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# MULTIPLE SOLUTIONS FOR A NONHOMOGENEOUS SCHRÖDINGER-POISSON SYSTEM WITH CRITICAL EXPONENT

Li-Jun Zhu[1] and Jia-Feng Liao[1,2,†]

**Abstract** In this paper, a nonhomogeneous Schrödinger-Poisson system with critical exponent was considered. By using the Mountain Pass Theorem and variational method, two positive solutions were obtained for the system which generalize and improve some recent results in the literature.

**Keywords** Schrödinger-Poisson system, critical exponent, variational method, mountain pass theorem.

**MSC(2010)** 35B33, 35J20, 35J60.

## 1. Introduction and Main Result

In this paper, we study the existence and multiplicity of positive solutions for the following nonhomogeneous Schrödinger-Poisson system with critical exponent

$$\begin{cases} -\Delta u + u + \eta l(x)\phi u = \lambda f(x) + u^5, & x \in \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\eta \in \mathbb{R}\backslash\{0\}, \lambda > 0$. We will make the following assumptions on $f$ and $l$:
$(H_f)$ $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$, $f \geq 0$, and $f \not\equiv 0$,
$(H_l)$ $l \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $l \geq 0$, and $l \not\equiv 0$.

It is well known that the Schrödinger-Poisson system stems from quantum mechanics models and semiconductor theory and it has been studied extensively. From a physical standpoint, Schrödinger-Poisson systems describe systems of identical charged particles interacting each other if magnetic effects could be ignored and their solutions are standing waves. For more details about the mathematical and physical background of Schrödinger-Poisson system, please refer to the papers [1–3] and the references therein.

The general form of the Schrödinger-Poisson system with critical exponent is as follows

$$\begin{cases} -\Delta u + u + l(x)\phi u = g(x,u) + u^5, & x \in \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

[†]The corresponding author. Email: liaojiafeng@163.com(J.-F. Liao)
[1]School of Mathematics and Information, China West Normal University, Nanchong, Sichuan 637009, China
[2]College of Mathematics Education, China West Normal University, Nanchong, Sichuan 637009, China

The system (1.2) has been extensively studied, for example: [9–15, 19, 20, 24]. Particularly, when $g(x,u)$ is superlinear, Huang and Rocha [10] studied system (1.2) in case of $g(x,u) = \mu h(x)|u|^{q-2}u$ with $2 \leq q < 6$ and established a positive solution by using variational methods. Recently, Lei et al [12] considered the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \eta\phi u = \lambda f(x)u^{q-1} + u^5, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $1 < q < 2, \eta \in \mathbb{R}\backslash\{0\}, \lambda > 0$ is a real parameter and $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ is a nonzero nonnegative function. Using the variational methods, they obtained that there exists a positive constant $\lambda_*$ such that for all $\lambda \in (0, \lambda_*)$, the system has at least two positive solutions.

Compared to the homogeneous case, there are a few papers concerning the nonhomogeneous case. The nonhomogeneous system with mass $m$ is acquired by coupling together the standing waves of nonlinear Schrödinger equation coupled with Maxwell's equations. The form of nonlinear Schrödinger type equation is as follow

$$ih\frac{\partial u}{\partial t} = -\frac{h^2}{2m}\Delta u - |u|^{p-2}u - g(x)e^{iwt}, x \in \mathbb{R}^3,$$

where $u$ is the wave function, $h$ is the Plancks constant, $m$ is the mass of the particle, $e$ is the electric charge and $\omega$ is the phase of the wave. The interaction of $u$ with the electromagnetic field is described by the minimal coupling rule. For more details as regards the relevance physical of the nonhomogeneous Schrödinger-Poisson system, we can refer to [6, 17]. Recently, Ye [21] studied the following a class of nonhomogeneous Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u) + h(x), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter and $0 \leq h(x) = h(|x|) \in L^2(\mathbb{R}^3)$, $f$ satisfies the following hypotheses:

$(f_1)$ $f \in C(\mathbb{R}, \mathbb{R}^+), f(0) = 0, f(t) \equiv 0$ for $t < 0$ and there exist $a > 0$ and $p \in (2, 6)$ such that

$$f(t) \leq a(1 + |t|^{p-1}), \forall t \in \mathbb{R}.$$

$(f_2)$ $\lim\limits_{t\to 0} f(t)/t = 0$.

