0, \xi):\ |\text{Im}\ \theta_0| \le \rho_0,\ |\text{Im}\ r_0| \le s_0,\ |\text{Im}\ \xi| \le \sigma_0\},$$

where $0 < a_0 < 2$. We also define $\rho_\nu = (\frac{1}{2} + \frac{1}{2^{\nu+1}})\rho_0$, $s_\nu = \frac{1}{2^{7\nu+2}} s_0$, $\sigma_\nu = \frac{1}{2^{7\nu+2}}\sigma_0$, $(\nu = 1, 2, \cdots)$. And χ is an integer that satisfies $2 < (\frac{4}{3})^\chi < 4$, for instance, $\chi = 3$ or 4.

The mapping at 0-th step is

$$\mathscr{F}_0: \begin{cases} \theta_0^1 = \theta_0 + \omega_0(r_0,\xi) + f_0, \\ r_0^1 = r_0 + g_0, \end{cases} \tag{2.1}$$

where $(\theta_0, r_0, \xi) \in \mathcal{D}(\rho_0, s_0, \sigma_0)$, $\omega_0 = \omega$ is the original frequency vector, and $f_0 = f$, $g_0 = g$ are original perturbations.

From (1.3) we have

$$\| f_0(\theta_1, r_1, \xi) \|_{\mathcal{D}(\rho_0,s_0,\sigma_0)} + \| g_0(\theta_1, r_1, \xi) \|_{\mathcal{D}(\rho_0,s_0,\sigma_0)} \le s_0^K \delta_0^K \mu_0.$$

For $0 \le \kappa, j, h \le K$, $\kappa+j+h=K$, denote $\partial^K(\cdot)=\partial_\theta^\kappa\partial_r^j\partial_\xi^h(\cdot)$ throughout this paper. Then taking use of Cauchy estimate on $\hat{\mathcal{D}}(\rho_1,s_1,\sigma_1)=\mathcal{D}(\rho_1+\frac{3}{4}(\rho_0-\rho_1),s_1,\sigma_1)$, we get

$$\begin{aligned}
&\| \partial^K f_0(\theta_1,r_1,\xi) \|_{\hat{\mathcal{D}}(\rho_1,s_1,\sigma_1)} + \| \partial^K g_0(\theta_1,r_1,\xi) \|_{\hat{\mathcal{D}}(\rho_1,s_1,\sigma_1)}\\
&= \| \partial_{\theta_1}^\kappa\partial_{r_1}^j\partial_\xi^h f_0(\theta_1,r_1,\xi) \|_{\hat{\mathcal{D}}(\rho_1,s_1,\sigma_1)} + \| \partial_{\theta_1}^\kappa\partial_{r_1}^j\partial_\xi^h g_0(\theta_1,r_1,\xi) \|_{\hat{\mathcal{D}}(\rho_1,s_1,\sigma_1)}\\
&\le \frac{4^\kappa \kappa! j! h! (\| f_0(\theta_1,r_1,\xi) \|_{\mathcal{D}(\rho_0,s_0,\sigma_0)} + \| g_0(\theta_1,r_1,\xi) \|_{\mathcal{D}(\rho_0,s_0,\sigma_0)})}{(\rho_0-\rho_1)^\kappa (s_0-s_1)^j(\sigma_0-\sigma_1)^h}\\
&\le \cdot \frac{s_0^{K-j}\delta_0^K\mu_0}{(\rho_0-\rho_1)^\kappa(\sigma_0-\sigma_1)^h}.
\end{aligned}$$

2.2. Description of the ν-th KAM step

In this subsection, we will show one cycle of KAM steps from ν-th step to $\nu+1$. For any $\nu = 1,2,\cdots$, set

$$\begin{aligned}
&\epsilon_\nu = s_\nu^K\delta_\nu^K\mu_\nu, \qquad s_\nu=\frac{1}{2^{7\nu+2}}s_0, \qquad \delta_\nu=(\frac{1}{2}+\frac{1}{2^{\nu+1}})\delta_0, \qquad \mu_\nu=\mu_{\nu-1}^{\frac{4}{3}},\\
&\rho_\nu=(\frac{1}{2}+\frac{1}{2^{\nu+1}})\rho_0, \qquad \sigma_\nu=\frac{1}{2^{7\nu+2}}\sigma_0, \qquad N_{\nu+1}=([\log\frac{1}{\mu_\nu}]+1)^{3\chi},\\
&\mathcal{D}_j=\mathcal{D}(\rho_{\nu+1}+\frac{j-1}{4}(\rho_\nu-\rho_{\nu+1}), js_{\nu+1}, j\sigma_{\nu+1}), \quad j=1,2,3,4,\\
&\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})=\mathcal{D}(\rho_{\nu+1}+\frac{3}{4}(\rho_\nu-\rho_{\nu+1}),s_{\nu+1},\sigma_{\nu+1}),\\
&\mathcal{D}_1=\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1}).
\end{aligned}$$

Suppose that after ν KAM steps, we have arrived at the following mapping

$$\mathscr{F}_\nu : \begin{cases} \theta_\nu^1 = \theta_\nu + \omega_\nu(r_\nu,\xi) + f_\nu(\theta_\nu,r_\nu,\xi),\\ r_\nu^1 = r_\nu + g_\nu(\theta_\nu,r_\nu,\xi), \end{cases} \tag{2.2}$$

where $(\theta_\nu,r_\nu,\xi)\in\mathcal{D}(\rho_\nu,s_\nu,\sigma_\nu)=\{(\theta_\nu,r_\nu,\xi): |\mathrm{Im}\ \theta_\nu|\le\rho_\nu,\ |\mathrm{Im}\ r_\nu|\le s_\nu,\ |\mathrm{Im}\ \xi|\le\sigma_\nu\}$. Introduce a transformation $\mathscr{U}_{\nu+1}$:

$$\begin{cases} \theta_\nu = \theta_{\nu+1} + U_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi),\\ r_\nu = r_{\nu+1} + V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi), \end{cases} \tag{2.3}$$

which satisfies

$$\mathscr{U}_{\nu+1}\circ\mathscr{F}_{\nu+1}=\mathscr{F}_\nu\circ\mathscr{U}_{\nu+1}. \tag{2.4}$$

From (2.4) we have

$$\begin{aligned}
&\theta_{\nu+1}+\omega_{\nu+1}(r_{\nu+1},\xi)+f_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)\\
&+U_{\nu+1}(\theta_{\nu+1}+\omega_{\nu+1}(r_{\nu+1},\xi)+f_{\nu+1},r_{\nu+1}+g_{\nu+1},\xi)\\
&=\theta_\nu+\omega_\nu(r_\nu,\xi)+f_\nu(\theta_\nu,r_\nu,\xi)\\
&=\theta_{\nu+1}+U_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)+\omega_\nu(r_\nu,\xi)+f_\nu(\theta_{\nu+1}+U_{\nu+1},r_{\nu+1}+V_{\nu+1},\xi),\\
&r_{\nu+1}+g_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)+V_{\nu+1}(\theta_{\nu+1}+\omega_{\nu+1}(r_{\nu+1},\xi)+f_{\nu+1},r_{\nu+1}+g_{\nu+1},\xi)\\
&=r_\nu+g_\nu(\theta_\nu,r_\nu,\xi)
\end{aligned}$$

$$= r_{\nu+1} + V_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) + g_\nu(\theta_{\nu+1} + U_{\nu+1}, r_{\nu+1} + V_{\nu+1}, \xi),$$

i.e.

$$\begin{aligned}
&\omega_{\nu+1}(r_{\nu+1}, \xi) + f_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi)\\
&+U_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1} + f_{\nu+1}, r_{\nu+1} + g_{\nu+1}, \xi)\\
&= U_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) + \omega_\nu(r_\nu, \xi) + f_\nu(\theta_{\nu+1} + U_{\nu+1}, r_{\nu+1} + V_{\nu+1}, \xi), \quad (2.5)\\
&g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) + V_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1} + f_{\nu+1}, r_{\nu+1} + g_{\nu+1}, \xi)\\
&= V_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) + g_\nu(\theta_{\nu+1} + U_{\nu+1}, r_{\nu+1} + V_{\nu+1}, \xi). \quad (2.6)
\end{aligned}$$

2.2.1. Estimates of the remainders

Consider the Fourier series expansion of $f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi)$,

$$f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) = \sum_{k\in\mathbb{Z}^n} f_{k,\nu}(r_{\nu+1}, \xi) e^{i\langle k, \theta_{\nu+1}\rangle}. \quad (2.7)$$

The operators of truncation and remainder are respectively

$$\Gamma_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) = \sum_{0<|k|\le N_{\nu+1}} f_{k,\nu}(r_{\nu+1}, \xi) e^{i\langle k, \theta_{\nu+1}\rangle}, \quad (2.8)$$

$$R_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) = \sum_{|k|> N_{\nu+1}} f_{k,\nu}(r_{\nu+1}, \xi) e^{i\langle k, \theta_{\nu+1}\rangle}, \quad (2.9)$$

where $f_{k,\nu}(r_{\nu+1}, \xi) = \int_{\mathbb{T}^n} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) e^{-i\langle k, \theta_{\nu+1}\rangle}\, d\theta_{\nu+1}$ is the Fourier coefficient of $f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi)$.

Lemma 2.1. *If*

$$(H1) \qquad \int_{N_{\nu+1}}^{+\infty} l^n e^{-l\frac{\rho_\nu - \rho_{\nu+1}}{4}}\, dl \le \mu_\nu, \quad (2.10)$$

then

$$\| R_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_3} \le s_\nu^K \delta_\nu^K \mu_\nu^2. \quad (2.11)$$

Proof. By a direct computation, we have

$$\begin{aligned}
&\| R_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_3}\\
&\le \Big| \sum_{|k|>N_{\nu+1}} f_{k,\nu}(r_{\nu+1}, \xi) e^{i\langle k, \theta_{\nu+1}\rangle} \Big|\\
&\le \sum_{|k|>N_{\nu+1}} |f_{k,\nu}(r_{\nu+1}, \xi) e^{i\langle k, \theta_{\nu+1}\rangle}|\\
&\le \sum_{|k|>N_{\nu+1}} |f_{k,\nu}(r_{\nu+1}, \xi)|\, |e^{i\langle k, \theta_{\nu+1}\rangle}|\\
&\le \| f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_4} \sum_{|k|>N_{\nu+1}} e^{-|k|(\rho_{\nu+1} + \frac{3}{4}(\rho_\nu - \rho_{\nu+1}))} e^{|k|(\rho_{\nu+1} + \frac{1}{2}(\rho_\nu - \rho_{\nu+1}))}\\
&\le \| f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_4} \sum_{|k|>N_{\nu+1}} e^{-|k|\frac{\rho_\nu - \rho_{\nu+1}}{4}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \| f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_4} \sum_{|k|>N_{\nu+1}} |k|^n e^{-|k|\frac{\rho_\nu-\rho_{\nu+1}}{4}} \\
&\leq \| f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_4} \int_{N_{\nu+1}}^{+\infty} l^n e^{-l\frac{\rho_\nu-\rho_{\nu+1}}{4}}\, dl \\
&\leq s_\nu^K \delta_\nu^K \mu_\nu^2.
\end{aligned}
$$

□

Remark 2.1. On the third inequality, we notice that the coefficients $f_{k,\nu}(r_{\nu+1}, \xi)$ decay exponentially, that is,

$$
|f_{k,\nu}(r_{\nu+1}, \xi)| \leq \| f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_4} e^{-|k|(\rho_{\nu+1}+\frac{3}{4}(\rho_\nu-\rho_{\nu+1}))}. \tag{2.12}
$$

From Cauchy estimate we get

$$
\begin{aligned}
&\| \partial^K R_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\
&= \| \partial^\kappa_{\theta_{\nu+1}} \partial^j_{r_{\nu+1}} \partial^h_\xi R_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\
&\leq \cdot \frac{\| R_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu)}}{(\frac{\rho_\nu-\rho_{\nu+1}}{4})^\kappa (s_\nu - s_{\nu+1})^j (\sigma_\nu - \sigma_{\nu+1})^h} \\
&\leq \cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h s_{\nu+1}^j} \\
&\leq \cdot \frac{s_\nu^{K-j} \delta_\nu^K \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h}.
\end{aligned} \tag{2.13}
$$

Remark 2.2. Note

$$
\int_{N_{\nu+1}}^{+\infty} l^n e^{-l\frac{\rho_\nu-\rho_{\nu+1}}{4}}\, dl = \sum_{i=0}^{n} \frac{4^{i+1} \overbrace{n(n-1)\cdots(n-i+1)}^{i}}{(\rho_\nu - \rho_{\nu+1})^{i+1}} N_{\nu+1}^{n-i} e^{-N_{\nu+1}\frac{\rho_\nu-\rho_{\nu+1}}{4}}.
$$

Then in order to get

$$
\int_{N_{\nu+1}}^{+\infty} l^n e^{-l\frac{\rho_\nu-\rho_{\nu+1}}{4}}\, dl < \mu_\nu,
$$

it suffices to hold

$$
\frac{4^{n+1}(n+1)!}{(\rho_\nu - \rho_{\nu+1})^{n+1}} N_{\nu+1}^n e^{-N_{\nu+1}\frac{\rho_\nu-\rho_{\nu+1}}{4}} < \mu_\nu,
$$

i.e.

$$
\begin{aligned}
&(n+1)\log 4 + \log(n+1)! + (\nu+2)(n+1)\log 2 - (n+1)\log\rho_0 \\
&+3\chi n \log([\log\frac{1}{\mu_\nu}] + 1) - \frac{1}{2^{\nu+4}}\rho_0([\log\frac{1}{\mu_\nu}] + 1)^{3\chi} \\
&< \log\mu_\nu.
\end{aligned}
$$

Since $2 < (\frac{4}{3})^\chi < 4$, we get

$$
\frac{\rho_0}{2^{\nu+4}}([\log\frac{1}{\mu_\nu}] + 1)^\chi \geq \frac{\rho_0}{2^{\nu+4}}\left((\frac{4}{3})^\nu [\log\frac{1}{\mu_0}]\right)^\chi
$$

$$\begin{aligned}
&\geq \frac{\rho_0}{2^{\nu+4}}(\frac{4}{3})^{\chi\nu}[\log\frac{1}{\mu_0}]^{\chi}\\
&\geq -\frac{\rho_0}{2^{\nu+4}}(\frac{4}{3})^{\chi\nu}[\log\mu_0]^{\chi}\\
&\geq -\frac{\rho_0}{2^{4}}[\log\mu_0]^{\chi}\\
&\geq 1.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(n+1)\log 4+\log(n+1)!-(n+1)\log\rho_0+(n+1)(\nu+2)\log 2\\
&+3n\chi\log([\log\frac{1}{\mu_\nu}]+1)-\frac{\rho_0}{2^{\nu+4}}([\log\frac{1}{\mu_\nu}]+1)^{3\chi}\\
\leq\ &(n+1)\log 4+\log(n+1)!-(n+1)\log\rho_0+(n+1)(\nu+2)\log 2\\
&+3n\chi\log([\log\frac{1}{\mu_\nu}]+1)-[\log\frac{1}{\mu_\nu}]^{2\chi}\\
\leq\ &-\log\frac{1}{\mu_\nu}.
\end{aligned}$$

2.2.2. Homological equations

Before considering the homological equations in KAM steps, it is necessary to take time to consider the Diophantine condition in ν-th step to make sure that $\mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu)$ is well defined.

Lemma 2.2. *There exist positive constants M_0, M_1, and M_2 such that $|\omega_\nu|$, $|\frac{\partial\omega_\nu}{\partial r_\nu}|$, and $|\frac{\partial\omega_\nu}{\partial\xi}|$ are bounded by them on $\mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu)$ respectively. If*

$$(H2)\qquad (M_1+M_2)([log\frac{1}{\mu_0}]+1)^{3\chi}\delta_0^{\frac{2}{a_0}-1}\leq\frac{1}{|k|^\tau},$$

the frequency ω_ν satisfies

$$|\langle k,\omega_\nu(r_\nu,\xi)\rangle-k_0|\geq\frac{\delta_\nu}{2|k|^\tau},\qquad\forall\ |k|\in(0,N_{\nu+1}],\qquad\forall\ k_0\in\mathbb{Z}.\tag{2.14}$$

Proof. Due to

$$\omega_\nu(r_\nu,\xi)=\omega_0(r_0,\xi)+\sum_{l=0}^{\nu-1}\bar{f}_l(r_{l+1},\xi),\tag{2.15}$$

the $\bar{f}_l(r_{l+1},\xi)$ above is the average of $f_l(\theta_{l+1},r_{l+1},\xi)$ and satisfies

$$\begin{aligned}
|\bar{f}_l(r_{l+1},\xi)|&=\frac{1}{(2\pi)^n}|\int_{\mathbb{T}^n}f_l(\theta_{l+1},r_{l+1},\xi)\,d\theta_{l+1}|\\
&\leq\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\parallel f_l(\theta_{l+1},r_{l+1},\xi)\parallel_{\mathcal{D}(\rho_l,s_l,\sigma_l)}\ d\theta_{l+1}\\
&\leq s_l^K\delta_l^K\mu_l,
\end{aligned}$$

and then ω_ν is well defined in its domain with

$$|\omega_\nu|\leq|\omega_0|+\sum_{l=0}^{\nu-1}|\bar{f}_l(\theta_{l+1},r_{l+1},\xi)|:=M_0.$$

Therefore, the bounds of $|\frac{\partial\omega_\nu}{\partial r_\nu}|$ and $|\frac{\partial\omega_\nu}{\partial\xi}|$ are respectively,

$$
\begin{aligned}
|\frac{\partial\omega_\nu}{\partial r_\nu}| &\le |\frac{\partial\omega_0}{\partial r_\nu}| + \sum_{l=0}^{\nu-1} |\frac{\partial \bar{f}_l(r_l,\xi)}{\partial r_\nu}| \\
&\le |\frac{\partial\omega_0}{\partial r_\nu}| + \sum_{l=0}^{\nu-1} \frac{s_l^K \delta_l^K \mu_l}{s_l - s_{l+1}} \\
&:= M_1, \\
|\frac{\partial\omega_\nu}{\partial\xi}| &\le |\frac{\partial\omega_0}{\partial\xi}| + \sum_{l=0}^{\nu-1} |\frac{\partial \bar{f}_l(r_l,\xi)}{\partial\xi}| \\
&\le |\frac{\partial\omega_0}{\partial\xi}| + \sum_{l=0}^{\nu-1} \frac{s_l^K \delta_l^K \mu_l}{\sigma_l - \sigma_{l+1}} \\
&:= M_2.
\end{aligned}
$$

We see that

$$
\begin{aligned}
N_{\nu+1} &= ([\log\frac{1}{\mu_\nu}] + 1)^{3\chi} \\
&= \left((\frac{4}{3})^\nu [\log\frac{1}{\mu_0}] + 1 \right)^{3\chi} \\
&\le (\frac{4}{3})^{3\chi\nu} ([\log\frac{1}{\mu_0}] + 1)^{3\chi} \\
&\le 2^{6\nu} ([\log\frac{1}{\mu_0}] + 1)^{3\chi}.
\end{aligned}
$$