$(f_3)$ $\lim\limits_{t\to\infty} f(t)/t = +\infty$.

She proved that system (1.3) has at least two positive solutions with the aid of Ekeland's variational principle, Jeanjean's monotone method, Pohožaev's identity and the mountain pass theorem. However, the author did not consider the case of the critical exponent. Indeed, when the nonlinear term contains the critical exponential term, it is more difficult to study system (1.3).

Our paper is mainly inspired by [12, 21]. Up to now, there was no information about system (1.1). Therefore, in this paper, we will study the existence of multiple solutions of system (1.1) with $\eta \in \mathbb{R}\backslash\{0\}$ by using the Mountain Pass Theorem and variational method.

Our main result can be described as follows.

**Theorem 1.1.** *Assume $\eta \in \mathbb{R}\backslash\{0\}$ and conditions $(H_f), (H_l)$ hold. Then there exists a positive constant $\lambda^*$ such that for all $\lambda \in (0, \lambda^*)$, system* (1.1) *has at least two positive solutions $(u, \phi_u)$* in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

**Remark 1.1.** Compared to [21], on the one hand, our paper gets rid of the restriction of the coefficient of the nonlocal term. On the other hand, we consider the critical system.

This paper is organized as follows. In section 2, we present some notations and prove some useful preliminary lemmas which pave the way for getting two positive solutions. Then we give the proof of Theorem 1.1.

## 2. Proof of Theorem 1.1

Throughout this paper, we make use of the following notations:

- $|u|_s = (\int_{\mathbb{R}^3} |u|^s dx)^{\frac{1}{s}}$ is the usual Lebesgue space $L^s(\mathbb{R}^3)$ norm.
- $L^\infty(\mathbb{R}^3)$ is equipped with the norm $\|u\|_\infty =$esssup$|u|$.
- The norm of $H^1(\mathbb{R}^3)$ by $\|u\| = (\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx)^{\frac{1}{2}}$. $H^{-1}$ is the dual space of $H^1$.
- $B_r(0)$ (respectively, $\partial B_r(0)$) the closed ball (respectively, the sphere) of center zero and radius $r$ i.e $B_r(0) = \{u \in H^1(\mathbb{R}^3) : \|u\| \le r\}, \partial B_r(0) = \{u \in H^1(\mathbb{R}^3) : \|u\| = r\}$.
- $C, C_i (i = 1, 2, ...)$ denote various positive constants, which may vary from line to line.
- For each $p \in [2, 6)$, by the Sobolev constants, we denote

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^3)\backslash\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{|u|_6^2}; \qquad S_p := \inf_{u\in H^1(\mathbb{R}^3)\backslash\{0\}} \frac{\|u\|^2}{|u|_p^2}.$$

As we all known that system (1.1) can be reduced to a nonlinear Schrödinger equation with nonlocal term. Indeed, the Lax-Milgram theorem implies that for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = l(x)u^2.$$

We substitute $\phi_u$ to the first equation of system (1.1), then system (1.1) can be transformed into the following equation

$$-\Delta u + u + \eta l(x)\phi_u u = \lambda f(x) + u^5, \quad x \in \mathbb{R}^3. \tag{2.1}$$

According to [10] or [23], we have the following conclusions.

**Lemma 2.1.** *For every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = l(x)u^2$$

*and the following results hold*

*(1)* $\|\phi_u\|^2 = \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx$,

*(2)* $\phi_u \ge 0$, *moreover* $\phi_u > 0$ *when* $u \ne 0$,

*(3)* $\int_{\mathbb{R}^3} l(x)\phi_u u^2dx = \int_{\mathbb{R}^3} |\nabla\phi_u|^2dx \le C\|u\|^4$,

*(4)* $F : H^1(\mathbb{R}^3) \to \mathbb{R}$ *is well defined with* $F(u) = \eta\int_{\mathbb{R}^3} l(x)\phi_u u^2dx$, *assume that* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *and* $F(u_n) \to F(u)$ *in* $H^1(\mathbb{R}^3)$,

*(5)* $F$ *is* $C^1$ *and*

$$\langle F'(u), v\rangle = 4\eta\int_{\mathbb{R}^3} l(x)\phi_u uvdx, \quad \forall v \in H^1(\mathbb{R}^3).$$

The Euler functional of equation (2.1) is defined by $I_\lambda$: $H^1(\mathbb{R}^3) \to \mathbb{R}$, that is,

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\eta\int_{\mathbb{R}^3} l(x)\phi_u u^2dx - \lambda\int_{\mathbb{R}^3} f(x)udx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6dx.$$

From Lemma 2.1, we can deduce that the functional $I_\lambda$ is of class $C^1$ and its critical points are weak solutions of equation (2.1). Moreover, we can obtain that

$$\langle I'_\lambda(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u\cdot\nabla v + uv)dx + \eta\int_{\mathbb{R}^3} l(x)\phi_u uvdx - \lambda\int_{\mathbb{R}^3} f(x)vdx - \int_{\mathbb{R}^3}|u|^5vdx,$$

for any $v \in H^1(\mathbb{R}^3)$.

**Lemma 2.2.** *There exist* $\Lambda_0, \rho > 0$ *such that for each* $\lambda \in (0, \Lambda_0)$, *then it holds*

$$d := \inf_{u\in B_\rho(0)} I_\lambda(u) < 0 \quad \text{and} \quad I_\lambda|_{\partial B_\rho(0)} > 0. \tag{2.2}$$

**Proof.** When $\eta > 0$, by the Sobolev and Hölder inequalities, we obtain

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\eta\int_{\mathbb{R}^3} l(x)\phi_u u^2dx - \lambda\int_{\mathbb{R}^3} f(x)udx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx \\ &\ge \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^3} f(x)udx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx \\ &\ge \frac{1}{2}\|u\|^2 - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\|u\| - \frac{1}{6S_6^3}\|u\|^6 \\ &= \|u\|(\frac{1}{2}\|u\| - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}} - \frac{1}{6S_6^3}\|u\|^5). \end{aligned} \tag{2.3}$$

Set $g(t) = \frac{1}{2}t - \frac{1}{6S_6^3}t^5$, we can easily calculate that there exists a positive constant $\rho_1 = (\frac{3}{5}S_6^3)^{\frac{1}{4}}$ such that $\max_{t>0} g(t) = g(\rho_1) > 0$. Let $\lambda_* = \frac{S_6^{\frac{1}{2}}}{2|f|_{\frac{6}{5}}}g(\rho_1)$, we have $I_\lambda|_{\|u\|=\rho_1} \ge \frac{g(\rho_1)}{2}\rho_1$ for any $\lambda \in (0, \lambda_*)$.