Moreover, for $(\theta_{\nu+1}, r_{\nu+1}, \xi)$ and $(\theta'_{\nu+1}, r'_{\nu+1}, \xi')$ belonging to $\mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu)$, we have

$$
\begin{aligned}
& |\langle k, \omega_\nu(r_\nu,\xi) - \omega_\nu(r'_\nu,\xi')\rangle| \\
\le\ & |\langle k, \omega_\nu(r_\nu,\xi) - \omega_\nu(r'_\nu,\xi) + \omega_\nu(r'_\nu,\xi) - \omega_\nu(r'_\nu,\xi')\rangle| \\
\le\ & |\langle k, \omega_\nu(r_\nu,\xi) - \omega_\nu(r'_\nu,\xi)\rangle| + |\langle k, \omega_\nu(r'_\nu,\xi) - \omega_\nu(r'_\nu,\xi')\rangle| \\
\le\ & |k|\ |\omega_\nu(r_\nu,\xi) - \omega_\nu(r'_\nu,\xi)| + |k|\ |\omega_\nu(r'_\nu,\xi) - \omega_\nu(r'_\nu,\xi')| \\
\le\ & |k|\ |\frac{\partial\omega_\nu}{\partial r_\nu}|\ |r_\nu - r'_\nu| + |k|\ |\frac{\partial\omega_\nu}{\partial\xi}|\ |\xi - \xi'| \\
\le\ & M_1 s_\nu |k| + M_2 \sigma_\nu |k| \\
\le\ & N_{\nu+1}(M_1 s_\nu + M_2 \sigma_\nu) \\
\le\ & 2^{6\nu}([\log\frac{1}{\mu_0}] + 1)^{3\chi} \frac{1}{2^{7\nu+2}}(M_1 s_0 + M_2 \sigma_0) \\
=\ & \frac{1}{2^{\nu+2}}([\log\frac{1}{\mu_0}] + 1)^{3\chi}(M_1 \epsilon_0^{\frac{a_0}{4K}\frac{2}{a_0}} + M_2 \epsilon_0^{\frac{a_0}{4K}\frac{2}{a_0}}) \\
=\ & \frac{\delta_0}{2^{\nu+2}}([\log\frac{1}{\mu_0}] + 1)^{3\chi}(M_1 + M_2)\delta_0^{\frac{2}{a_0}-1} \\
\le\ & (\frac{1}{2^2} + \frac{1}{2^{\nu+2}})\delta_0([\log\frac{1}{\mu_0}] + 1)^{3\chi}(M_1 + M_2)\delta_0^{\frac{2}{a_0}-1} \\
\le\ & \frac{\delta_\nu}{2|k|^\tau}.
\end{aligned}
$$

From the Diophantine condition in ν-th step, one has

$$\begin{aligned}|\langle k,\omega_\nu(r_\nu,\xi)\rangle-k_0| &= |\langle k,\omega_\nu(r'_\nu,\xi')+\omega_\nu(r_\nu,\xi)-\omega_\nu(r'_\nu,\xi')\rangle-k_0|\\ &\ge |\langle k,\omega_\nu(r'_\nu,\xi')\rangle-k_0|-|\langle k,\omega_\nu(r_\nu,\xi)-\omega_\nu(r'_\nu,\xi')\rangle|\\ &\ge \frac{\delta_\nu}{|k|^\tau}-\frac{\delta_\nu}{2|k|^\tau}\\ &\ge \frac{\delta_\nu}{2|k|^\tau}.\end{aligned}$$

□

From (2.5), (2.6), and (2.15) we now get homological equations

$$\begin{cases}U_{\nu+1}(\theta_{\nu+1}+\omega_{\nu+1},r_{\nu+1},\xi)-U_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)=\Gamma_{N_{\nu+1}}f_\nu(\theta_{\nu+1},r_{\nu+1},\xi),\\ V_{\nu+1}(\theta_{\nu+1}+\omega_{\nu+1},r_{\nu+1},\xi)-V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)=\Gamma_{N_{\nu+1}}g_\nu(\theta_{\nu+1},r_{\nu+1},\xi).\end{cases} \tag{2.16}$$

Then the following lemma yields the solutions of (2.16) and their estimates.

Lemma 2.3. *Assume that*

$$(H3)\qquad s_\nu^K\delta_\nu^{K-1}\mu_\nu\le\frac{1}{4N_{\nu+1}^{\tau+1}}.$$

Then the second equation of (2.16) *has a unique analytic solution* $V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)$ *of zero average on* $\mathcal{D}(\rho_\nu,s_\nu,\sigma_\nu)$ *and the estimate*

$$\parallel V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)\parallel_{\mathcal{D}_3}\le\cdot\frac{\parallel g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)\parallel_{\mathcal{D}_4}}{\delta_\nu(\rho_\nu-\rho_{\nu+1})^{\tau+n+1}} \tag{2.17}$$

holds.

Proof. The Fourier expansion of $V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)$ is

$$V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)=\sum_{k\in\mathbb{Z}^n}V_{k,\nu+1}(r_{\nu+1},\xi)e^{i\langle k,\theta_{\nu+1}\rangle},$$

and the truncation of $V_{\nu+1}$ is still represented by $V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)$, that is,

$$V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)=\sum_{0<|k|\le N_{\nu+1}}V_{k,\nu+1}(r_{\nu+1},\xi)e^{i\langle k,\theta_{\nu+1}\rangle}.$$

Let the Fourier expansion, truncation, and remainder of $g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)$ be respectively

$$\begin{aligned}g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)&=\sum_{k\in\mathbb{Z}^n}g_{k,\nu}(r_{\nu+1},\xi)e^{i\langle k,\theta_{\nu+1}\rangle},\\ \Gamma_{N_{\nu+1}}g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)&=\sum_{0<|k|\le N_{\nu+1}}g_{k,\nu}(r_{\nu+1},\xi)e^{i\langle k,\theta_{\nu+1}\rangle},\\ R_{N_{\nu+1}}g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)&=\sum_{|k|>N_{\nu+1}}g_{k,\nu}(r_{\nu+1},\xi)e^{i\langle k,\theta_{\nu+1}\rangle},\end{aligned}$$

where $g_{k,\nu}(r_{\nu+1},\xi)=\int_{\mathbb{T}^n}g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)e^{-i\langle k,\theta_{\nu+1}\rangle}\,d\theta_{\nu+1}$ is the Fourier coefficient of $g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)$.

Putting them into the second equation of (2.16), we get

$$\sum_{0<|k|\le N_{\nu+1}} V_{k,\nu+1}e^{i\langle k,\theta_{\nu+1}+\omega_{\nu+1}\rangle} - \sum_{0<|k|\le N_{\nu+1}} V_{k,\nu+1}e^{i\langle k,\theta_{\nu+1}\rangle} = \sum_{0<|k|\le N_{\nu+1}} g_{k,\nu}e^{i\langle k,\theta_{\nu+1}\rangle},$$

i.e.

$$\sum_{0<|k|\le N_{\nu+1}} V_{k,\nu+1}e^{i\langle k,\theta_{\nu+1}\rangle}(e^{i\langle k,\omega_{\nu+1}\rangle}-1) = \sum_{0<|k|\le N_{\nu+1}} g_{k,\nu}e^{i\langle k,\theta_{\nu+1}\rangle},$$

that is

$$V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi) = \sum_{0<|k|\le N_{\nu+1}} \frac{g_{k,\nu}(r_{\nu+1},\xi)e^{i\langle k,\theta_{\nu+1}\rangle}}{e^{i\langle k,\omega_{\nu+1}(r_{\nu+1},\xi)\rangle}-1}.$$

Then

$$\begin{aligned}
&\| V_{\nu+1}(\theta_{\nu+1},r_{\nu+1},\xi)\|_{\mathcal{D}_3}\\
\le &\sum_{0<|k|\le N_{\nu+1}} \frac{|g_{k,\nu}(r_{\nu+1},\xi)|}{|e^{i\langle k,\omega_{\nu+1}(r_{\nu+1},\xi)\rangle}-1|}|e^{i\langle k,\theta_{\nu+1}\rangle}|\\
\le &\sum_{0<|k|\le N_{\nu+1}} \frac{|g_{k,\nu}(r_{\nu+1},\xi)|}{|e^{i\langle k,\omega_{\nu+1}(r_{\nu+1},\xi)\rangle}-1|}e^{|k|(\rho_{\nu+1}+\frac{1}{2}(\rho_\nu-\rho_{\nu+1}))}\\
\le &\frac{2\pi}{\delta_\nu}\| g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)\|_{\mathcal{D}_4}\sum_{0<|k|\le N_{\nu+1}} |k|^\tau e^{-|k|\frac{\rho_\nu-\rho_{\nu+1}}{4}}\\
\le &\frac{2\pi}{\delta_\nu}\| g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)\|_{\mathcal{D}_4}\int_0^{N_{\nu+1}} l^{\tau+n}e^{-l\frac{\rho_\nu-\rho_{\nu+1}}{4}}\,dl\\
\le &\frac{2\pi}{\delta_\nu}\| g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)\|_{\mathcal{D}_4}\int_0^{\infty} (\frac{4}{\rho_\nu-\rho_{\nu+1}})^{\tau+n}x^{\tau+n}e^{-x}\,d(\frac{4x}{\rho_\nu-\rho_{\nu+1}})\\
\le &\frac{2\pi}{\delta_\nu}\| g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)\|_{\mathcal{D}_4}\frac{4^{\tau+n+1}}{(\rho_\nu-\rho_{\nu+1})^{\tau+n+1}}\int_0^\infty x^{\tau+n}e^{-x}\,dx\\
= &\frac{2\pi}{\delta_\nu}\| g_\nu(\theta_{\nu+1},r_{\nu+1},\xi)\|_{\mathcal{D}_4}\frac{4^{\tau+n+1}}{(\rho_\nu-\rho_{\nu+1})^{\tau+n+1}}\Gamma(\tau+n+1)\\
\le &\frac{4^{\tau+n+2}\pi(\tau+n)!}{\delta_\nu(\rho_\nu-\rho_{\nu+1})^{\tau+n+1}}\| g_\nu(\theta_{\nu+1+},r_{\nu+1},\xi)\|_{\mathcal{D}_4}.
\end{aligned}$$