When $\eta < 0$, by the Sobolev and Hölder inequalities, we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\eta\int_{\mathbb{R}^3} l(x)\phi_u u^2dx - \lambda\int_{\mathbb{R}^3} f(x)udx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx \\ &= \frac{1}{2}\|u\|^2 - \frac{-\eta}{4}\int_{\mathbb{R}^3} l(x)\phi_u u^2dx - \lambda\int_{\mathbb{R}^3} f(x)udx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{-\eta}{4}\|l\|_\infty\int_{\mathbb{R}^3}\phi_u u^2dx - \lambda\int_{\mathbb{R}^3} f(x)udx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx \\ &\ge \frac{1}{2}\|u\|^2 - C\|u\|^4 - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\|u\| - \frac{1}{6S_6^3}\|u\|^6 \\ &= \|u\|(\frac{1}{2}\|u\| - C\|u\|^3 - \lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}} - \frac{1}{6S_6^3}\|u\|^5). \end{aligned} \tag{2.4}$$

Set $g(t) = \frac{1}{2}t - Ct^3 - \frac{1}{6S_6^3}t^5$, we see that there exists a constant $\rho_2 > 0$ such that $\max\limits_{t>0} g(t) = g(\rho_2) > 0$. Let $\lambda_{**} = \frac{S_6^{\frac{1}{2}}}{2|f|_{\frac{6}{5}}} g(\rho_2)$, we have $I_\lambda|_{\|u\|=\rho_2} \geq \frac{g(\rho_2)}{2}\rho_2$ for any $\lambda \in (0, \lambda_{**})$. Thus, set $\Lambda_0 = \min\{\lambda_*, \lambda_{**}\}, \rho = \min\{\rho_1, \rho_2\}$, then it follows that there exists a positive constant $\alpha = \min\{\frac{g(\rho_1)}{2}\rho_1, \frac{g(\rho_2)}{2}\rho_2\}$ such that $I_\lambda \geq \alpha$ for all $\|u\| = \rho$. Moreover, by (2.3) and (2.4), $d = \inf\limits_{u \in B_\rho(0)} I_\lambda(u)$ is well defined. Furthermore, for any $u \in H^1(\mathbb{R}^3)$, it holds

$$\lim_{t \to 0^+} \frac{I_\lambda(tu)}{t} = -\lambda \int_{\mathbb{R}^3} f(x)udx.$$

Thus, there exists $u_0 > 0$ such that $\|u_0\| < \rho$ and $I_\lambda(u_0) < 0$. Consequently, $d = \inf\limits_{u \in B_\rho(0)} I_\lambda(u) < 0$. The proof is complete. □

**Theorem 2.1.** *Suppose* $0 < \lambda < \Lambda_0$ ($\Lambda_0$ *defined in Lemma 2.2). Then system* (1.1) *has a positive solution* $(u_*, \phi_{u_*}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *satisfying* $I_\lambda(u_*) < 0$.

**Proof.** By Lemma 2.2, there exist $\alpha > 0, \rho > 0$ such that when $\lambda \in (0, \Lambda_0)$, for any $\|u\| = \rho$, we have $I_\lambda(u) \geq \alpha > 0$ and $d = \inf\limits_{u \in B_\rho(0)} I_\lambda(u) < 0$. There exists a minimization sequence $\{u_n\} \subset B_\rho(0)$. Thanks to $I_\lambda(|u|) \leq I_\lambda(u)$, we can assume from the beginning that $u_n \geq 0$ in $\mathbb{R}^3$. Since $\{u_n\} \subset B_\rho(0)$, it's easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. So there exist a subsequence (still denoted by itself) and $u_* \in H^1(\mathbb{R}^3)$ with $u_* \geq 0$ such that

$$\begin{cases} u_n \rightharpoonup u_* \quad \text{in } H^1(\mathbb{R}^3) \\ u_n(x) \to u_*(x) \quad a.e \text{ in } \mathbb{R}^3 \\ u_n \rightharpoonup u_* \quad \text{in } L^q_{loc}(\mathbb{R}^3) \quad 2 \leq q \leq 6. \end{cases} \tag{2.5}$$

Set $w_n = u_n - u_*$, so $w_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. The Brézis-Lieb Lemma ( [7] or [18]) implies that

$$\begin{cases} \|u_n\|^2 = \|w_n\|^2 + \|u_*\|^2 + o_n(1), \\ \int_{\mathbb{R}^3} |u_n|^6 dx = \int_{\mathbb{R}^3} |w_n|^6 dx + \int_{\mathbb{R}^3} |u_*|^6 dx + o_n(1). \end{cases} \tag{2.6}$$

Since $u_n \rightharpoonup u_*$ in $L^6(\mathbb{R}^3)$ and $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we have

$$\lambda \int_{\mathbb{R}^3} f(x)u_n dx = \lambda \int_{\mathbb{R}^3} f(x)u_* dx + o_n(1). \tag{2.7}$$

By (2.2), for an appropriate constant $\rho$, we can deduce that

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx \geq 0, \quad \text{for } \ u_n \in B_\rho(0). \tag{2.8}$$

If $u_* = 0$, then $w_n = u_n$, which follows that $w_n \in B_\rho(0)$. If $u_* \neq 0$, we also get $w_n \in B_\rho(0)$ for $n$ large sufficiently. From (2.8), one has

$$\frac{1}{2}\|w_n\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx \geq 0. \tag{2.9}$$

Therefore, by Lemma 2.1, it follows from (2.5)-(2.7) and (2.9) that

$$\begin{aligned} d &= I_\lambda(u_n) + o_n(1) \\ &= I_\lambda(u_*) + \frac{1}{2}\|w_n\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx + o_n(1) \\ &\geq I_\lambda(u_*) + o_n(1) \end{aligned}$$

as $n \to \infty$, it holds that $d \geq I_\lambda(u_*)$. Since $B_\rho(0)$ is closed and convex, thus $u_* \in B_\rho(0)$, we obtain $d \leq I_\lambda(u_*)$. Hence, one has $I_\lambda(u_*) = d < 0$ and $u_* \neq 0$. It follows that $u_*$ is a local minimizer of $I_\lambda$. By using the strong maximum principle, we get $u_* > 0$. So $u_*$ is positive solution of equation (1.2) with $I_\lambda(u_*) < 0$. Therefore, we can conclude that $(u_*, \phi_{u_*})$ is a positive solution of system (1.1). This completes the proof of Theorem 2.1. □

**Lemma 2.3.** *The functional $I_\lambda$ satisfies the $(PS)_c$ condition provided $c < \frac{1}{3}S^{\frac{3}{2}} - D\lambda^2$, where $D = \frac{9}{8}(|f|_{\frac{6}{5}} S_6^{-\frac{1}{2}})^2$.*

**Proof.** Let $\{u_n\} \subset H^1(\mathbb{R}^3)$ be a $(PS)_c$ sequence of $I_\lambda$, that is,

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0, \qquad \text{as} \quad n \to \infty. \tag{2.10}$$

We claim that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. For $n$ large enough and combining with (2.10), one gets that

$$\begin{aligned} c + 1 + o(\|u_n\|) &\geq I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 - \frac{3}{4}\lambda\int_{\mathbb{R}^3} f(x)u_n dx + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{3}{4}\lambda S_6^{-\frac{1}{2}}|f|_{\frac{6}{5}}\|u_n\| \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, still denoted by $\{u_n\}$ and there exists $v \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$ as $n \to \infty$ and satisfies (2.5). Set $w_n = u_n - v$, if $\|w_n\|^2 \to 0$, then the conclusion holds. Otherwise, there exists a subsequence (still denoted by itself) such that $\lim_{n\to\infty} \|w_n\|^2 = \kappa > 0$. From (2.10), for any $\varphi \in H^1(\mathbb{R}^3)$, we have $\langle I'_\lambda(u_n), \varphi\rangle \to 0$. By Lemma 2.1 and (2.6), as $n \to \infty$ , it follows that

$$\begin{aligned} &\int_{\mathbb{R}^3} (\nabla v \cdot \nabla\varphi + v\varphi)dx + \eta\int_{\mathbb{R}^3} l(x)\phi_v(x)v\varphi dx \\ &- \lambda\int_{\mathbb{R}^3} f(x)\varphi dx - \int_{\mathbb{R}^3} |v|^5\varphi dx = 0. \end{aligned} \tag{2.11}$$