□

Remark 2.3. In order to estimate $|e^{i\langle k,\omega_{\nu+1}(r_{\nu+1},\xi)\rangle}-1|$, we first consider $|xe^{i\phi}-y|$ for $x,y\in\mathbb{N}$ and $\phi\in\mathbb{R}$.

$$\begin{aligned}
|xe^{i\phi}-y|^2 &= |x\cos\phi+ix\sin\phi-y|^2\\
&= |x\cos\phi-y|^2+x^2\sin^2\phi\\
&= x^2\cos^2\phi+y^2-2xy\cos\phi+x^2\sin^2\phi\\
&= x^2+y^2-2xy\cos\phi\\
&= x^2\cos^2\frac{\phi}{2}+x^2\sin^2\frac{\phi}{2}+y^2\cos^2\frac{\phi}{2}+y^2\sin^2\frac{\phi}{2}\\
&\quad -2xy\cos^2\frac{\phi}{2}+2xy\sin^2\frac{\phi}{2}\\
&= |x-y|^2\cos^2\frac{\phi}{2}+|x+y|^2\sin^2\frac{\phi}{2}
\end{aligned}$$

$$\geq |x+y|^2 \sin^2 \frac{\phi}{2}$$
$$= |x+y|^2 \sin^2 \frac{\phi - 2\pi l}{2}, \quad (l \in \mathbb{Z}).$$

Therefore,

$$|xe^{i\phi} - y| \geq |x+y| \sin \frac{\phi - 2\pi l}{2}.$$

Since $|\sin\varphi| \geq \frac{2}{\pi}|\varphi|$, if $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, taking $x, y = 1$, $\phi = \langle k, \omega_{\nu+1}(r_{\nu+1}, \xi)\rangle$, $2\pi l = k_0$, we obtain

$$|e^{i\langle k, \omega_{\nu+1}(r_{\nu+1},\xi)\rangle} - 1| \geq 2|\sin \frac{\langle k, \omega_{\nu+1}(r_{\nu+1}, \xi)\rangle - k_0}{2}|$$
$$\geq \frac{4}{\pi}|\frac{\langle k, \omega_{\nu+1}(r_{\nu+1}, \xi)\rangle - k_0}{2}|,$$

where $k_0 \in \mathbb{Z}$ satisfies $|\frac{\langle k,\omega_{\nu+1}(r_{\nu+1},\xi)\rangle - k_0}{2}| \leq \frac{\pi}{2}$.

From Lemma 2.2 we have $|\langle k, \omega_\nu(r_\nu, \xi)\rangle - k_0| \geq \frac{\delta_\nu}{2|k|^\tau}$. Also $(H3)$ implies that

$$|\langle k, \bar{f}_\nu(r_\nu, \xi)\rangle| \leq |k|\ |\bar{f}_\nu(r_\nu, \xi)| \leq N_{\nu+1} s_\nu^K \delta_\nu^K \mu_\nu$$
$$\leq N_{\nu+1} \frac{\delta_\nu}{4N_{\nu+1}^{\tau+1}} \leq \frac{\delta_\nu}{4N_{\nu+1}^\tau} \leq \frac{\delta_\nu}{4|k|^\tau}.$$

Then

$$|\langle k, \omega_{\nu+1}(r_{\nu+1}, \xi)\rangle - k_0| = |\langle k, \omega_\nu(r_\nu, \xi) + \bar{f}_\nu(r_\nu, \xi)\rangle - k_0|$$
$$= |\langle k, \omega_\nu(r_\nu, \xi)\rangle + \langle k, \bar{f}_\nu(r_{\nu+1}, \xi)\rangle - k_0|$$
$$\geq |\langle k, \omega_\nu(r_\nu, \xi)\rangle - k_0| - |\langle k, \bar{f}_\nu(r_{\nu+1}, \xi)\rangle|$$
$$\geq \frac{\delta_\nu}{2|k|^\tau} - \frac{\delta_\nu}{4|k|^\tau}$$
$$\geq \frac{\delta_\nu}{4|k|^\tau},$$

and therefore,

$$|e^{i\langle k, \omega_{\nu+1}(r_{\nu+1},\xi)\rangle} - 1| \geq \frac{\delta_\nu}{2\pi|k|^\tau}.$$

Remark 2.4. Note that $\int_0^{+\infty} x^{\tau+n} e^{-x}\, dx$ is Γ function and $\Gamma(\tau+n+1) = (\tau+n)!$. Then

$$\sum_{0<|k|\leq N_{\nu+1}} |k|^\tau e^{-|k|\frac{\rho_\nu - \rho_{\nu+1}}{4}} \leq \int_0^{N_{\nu+1}} l^{\tau+n} e^{-l\frac{\rho_\nu-\rho_{\nu+1}}{4}}\, dl$$
$$\leq \int_0^{+\infty} (\frac{4}{\rho_\nu - \rho_{\nu+1}})^{\tau+n} x^{\tau+n} e^{-x}\, d(\frac{4}{\rho_\nu - \rho_{\nu+1}} x)$$
$$\leq \frac{4^{\tau+n+1}}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} \int_0^{+\infty} x^{\tau+n} e^{-x}\, dx$$
$$= \frac{4^{\tau+n+1}}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} \Gamma(\tau+n+1)$$
$$= \frac{4^{\tau+n+1}}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} (\tau+n)!.$$

From Cauchy inequality we get

$$\begin{aligned}
&\| \partial^K V_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\
&= \| \partial^{\kappa}_{\theta_{\nu+1}} \partial^{j}_{r_{\nu+1}} \partial^{h}_{\xi} V_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\
&\leq \cdot \frac{\| V_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu)}}{(\frac{\rho_\nu - \rho_{\nu+1}}{4})^{\kappa} (s_\nu - s_{\nu+1})^j (\sigma_\nu - \sigma_{\nu+1})^h} \\
&\leq \cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1+\kappa} (\sigma_\nu - \sigma_{\nu+1})^h \delta_\nu s_{\nu+1}^j} \\
&\leq \cdot \frac{s_\nu^{K-j} \delta_\nu^{K-1} \mu_\nu}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1+\kappa} (\sigma_\nu - \sigma_{\nu+1})^h}. \qquad (2.18)
\end{aligned}$$

In an analogous manner we obtain

$$\| U_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_3} \leq \cdot \frac{\| f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_4}}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+1}}, \qquad (2.19)$$

and

$$\begin{aligned}
&\| \partial^K U_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\
&= \| \partial^{\kappa}_{\theta_{\nu+1}} \partial^{j}_{r_{\nu+1}} \partial^{h}_{\xi} U_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\
&\leq \cdot \frac{\| U_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu)}}{(\frac{\rho_\nu - \rho_{\nu+1}}{4})^{\kappa} (s_\nu - s_{\nu+1})^j (\sigma_\nu - \sigma_{\nu+1})^h} \\
&\leq \cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1+\kappa} (\sigma_\nu - \sigma_{\nu+1})^h \delta_\nu s_{\nu+1}^j} \\
&\leq \cdot \frac{s_\nu^{K-j} \delta_\nu^{K-1} \mu_\nu}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1+\kappa} (\sigma_\nu - \sigma_{\nu+1})^h}. \qquad (2.20)
\end{aligned}$$

Remark 2.5. In order to make sure that

$$\mathscr{U}_{\nu+1} : \mathcal{D}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1}) \subset \mathcal{D}_3 \to \mathcal{D}_4 \subset \mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu),$$

it requests

$$\begin{aligned}
|r_\nu - r_{\nu+1}| &= \| V_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_3} \\
&\leq 4s_{\nu+1} - 3s_{\nu+1} \\
&= s_{\nu+1}, \qquad (2.21)
\end{aligned}$$

and

$$|\theta_\nu - \theta_{\nu+1}| = \| U_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}_3} \leq \frac{\rho_\nu - \rho_{\nu+1}}{4}. \qquad (2.22)$$

That is,

$(H4)$
$$\frac{4^{\tau+n+2}\pi(\tau+n)!}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} s_\nu^K \delta_\nu^K \mu_\nu \leq s_{\nu+1},$$

and

$(H5)$
$$\frac{4^{\tau+n+2}\pi(\tau+n)!}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} s_\nu^K \delta_\nu^K \mu_\nu \leq \frac{\rho_\nu - \rho_{\nu+1}}{4}.$$

2.2.3. Estimates of new perturbations

The following lemma shows the estimates of new perturbations in one cycle of the iteration.

Lemma 2.4. *Assume*

$$(H6) \qquad 2^{7K}(1+\frac{1}{2^{\nu+1}})^K \mu_\nu^{\frac{2}{3}} \ll 1.$$

Then

$$\| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} + \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \leq s_{\nu+1}^K \delta_{\nu+1}^K \mu_{\nu+1}, \tag{2.23}$$

and

$$\begin{aligned} &\| \partial^K f_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} + \| \partial^K g_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\ \leq\ & \frac{s_{\nu+1}^{K-j}\delta_{\nu+1}^K \mu_{\nu+1}}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h} \end{aligned} \tag{2.24}$$

holds.

Proof. We have already obtained the expressions of $f_{\nu+1}$ and $g_{\nu+1}$ from (2.5) and (2.6), that is,

$$\begin{aligned} & f_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) + U_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi) + f_{\nu+1}, r_{\nu+1} + g_{\nu+1}, \xi) \\ & - U_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi), r_{\nu+1} + g_{\nu+1}, \xi) \\ & + U_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi), r_{\nu+1} + g_{\nu+1}, \xi) \\ & - U_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi), r_{\nu+1}, \xi) \\ = & f_\nu(\theta_{\nu+1} + U_{\nu+1}, r_{\nu+1} + V_{\nu+1}, \xi) - f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \\ & + R_{N_{\nu+1}} f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi), \end{aligned} \tag{2.25}$$

$$\begin{aligned} & g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) + V_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi) + f_{\nu+1}, r_{\nu+1} + g_{\nu+1}, \xi) \\ & - V_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi), r_{\nu+1} + g_{\nu+1}, \xi) \\ & + V_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi), r_{\nu+1} + g_{\nu+1}, \xi) \\ & - V_{\nu+1}(\theta_{\nu+1} + \omega_{\nu+1}(r_{\nu+1}, \xi), r_{\nu+1}, \xi) \\ = & g_\nu(\theta_{\nu+1} + U_{\nu+1}, r_{\nu+1} + V_{\nu+1}, \xi) - g_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) + \bar{g}_\nu(r_{\nu+1}, \xi) \\ & + R_{N_{\nu+1}} g_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi). \end{aligned} \tag{2.26}$$

Then $f_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi)$ and $g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi)$ can be solved by the Implicit Function Theorem. From

$$\begin{aligned} & \| \partial_{\theta_{\nu+1}} U_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\ & + \| \partial_{r_{\nu+1}} U_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\ & + \| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\ \leq & \| \partial_{\theta_{\nu+1}} f_\nu \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| U_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\ & + \| \partial_{r_{\nu+1}} f_\nu \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| V_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\ & + \| R_{N_{\nu+1}} f_\nu \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})}, \end{aligned}$$

we obtain

$$\begin{aligned}
&\| \partial_{\theta_{\nu+1}} U_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \| \partial_{r_{\nu+1}} U_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\le &\cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu}{\rho_\nu - \rho_{\nu+1}} \frac{s_\nu^K \delta_\nu^K \mu_\nu}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} + \cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu}{s_\nu - s_{\nu+1}} \frac{s_\nu^K \delta_\nu^K \mu_\nu}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} \\
&+ s_\nu^K \delta_\nu^K \mu_\nu^2 \\
\le &\cdot \frac{s_\nu^{2K} \delta_\nu^{2K-1} \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+2}} + \cdot \frac{s_\nu^{2K-1} \delta_\nu^{2K-1} \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} + s_\nu^K \delta_\nu^K \mu_\nu^2.
\end{aligned}$$

The intersection property is written as

$$\begin{aligned}
\mathscr{F}_{\nu+1} \mathcal{T}_0 \cap \mathcal{T}_0 &= \mathscr{U}_{\nu+1}^{-1} \circ \mathscr{F}_\nu \circ \mathscr{U}_{\nu+1} \mathcal{T}_0 \cap \mathcal{T}_0 \\
&= \mathscr{U}_{\nu+1}^{-1} \circ (\mathscr{F}_\nu \circ \mathscr{U}_{\nu+1} \mathcal{T}_0 \cap \mathscr{U}_{\nu+1} \mathcal{T}_0) \\
&\ne \varnothing,
\end{aligned}$$

where $\mathcal{T}_0$ is the initial torus of the mapping. As for $\| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})}$, taking use of the intersection property of twist mapping, there exists a $\theta_{\nu+1}^0$ such that $g_{\nu+1}(\theta_{\nu+1}^0, r_{\nu+1}^0, \xi) = 0$ for each $r_{\nu+1}^0$. Therefore,

$$\begin{aligned}
&\sup \| g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
= &\sup \| g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) - g_{\nu+1}(\theta_{\nu+1}^0, r_{\nu+1}^0, \xi) \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\le &\, osc(g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi)) \\
= &\, osc(g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) - h) \\
\le &\, 2 \sup \| g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) - h \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})},
\end{aligned}$$

where $osc(g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi))$ denotes the oscillation of function $g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi)$. Specially, taking $h = \bar{g}_\nu(r_{\nu+1}, \xi)$, we have

$$\begin{aligned}
&\frac{1}{2} \| g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\le &\sup \| g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) - \bar{g}_\nu(r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} .
\end{aligned}$$

Therefore, from

$$\begin{aligned}
&\| \partial_{\theta_{\nu+1}} V_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \| \partial_{r_{\nu+1}} V_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \frac{1}{2} \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\le &\| \partial_{\theta_{\nu+1}} V_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \| \partial_{r_{\nu+1}} V_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \| g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) - \bar{g}_\nu(r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\le &\| \partial_{\theta_{\nu+1}} g_\nu \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| U_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \| \partial_{r_{\nu+1}} g_\nu \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \| V_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \| R_{N_{\nu+1}} g_\nu \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})},
\end{aligned}$$

we have

$$\begin{aligned}
&2 \parallel \partial_{\theta_{\nu+1}} V_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \parallel f_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+2 \parallel \partial_{r_{\nu+1}} V_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \parallel g_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \parallel g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\leq\, &2 \parallel \partial_{\theta_{\nu+1}} V_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \parallel f_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+2 \parallel \partial_{r_{\nu+1}} V_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \parallel g_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+2 \parallel g_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi) - \bar{g}_{\nu}(r_{\nu+1}, \xi) \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\leq\, &2 \parallel \partial_{\theta_{\nu+1}} g_{\nu} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \parallel U_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+2 \parallel \partial_{r_{\nu+1}} g_{\nu} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \parallel V_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+2 \parallel R_{N_{\nu+1}} g_{\nu}(\theta_{\nu+1}, r_{\nu+1}, \xi) \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\leq\, &\cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu}{\rho_\nu - \rho_{\nu+1}} \frac{s_\nu^K \delta_\nu^K \mu_\nu}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} \\
&+ \cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu}{s_\nu - s_{\nu+1}} \frac{s_\nu^K \delta_\nu^K \mu_\nu}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} \\
&+ \cdot s_\nu^K \delta_\nu^K \mu_\nu^2 \\
\leq\, &\cdot \frac{s_\nu^{2K} \delta_\nu^{2K-1} \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+2}} + \cdot \frac{s_\nu^{2K-1} \delta_\nu^{2K-1} \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} \\
&+ \cdot s_\nu^K \delta_\nu^K \mu_\nu^2.
\end{aligned}$$

If

$$(H7) \qquad 1 + \frac{3\pi 4^{\tau+n+2} (\tau+n)!}{\delta_\nu (\rho_\nu - \rho_{\nu+1})^{\tau+n+2}} s_\nu^K \delta_\nu^K \mu_\nu \leq 2,$$

$$(H8) \qquad 1 + \frac{3\pi 4^{\tau+n+2} (\tau+n)!}{\delta_\nu (s_\nu - s_{\nu+1})(\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} s_\nu^K \delta_\nu^K \mu_\nu \leq 2,$$

the estimate of new perturbation is

$$\begin{aligned}
&\parallel f_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} + \parallel g_{\nu+1} \parallel_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
\leq\, &\cdot \frac{s_\nu^{2K} \delta_\nu^{2K-1} \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+2}} + \cdot \frac{s_\nu^{2K-1} \delta_\nu^{2K-1} \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1}} + \cdot s_\nu^K \delta_\nu^K \mu_\nu^2 \\
\leq\, &\cdot s_\nu^K \delta_\nu^K \mu_\nu^2 \\
\leq\, &\cdot s_{\nu+1}^K \delta_{\nu+1}^K \mu_{\nu+1} \left(\frac{s_\nu}{s_{\nu+1}}\right)^K \left(\frac{\delta_\nu}{\delta_{\nu+1}}\right)^K \mu_\nu^{\frac{2}{3}} \\
\leq\, &\cdot s_{\nu+1}^K \delta_{\nu+1}^K \mu_{\nu+1} 2^{7K} \left(1 + \frac{1}{2^{\nu+1}}\right)^K \mu_\nu^{\frac{2}{3}} \\
\leq\, &s_{\nu+1}^K \delta_{\nu+1}^K \mu_{\nu+1}.
\end{aligned}$$

Cauchy estimate yields that

$$\begin{aligned}
&\parallel \partial^K f_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} + \parallel \partial^K g_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
=\, &\parallel \partial_{\theta_{\nu+1}}^\kappa \partial_{r_{\nu+1}}^j \partial_\xi^h f_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&+ \parallel \partial_{\theta_{\nu+1}}^\kappa \partial_{r_{\nu+1}}^j \partial_\xi^h g_{\nu+1} \parallel_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})}
\end{aligned}$$

$$\leq \cdot \frac{\| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} + \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})}}{(\rho_\nu - \rho_{\nu+1})^\kappa (s_\nu - s_{\nu+1})^j (\sigma_\nu - \sigma_{\nu+1})^h},$$

that is,

$$\begin{aligned}
&\| \partial^K f_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} + \| \partial^K g_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1},s_{\nu+1},\sigma_{\nu+1})} \\
&\leq \cdot \frac{s_\nu^K \delta_\nu^K \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^\kappa s_{\nu+1}^j (\sigma_\nu - \sigma_{\nu+1})^h} \\
&\leq \cdot \frac{s_\nu^{K-j} \delta_\nu^K \mu_\nu^2}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h} \\
&\leq \cdot \frac{s_{\nu+1}^{K-j} \delta_{\nu+1}^K \mu_{\nu+1}}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h} (\frac{s_\nu}{s_{\nu+1}})^{K-j} (\frac{\delta_\nu}{\delta_{\nu+1}})^K \mu_\nu^{\frac{2}{3}} \\
&\leq \cdot \frac{s_{\nu+1}^{K-j} \delta_{\nu+1}^K \mu_{\nu+1}}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h} 2^{7K-7j} (1 + \frac{1}{2^{\nu+1}})^K \mu_\nu^{\frac{2}{3}} \\
&\leq \frac{s_{\nu+1}^{K-j} \delta_{\nu+1}^K \mu_{\nu+1}}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h}.
\end{aligned}$$

□

Remark 2.6. It allows us to consider the problem in the whole domain in view of Whitney smoothness in a standard manner, and thus we omit the details.