Taking the test function $\varphi = v$ in (2.11), then it holds that

$$\|v\|^2 + \eta\int_{\mathbb{R}^3} l(x)\phi_v v^2 dx - \lambda\int_{\mathbb{R}^3} f(x)v dx - \int_{\mathbb{R}^3} |v|^6 dx = 0. \tag{2.12}$$

From (2.10), we have $\langle I'_\lambda(u_n), u_n\rangle \to 0$. By Lemma 2.1, (2.6) and (2.7), we obtain

$$\begin{aligned} o_n(1) =& \|v\|^2 + \eta\int_{\mathbb{R}^3} l(x)\phi_v v^2 dx - \lambda\int_{\mathbb{R}^3} f(x)v dx \\ &- \int_{\mathbb{R}^3} |v|^6 dx + \|w_n\|^2 - \int_{\mathbb{R}^3} |w_n|^6 dx. \end{aligned} \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\|w_n\|^2 - \int_{\mathbb{R}^3} |w_n|^6 dx = o_n(1). \tag{2.14}$$

By the Sobolev inequality, we have

$$|w_n|_6^2 \le S^{-1} \int_{\mathbb{R}^3} |\nabla w_n|^2 dx \le S^{-1} \|w_n\|^2.$$

Consequently, we can obtain $\kappa \ge S^{\frac{3}{2}}$.

On the one hand, by (2.12), Hölder inequality, Young inequality and Sobolev inequality, it holds that

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\|v\|^2 + \frac{1}{4}\eta \int_{\mathbb{R}^3} l(x)\phi_v v^2 dx - \lambda \int_{\mathbb{R}^3} f(x) v dx - \frac{1}{6}\int_{\mathbb{R}^3} |v|^6 dx \\ &= \frac{1}{4}\|v\|^2 - \frac{3}{4}\lambda \int_{\mathbb{R}^3} f(x) v dx + \frac{1}{12}\int_{\mathbb{R}^3} |v|^6 dx \\ &\ge \frac{1}{4}\|v\|^2 - \frac{3}{4}\lambda S_6^{-\frac{1}{2}} |f|_{\frac{6}{5}} \|v\| \\ &\ge \frac{1}{4}\|v\|^2 - \left[\frac{1}{4}\|v\|^2 + \frac{9}{8}\left(\lambda |f|_{\frac{6}{5}} S_6^{-\frac{1}{2}}\right)^2\right] \\ &\ge \frac{1}{4}\|v\|^2 - \frac{1}{4}\|v\|^2 - \frac{9}{8}\left(\lambda |f|_{\frac{6}{5}} S_6^{-\frac{1}{2}}\right)^2 \\ &\ge -D\lambda^2 \end{aligned} \tag{2.15}$$

where $D = \frac{9}{8}|f|_{\frac{6}{5}}^2 S_6^{-1}$.

On the other hand, it follows from (2.7), (2.10) and (2.14) that

$$\begin{aligned} I_\lambda(v) &= I_\lambda(u_n) - \frac{1}{2}\|w_n\|^2 + \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx + o_n(1) \\ &= I_\lambda(u_n) - \frac{1}{3}\|w_n\|^2 + o_n(1) \\ &= c - \frac{1}{3}\kappa + o_n(1) \\ &< c - \frac{1}{3} S^{\frac{3}{2}} \\ &< -D\lambda^2 \end{aligned}$$

which contradicts (2.15). Therefore $\kappa = 0$. The proof is complete. □

We know that the extremal function

$$U(x) = \frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, x \in \mathbb{R}^3$$

solves

$$-\Delta u = u^5 \quad \text{in } \mathbb{R}^3 \backslash \{0\}$$

and $|\nabla U|_2^2 = |U|_6^6 = S^{\frac{3}{2}}$. We choose a function $\zeta \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \zeta \le 1$ in $\mathbb{R}^3$. $\zeta(x) = 1$ near $x = 0$ and it is radially symmetric. We define

$$u_\varepsilon(x) = \zeta(x) U(x).$$

Besides, since $(u_*, \phi_{u_*})$ is a positive solution of system (1.1), by a standard method, we can obtain that there exist $m, M > 0$ such that $m \le u_* \le M$ for each $x \in supp\zeta$.