3. Proof of Theorem 1.1

In this subsection, Iteration lemma is used to prove Theorem 1.1. Let s_0, δ_0, μ_0, ρ_0, σ_0, ϵ_0, f_0, g_0, $\mathscr{F}_0$, $\mathcal{D}(\rho_0, s_0, \sigma_0)$ be given in subsection 2.1, we define

$$\begin{aligned}
&\epsilon_\nu = s_\nu^K \delta_\nu^K \mu_\nu, \qquad s_\nu = \frac{1}{2^{7\nu+2}} s_0, \qquad \delta_\nu = (\frac{1}{2} + \frac{1}{2^{\nu+1}})\delta_0, \qquad \mu_\nu = \mu_{\nu-1}^{\frac{4}{3}}, \\
&\rho_\nu = (\frac{1}{2} + \frac{1}{2^{\nu+1}})\rho_0, \qquad \sigma_\nu = \frac{1}{2^{7\nu+2}}\sigma_0, \qquad N_{\nu+1} = ([\log\frac{1}{\mu_\nu}] + 1)^{3\chi}, \\
&\mathcal{D}_j = \mathcal{D}(\rho_{\nu+1} + \frac{j-1}{4}(\rho_\nu - \rho_{\nu+1}), j s_{\nu+1}, j\sigma_{\nu+1}), \quad j = 1,2,3,4, \\
&\hat{\mathcal{D}}(\rho_\nu, s_\nu, \sigma_\nu) = \mathcal{D}(\rho_{\nu+1} + \frac{3}{4}(\rho_\nu - \rho_{\nu+1}), s_{\nu+1}, \sigma_{\nu+1}),\ \mathcal{D}_1 = \mathcal{D}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1}),
\end{aligned}$$

for any $\nu = 1, 2, \cdots$.

The Iteration lemma is stated as follows.

3.1. Iteration Lemma

Lemma 3.1. *Assume* $(H1)$*-*$(H8)$*. Then for mapping* $\mathscr{F}_\nu$ *on* $\mathcal{D}(\rho_\nu, s_\nu, \sigma_\nu)$*, if*

$$\| f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_\nu,s_\nu,\sigma_\nu)} + \| g_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\mathcal{D}(\rho_\nu,s_\nu,\sigma_\nu)} \leq s_\nu^K \delta_\nu^K \mu_\nu,$$

the following hold for all $\nu = 1, 2, \cdots$.

(i) *There exists a transformation* $\mathscr{U}_{\nu+1}$*:*

$$\begin{cases} \theta_\nu = \theta_{\nu+1} + U_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi), \\ r_\nu = r_{\nu+1} + V_{\nu+1}(\theta_{\nu+1}, r_{\nu+1}, \xi), \end{cases}$$

such that

$$\mathscr{U}_{\nu+1} \circ \mathscr{F}_{\nu+1} = \mathscr{F}_\nu \circ \mathscr{U}_{\nu+1},$$

and

$$\| \partial^K U_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \leq \cdot \frac{s_\nu^{K-j} \delta_\nu^{K-1} \mu_\nu}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1+\kappa} (\sigma_\nu - \sigma_{\nu+1})^h}, \tag{3.1}$$

$$\| \partial^K V_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \leq \cdot \frac{s_\nu^{K-j} \delta_\nu^{K-1} \mu_\nu}{(\rho_\nu - \rho_{\nu+1})^{\tau+n+1+\kappa} (\sigma_\nu - \sigma_{\nu+1})^h}; \tag{3.2}$$

(*ii*)

$$\| f_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} + \| g_{\nu+1} \|_{\mathcal{D}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \leq s_{\nu+1}^K \delta_{\nu+1}^K \mu_{\nu+1}, \tag{3.3}$$

$$\begin{aligned} &\| \partial^K f_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} + \| \partial^K g_{\nu+1} \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\ \leq &\frac{s_{\nu+1}^{K-j} \delta_{\nu+1}^K \mu_{\nu+1}}{(\rho_\nu - \rho_{\nu+1})^\kappa (\sigma_\nu - \sigma_{\nu+1})^h}; \end{aligned} \tag{3.4}$$

(*iii*)

$$|\partial_{r_{\nu+1}}^j \partial_\xi^h (\omega_{\nu+1}(r_{\nu+1}, \xi) - \omega_\nu(r_\nu, \xi))| \leq \cdot \frac{s_\nu^{K-j} \delta_\nu^K \mu_\nu}{(\sigma_\nu - \sigma_{\nu+1})^h}. \tag{3.5}$$

Proof. It is easy to see that we can take sufficiently small ϵ such that $(H1)$-$(H8)$ hold. Now (i) follows from Lemma 2.3, (2.18), and (2.20), (ii) follows from Lemma 2.4. As for (iii), we have

$$\begin{aligned} &|\partial_{r_{\nu+1}}^j \partial_\xi^h (\omega_{\nu+1}(r_{\nu+1}, \xi) - \omega_\nu(r_\nu, \xi))| \\ \leq &\| \partial_{r_{\nu+1}}^j \partial_\xi^h \bar{f}_\nu(r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\ = &\frac{1}{(2\pi)^n} \| \partial_{r_{\nu+1}}^j \partial_\xi^h \Big(\int_{\mathbb{T}^n} f_\nu(\theta_{\nu+1}, r_{\nu+1}) \, d\theta_{\nu+1} \Big) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \\ \leq &\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \| \partial_{r_{\nu+1}}^j \partial_\xi^h f_\nu(\theta_{\nu+1}, r_{\nu+1}, \xi) \|_{\hat{\mathcal{D}}(\rho_{\nu+1}, s_{\nu+1}, \sigma_{\nu+1})} \, d\theta_{\nu+1} \\ \leq &\cdot \frac{s_\nu^{K-j} \delta_\nu^K \mu_\nu}{(\sigma_\nu - \sigma_{\nu+1})^h}. \end{aligned}$$

Above all, the proof is complete. □

3.2. Convergence

Denote $\mathscr{I}_\nu$ by

$$\mathscr{I}_\nu = \mathscr{U}_1 \circ \mathscr{U}_2 \circ \cdots \circ \mathscr{U}_\nu,$$

then

$$\mathscr{F}_\nu = \mathscr{I}_\nu^{-1} \circ \mathscr{F}_0 \circ \mathscr{I}_\nu.$$

Note that $\mathscr{I}_\nu$ has the form

$$\begin{cases} \theta_0 = \theta_\nu + P_\nu(\theta_\nu, r_\nu, \xi) = \theta_\nu + \sum\limits_{l=1}^{\nu} U_l(\theta_l, r_l, \xi), \\ r_0 = r_\nu + Q_\nu(\theta_\nu, r_\nu, \xi) = r_\nu + \sum\limits_{l=1}^{\nu} V_l(\theta_l, r_l, \xi), \end{cases}$$

and

$$\begin{aligned}
&\| P_l(\theta_l, r_l, \xi) \|_{\mathcal{D}(\rho_l, s_l, \sigma_l)} + \| Q_l(\theta_l, r_l, \xi) \|_{\mathcal{D}(\rho_l, s_l, \sigma_l)} \\
\le &\sum_{l=1}^{\nu} (\| U_l(\theta_l, r_l, \xi) \|_{\mathcal{D}(\rho_l, s_l, \sigma_l)} + \| V_l(\theta_l, r_l, \xi) \|_{\mathcal{D}(\rho_l, s_l, \sigma_l)}) \\
\le &\cdot \sum_{l=1}^{\nu} \frac{\| f_{l-1}(\theta_l, r_l, \xi) \|_{\mathcal{D}(\rho_l, s_l, \sigma_l)} + \| g_{l-1}(\theta_l, r_l, \xi) \|_{\mathcal{D}(\rho_l, s_l, \sigma_l)}}{\delta_{l-1}(\rho_{l-1} - \rho_l)^{\tau+n+1}} \\
\le &\cdot \sum_{l=1}^{\nu} \frac{s_{l-1}^K \delta_{l-1}^{K-1} \mu_{l-1}}{(\rho_{l-1} - \rho_l)^{\tau+n+1}} \\
\le &\cdot \frac{s_0^K \delta_0^{K-1} \mu_0}{\rho_0^{\tau+n+1}}.
\end{aligned}$$

Therefore, P_l and Q_l are uniformly convergent on $\mathcal{D}(\rho_\infty, s_\infty, \sigma_\infty)$. Let $P = \lim_{l\to\infty} P_l$, $Q = \lim_{l\to\infty} Q_l$, then $\mathscr{I}_\infty$ becomes

$$\begin{cases} \theta_0 = \theta_\infty + P(\theta, r, \xi), \\ r_0 = r_\infty + Q(\theta, r, \xi). \end{cases}$$

Moreover, from

$$\omega_{\nu+1}(r_{\nu+1}, \xi) = \omega_\nu(r_\nu, \xi) + \bar{f}_\nu(r_{\nu+1}, \xi),$$

and

$$|\partial_{r_{\nu+1}}^j \partial_\xi^h (\omega_{\nu+1}(r_{\nu+1}, \xi) - \omega_\nu(r_\nu, \xi))| \le \cdot \frac{s_\nu^{K-j} \delta_\nu^K \mu_\nu}{(\sigma_\nu - \sigma_{\nu+1})^h},$$

we get

$$\begin{aligned}
|\partial_{r_{\nu+1}}^j \partial_\xi^h (\omega_\infty(r_\infty, \xi) - \omega_0(r_0, \xi))| &\le |\partial_{r_{\nu+1}}^j \partial_\xi^h (\sum_{\nu=0}^{\infty} (\omega_{\nu+1}(r_{\nu+1}, \xi) - \omega_\nu(r_\nu, \xi))| \\
&\le \sum_{\nu=0}^{\infty} |\partial_{r_{\nu+1}}^j \partial_\xi^h (\omega_{\nu+1}(r_{\nu+1}, \xi) - \omega_\nu(r_\nu, \xi))| \\
&\le \cdot \sum_{\nu=0}^{\infty} \frac{s_\nu^{K-j} \delta_\nu^K \mu_\nu}{(\sigma_\nu - \sigma_{\nu+1})^h} \\
&\le \cdot \frac{s_0^{K-j} \delta_0^K \mu_0}{\sigma_0^h},
\end{aligned}$$

that is, ω_ν is uniformly convergent to $\omega_\infty(r_\infty, \xi)$ on $\mathcal{D}(\rho_\infty, s_\infty, \sigma_\infty)$. Therefore $\mathscr{I}_\infty$ transforms $\mathscr{F}_0$ into

$$\begin{cases} \theta_\infty^1 = \theta_\infty + \omega_\infty(r_\infty, \xi), \\ r_\infty^1 = r_\infty. \end{cases}$$

3.3. Measure estimates

The KAM steps are independent of the relationship between n and $m + p$, but measure estimates are not. For this purpose, let us consider the following three cases.

Case 1: $n = m + p$.

We assume $\tilde{\omega} = (\omega, -1)$, $\tilde{r} = (r, \xi, r_{n+1})$, $\tilde{k} = (k, k_0)$, $\tilde{D} = D \times G \times (1, 2)$, and

$$\tilde{D}_\delta = \{\tilde{r} \in \tilde{D} : \ |\langle \tilde{k}, \tilde{\omega} \rangle| \geq \delta |k|^{-\tau}, \ \forall \ \tilde{k} \in \mathbb{Z}^{n+1} \backslash \{0\}\},$$

then $\mathrm{meas}((D \times G) \backslash D_\delta)$ is equivalent to $\mathrm{meas}(\tilde{D} \backslash \tilde{D}_\delta)$.

Clearly on $\tilde{D}$,

$$\mathrm{rank}\{\partial_{\tilde{r}}^\alpha \tilde{\omega}(\tilde{r}) : \ 0 \leq |\alpha| \leq K - 1\} = n + 1,$$

we have

$$\mathrm{meas}(\tilde{D} \backslash \tilde{D}_\delta) \leq c_0 \delta^{\frac{1}{K}},$$

that is

$$\mathrm{meas}((D \times G) \backslash D_\delta) \leq c_0 \delta^{\frac{1}{K}},$$

where c_0 is a constant. See Appendix for details.

Case 2: $m + p < n$.

Assume $\tilde{r} = (r, \xi, r_{m+p+1}, \cdots, r_n, r_{n+1})$, $\tilde{\omega}(\tilde{r}) = \big(\omega, -1\big)$, $\tilde{k} = (k, k_0)$, then

$$\mathrm{rank}\{\partial_{\tilde{r}}^\alpha \tilde{\omega}(\tilde{r}) : \ 0 \leq |\alpha| \leq K - 1\} = n + 1,$$

and $\tilde{D} = D \times G \times \underbrace{(1, 2) \times \cdots \times (1, 2)}_{n+1-m-p}$, $\tilde{D}_\delta = D_\delta \times \underbrace{(1, 2) \times \cdots \times (1, 2)}_{n+1-m-p}$. Then

$$|\langle \tilde{k}, \tilde{\omega} \rangle| \geq \delta |k|^{-\tau}, \qquad \forall \ \tilde{k} \in \mathbb{Z}^{n+1} \backslash \{0\},$$

on $\tilde{D}_\delta$ is equivalent to the Diophantine condition

$$|\langle k, \omega \rangle - k_0| \geq \delta |k|^{-\tau}, \qquad \forall \ k \in \mathbb{Z}^n \backslash \{0\}, \quad \forall \ k_0 \in \mathbb{Z},$$

on D_δ.

Then from the estimates of measure, we obtain

$$\mathrm{meas}(D \backslash D_\delta) \leq c_0 \delta^{\frac{1}{K}}.$$

Case 3: $n < m + p$.

We have known that

$$D_\delta = \{(r, \xi) \in D \times G : \ |\langle k, \omega(r, \xi) \rangle - k_0| \geq \frac{\delta}{|k|^\tau}, \ \forall k \in \mathbb{Z}^n \backslash \{0\}, \ \forall k_0 \in \mathbb{Z}\}.$$

Assume

$$D_\delta^1 = \{(r, \xi) \in D \times G : \ |\langle k, \omega \rangle + \langle \check{k}, \check{r} \rangle - k_0| \geq \frac{\delta}{|k|^\tau}, \\ \forall \ k \in \mathbb{Z}^n \backslash \{0\}, \ \forall \ \check{k} \in \mathbb{Z}^{m+p-n}, \ \forall \ k_0 \in \mathbb{Z}\},$$

where $\check{k} = (k_{n+1}, k_{n+2}, \cdots, k_m, \cdots, k_{m+p})$, $\check{r} = (r_{n+1}, r_{n+2}, \cdots, r_m, \xi)$, if $n < m$, (resp. $\check{k} = (k_{n+1}, \cdots, k_{m+p})$, $\check{r} = (\xi_{n-m+1}, \cdots, \xi_p)$, if $m < n < m + p$). Obviously, $D_\delta^1 \subset D_\delta$, therefore,

$$\mathrm{meas}((D \times G) \backslash D_\delta) \leq \mathrm{meas}((D \times G) \backslash D_\delta^1).$$

From the non-degenerate condition (A2), we have

$$\operatorname{rank}\{\frac{\partial^\alpha \omega}{\partial (r,\xi)^\alpha}:\ 0\le |\alpha|\le K-1\}=n.$$

Let $\omega'=(\omega, r_{n+1}, r_{n+2},\cdots,r_m,\xi)$, (resp. $\omega'=(\omega,\xi_{n-m+1},\cdots,\xi_p)$), it is easy to see that

$$\operatorname{rank}\{\frac{\partial^\alpha \omega'}{\partial (r,\xi)^\alpha}:\ \ 0\le |\alpha|\le K-1\}=m+p.$$

If $\tilde\omega=(\omega',-1)$, $\tilde r=(r,\xi,r_{m+p+1})$, $\tilde k=(k,\check k,k_0)$, $\tilde D=D\times G\times(1,2)$ and $\tilde D_\delta=D^1_\delta\times(1,2)$, that is,

$$\tilde D_\delta=\{\tilde r\in\tilde D:\ |\langle\tilde k,\tilde\omega\rangle|\ge\frac{\delta}{|k|^\tau},\ \forall\ \tilde k\in\mathbb{Z}^{m+p+1}\backslash\{0\}\},$$

one has

$$\operatorname{rank}\{\partial^\alpha_{\tilde r}\tilde\omega(\tilde r):\ 0\le|\alpha|\le K-1\}=m+p+1.$$

Hence, from the Appendix we get

$$\operatorname{meas}(\tilde D\backslash\tilde D_\delta)\le c_0\delta^{\frac1K},$$

and

$$\operatorname{meas}((D\times G)\backslash D^1_\delta)=\operatorname{meas}(\tilde D\backslash\tilde D_\delta).$$

Therefore,

$$\operatorname{meas}((D\times G)\backslash D_\delta)\le\operatorname{meas}((D\times G)\backslash D^1_\delta)=\operatorname{meas}(\tilde D\backslash\tilde D_\delta)\le c_0\delta^{\frac1K}.$$

4. Appendix

Here we use the method of measure estimates mentioned in [15, p1787].

Assume $\tilde\omega=(\omega,-1)$, $\tilde r=(r,r_{n+1})$, $\tilde k=(k,k_0)$, $\tilde D=D\times(1,2)$, and

$$\tilde D_\delta=D_\delta\times(1,2)=\big\{\tilde r\in\tilde D:\ |\langle\tilde k,\tilde\omega\rangle|\ge\delta|k|^{-\tau},\ \forall\ \tilde k\in\mathbb{Z}^{n+1}\backslash\{0\}\big\}.$$

Then $\operatorname{meas}(D\backslash D_\delta)$ is equal to $\operatorname{meas}(\tilde D\backslash\tilde D_\delta)$.

Assume

$$\mathcal{R}=\big\{\tilde r\in\tilde D:\ |\langle\tilde k,\tilde\omega\rangle|\le\delta|k|^{-\tau},\ \forall\ \tilde k\in\mathbb{Z}^{n+1}\backslash\{0\}\},$$

and

$$z(\tilde r)=\langle\zeta,\tilde\omega\rangle,\qquad \zeta=\frac{\tilde k}{|\tilde k|}\in S^{n+1},$$

where S^{n+1} is a $(n+1)$-dimensional unit sphere. The Taylor expansion of $z(\tilde r)$ is

$$z(\tilde r)=\zeta^T\Lambda(\tilde r)\bar r,$$

where,

$$\Lambda(\tilde r)=\left(\tilde\omega(\tilde r),\partial_{\tilde r}\tilde\omega(\tilde r),\cdots,\partial^n_{\tilde r}\tilde\omega(\tilde r),\int_0^1(1-t)^{|n+1|}\partial^{n+1}_{\tilde r}\tilde\omega(\tilde r_0+(\tilde r_0+t\hat r_0))\,dt\right),$$
$$\bar r=(1,\hat r,\cdots,\hat r^n,\hat r^{n+1}),$$

$\hat{r} = \tilde{r} - \tilde{r}_0.$

By the non-degenerate condition (A2), $\operatorname{rank}(\Lambda(\tilde{r})) = n+1$ for $\tilde{r} \in \tilde{D}$ when $|\alpha| = K$. Then there exists an orthogonal matrix $O_{\tilde{r}_0} = (o_{ij})$ whose elements in different columns and rows are only one is 1 and others are 0, such that

$$\Lambda(\tilde{r}) O_{\tilde{r}_0} = \big(A(\tilde{r}), B(\tilde{r})\big),$$

where $\det A(\tilde{r}) \neq 0$ and $\tilde{r}$ is in the neighborhood $\bar{D} \subset \mathbb{R}^n$ of $\tilde{r}_0$. Therefore,

$$z(\tilde{r}) = \zeta^T \Lambda(\tilde{r}) \bar{r} = \zeta^T \Lambda(\tilde{r}) O_{\tilde{r}_0} O_{\tilde{r}_0}^{-1} \bar{r} = \big(\zeta^T A(\tilde{r}), \zeta^T B(\tilde{r})\big) O_{\tilde{r}_0}^{-1} \bar{r},$$

and

$$z^T(\tilde{r}) = \bar{r}^T O_{\tilde{r}_0} \big(\zeta^T \Lambda(\tilde{r}) O_{\tilde{r}_0}\big)^T = \bar{r}^T O_{\tilde{r}_0} \begin{pmatrix} A(\tilde{r})\zeta \\ B(\tilde{r})\zeta \end{pmatrix}.$$

From the non-degenerate condition

$$\operatorname{rank}\big\{\partial_{\tilde{r}}^{\alpha} z(\tilde{r}) : \ 0 \le |\alpha| \le K-1\big\} = 1,$$

which is equivalent to the non-degenerate condition that appears in this paper, we get

$$\operatorname{rank} \begin{pmatrix} A^T(\tilde{r})\zeta\zeta^T A(\tilde{r}) & A^T(\tilde{r})\zeta\zeta^T B(\tilde{r}) \\ B^T(\tilde{r})\zeta\zeta^T A(\tilde{r}) & B^T(\tilde{r})\zeta\zeta^T B(\tilde{r}) \end{pmatrix} = 1, \qquad \tilde{r} \in \bar{D}_{\tilde{r}_0}, \quad \zeta \in S^{n+1}.$$

Therefore, for all $\tilde{r}$ in the neighborhood of $\tilde{r}_0$, there exists an orthogonal matrix $O_{\tilde{r}}$ continuously depending on $\tilde{r}$ such that

$$O_{\tilde{r}}^{-1} \begin{pmatrix} A^T(\tilde{r})\zeta\zeta^T A(\tilde{r}) & A^T(\tilde{r})\zeta\zeta^T B(\tilde{r}) \\ B^T(\tilde{r})\zeta\zeta^T A(\tilde{r}) & B^T(\tilde{r})\zeta\zeta^T B(\tilde{r}) \end{pmatrix} O_{\tilde{r}} = \operatorname{diag}\big(0, \cdots, 0, \underbrace{\lambda(\tilde{r}, \zeta)}_{i-th}, 0, \cdots, 0\big),$$

where the position of λ depends on ζ, and

$$|(O_{\tilde{r}_0}^{-1} \bar{r})_i| = |(O_{\tilde{r}_0})_i \bar{r}| \ge \min_j |\hat{r}_j|^K.$$

Taking use of the Poincaré Separation Theorem, we have

$$\begin{aligned} z^T(\tilde{r}) z(\tilde{r}) &= \bar{r}^T O_{\tilde{r}_0} O_{\tilde{r}_0}^{-1} \Lambda^T(\tilde{r}) \zeta\zeta^T \Lambda(\tilde{r}) O_{\tilde{r}_0} O_{\tilde{r}_0}^{-1} \bar{r} \\ &= \bar{r}^T O_{\tilde{r}_0} \begin{pmatrix} A^T(\tilde{r})\zeta\zeta^T A(\tilde{r}) & A^T(\tilde{r})\zeta\zeta^T B(\tilde{r}) \\ B^T(\tilde{r})\zeta\zeta^T A(\tilde{r}) & B^T(\tilde{r})\zeta\zeta^T B(\tilde{r}) \end{pmatrix} O_{\tilde{r}_0}^{-1} \bar{r} \\ &\ge \bar{r}^T O_{\tilde{r}_0} \operatorname{diag}\big(0, \cdots, 0, \lambda(\tilde{r}, \zeta), 0, \cdots, 0\big) O_{\tilde{r}_0}^{-1} \bar{r} \\ &= \min \lambda(\tilde{r}, \zeta) |(O_{\tilde{r}_0} \bar{r})_i|^2 \\ &\ge \lambda_{\tilde{r}_0} (\min_j |\hat{r}_j|)^{2K}. \end{aligned}$$

Then

$$\operatorname{meas}\big(\{\tilde{r} \in \tilde{D} \cap \bar{D}_{\tilde{r}_0} | \ |\langle \zeta, \tilde{\omega} \rangle|^2 \le \delta^2 |k|^{-2\tau}\}\big)$$

$$
\begin{aligned}
&< \mathrm{meas}\big(\{\tilde r \in \tilde D \cap \bar D_{\tilde r_0} |\ \lambda_{\tilde r_0}(\min_j |\hat r_j|)^{2K+2} \le \delta^2 |k|^{-2\tau}\}\big)\\
&\le \frac{1}{\lambda_{\tilde r_0}} (\mathrm{diam}\tilde D)^n \delta^{\frac{1}{K}} |k|^{-\frac{\tau}{K}}\\
&\le c_0 \delta^{\frac{1}{K}},
\end{aligned}
$$

where diam$\tilde D$ is the diameter of area $\tilde D$ and c_0 is a constant dependents on $\tilde D$, diam$\tilde D$, n, and $\lambda_{\tilde r_0}$, that is,

$$\mathrm{meas}(\mathcal{R}) \le c_0 \delta^{\frac{1}{K}}.$$

Thus

$$\mathrm{meas}(\tilde D \backslash \tilde D_\delta) \le \mathrm{meas}(\mathcal{R}) \le c_0 \delta^{\frac{1}{K}},$$

due to $\tilde D \backslash \tilde D_\delta \subset \mathcal{R}$, and

$$\mathrm{meas}(D \backslash D_\delta) \le c_0 \delta^{\frac{1}{K}}.$$

Acknowledgements

We express our sincere thanks to Professor Yong Li for the meaningful suggestions and support. We also thank the reviewers for their comments, which are helpful for improving our manuscript.

References

[1] V. I. Arnol'd, *Proof of A. N. Kolmogorov's theorem on the preservation of quasi-periodic motions under small perturbations of the Hamiltonian*, Russian Math. Surveys, 1963, 18, 9–36.

[2] C. Cheng and Y. Sun, *Existence of invariant tori in three-dimensional measure-preserving mappings*, Celestial Mech. Dynam. Astronom., 1989/90, 47(3), 275–292.

[3] F. Cong, Y. Li and M. Huang, *Invariant tori for nearly twist mappings with intersection property*, Northeast. Math. J., 1996, 12(3), 280–298.

[4] L. Chierchia and C. E. Koudjinan, *V. I. Arnold's "Global" KAM theorem and geometric measure estimates*, Regul. Chaotic Dyn., 2021, 26(1), 61–88.

[5] R. C. Calleja, A. Celletti and R. de la Llave, *KAM quasi-periodic solutions for the dissipative standard map*, Commun. Nonlinear Sci. Numer. Simul., 2022, 106(106111), 1–29.

[6] M. R. Herman, *Sur les courbes invariantes par les difféomorphismes de l'anneau*, Astérisque, 1986, 2(144), 1–248.

[7] A. N. Kolmogorov, *On conservation of conditionally periodic motions for a small change in Hamilton's function*, Dokl. Akad. Nauk SSSR, 1954, 98, 527–530.

[8] C. E. Koudjinan, *A KAM theorem for finitely differentiable Hamiltonian systems*, J. Differential Equations, 2020, 269(6), 4720–4750.

[9] M. Levi and J. Moser, *A Lagrangian proof of the invariant curve theorem for twist mapping*, Smooth ergodic theory and its applications. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2001, 69, 733–746.

[10] Y. Li and Y. Yi, *Persistence of invariant tori in generalized Hamiltonian systems*, Ergodic Theory Dynam. Systems, 2002, 22(4), 1233–1261.

[11] J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen Math. Phys. KI. II, 1962 (1962), 1–20.

[12] J. Moser, *A rapidly convergent iteration method and non-linear partial differential equations. I*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1966, 20(3), 265–315.

[13] J. Moser, *A rapidly convergent iteration method and non-linear partial differential equations. II*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1966, 20(3), 499–535.

[14] J. Moser, *On the construction of almost periodic solutions for ordinary differential equations*, 1970 Proc. Internat. Conf. on Functional Analysis and Related Topics, 1969, 60–67.

[15] W. Qian, Y. Li and X. Yang, *Melnikov's conditions in matrices*, J. Dynam. Differential Equations, 2020, 32(4), 1779–1795.

[16] H. Rüssmann, *On the existence of invariant curves of twist mappings of an annulus*, Geometric dynamics, 677–718, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[17] N. V. Svanidze, *Small perturbations of an integrable dynamical system with an integral invariant*, Trudy Mat. Inst. Steklov., 1980, 147, 124–146.

[18] M. B. Sevryuk, *Partial preservation of the frequencies and Floquet exponents of invariant tori in KAM theory reversible context 2*, J. Math. Sci. (N. Y.), 2021, 253(5), 730–753.

[19] F. Trujillo, *Uniqueness properties of the KAM curve*, Discrete Contin. Dyn. Syst., 2021, 41(11), 5165–5182.

[20] Z. Xia, *Existence of invariant tori in volume-preserving diffeomorphisms*, Ergodic Theory Dynam. Systems, 1992, 12(3), 621–631.

[21] L. Yang and X. Li, *Existence of periodically invariant tori on resonant surfaces for twist mappings*, Discrete Contin. Dyn. Syst., 2020, 40(3), 1389–1409.

[22] E. Zehnder, *Generalized implicit function theorems with applications to some small divisor problems. I*, Comm. Pure Appl. Math., 1975, 28, 91–140.

[23] E. Zehnder, *Generalized implicit function theorems with applications to some small divisor problems. II*, Comm. Pure Appl. Math., 1976, 29(1), 49–111.