**Lemma 2.4.** *Under the conditions of Theorem 1.1, then there exist $\Lambda_1 > 0, u_\varepsilon \in H^1(\mathbb{R}^3)$ such that*

$$\sup_{t\ge 0} I_\lambda(u_* + tu_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}} - D\lambda^2, \quad \text{for all} \quad \lambda \in (0, \Lambda_1).$$

**Proof.** From [8], one has

$$\begin{cases} |u_\varepsilon|_6^6 = |U|_6^6 + O(\varepsilon^3) = S^{\frac{3}{2}} + O(\varepsilon^3), \\ \|u_\varepsilon\|^2 = |\nabla U|_2^2 + O(\varepsilon) = S^{\frac{3}{2}} + O(\varepsilon), \\ |u_\varepsilon|_p^p = O(\varepsilon^{\frac{p}{2}}), \quad 1 \le p < 3. \end{cases} \tag{2.16}$$

It is obvious that the following inequality

$$(a+b)^6 \ge a^6 + b^6 + 6a^5 b + 6ab^5$$

holds for each $a, b \ge 0$. Since $u_*$ is a positive solution of equation (2.1) with $I_\lambda(u_*) < 0$, by the above inequality, for all $t \ge 0$ we have

$$\begin{aligned}
&I_\lambda(u_* + tu_\varepsilon) \\
=&I_\lambda(u_*) + \frac{1}{2}t^2\|u_\varepsilon\|^2 + t\int_{\mathbb{R}^3}\left[\nabla u_* \cdot \nabla u_\varepsilon + u_* u_\varepsilon + \eta l(x)\phi_{u_*} u_* u_\varepsilon - u_*^5 u_\varepsilon - \lambda f(x) u_\varepsilon\right] dx \\
&+ \frac{1}{4}\eta\int_{\mathbb{R}^3} l(x)\left[\phi_{u_*+tu_\varepsilon}(u_* + tu_\varepsilon)^2 - \phi_{u_*} u_*^2 - 4t\phi_{u_*} u_* u_\varepsilon\right] dx \\
&- \frac{1}{6}\int_{\mathbb{R}^3}(|u_* + tu_\varepsilon|^6 - |u|_*^6 - 6tu_*^5 u_\varepsilon)dx \\
\le&\frac{1}{2}t^2\|u_\varepsilon\|^2 - \frac{1}{6}t^6\int_{\mathbb{R}^3}|u_\varepsilon|^6 dx - t^5\int_{\mathbb{R}^3} u_*|u_\varepsilon|^5 dx + g_\varepsilon(t) \\
\le&\frac{1}{2}t^2\|u_\varepsilon\|^2 - \frac{1}{6}t^6\int_{\mathbb{R}^3}|u_\varepsilon|^6 dx - t^5 m\int_{\mathbb{R}^3}|u_\varepsilon|^5 dx + g_\varepsilon(t),
\end{aligned}$$

where

$$g_\varepsilon(t) = \frac{1}{4}\eta\int_{\mathbb{R}^3} l(x)\left[\phi_{u_*+tu_\varepsilon}(u_* + tu_\varepsilon)^2 - \phi_{u_*} u_*^2 - 4t\phi_{u_*} u_* u_\varepsilon\right] dx.$$

According to [12], we can get that

$$g_\varepsilon(t) \le Ct^2\varepsilon + Ct^3\varepsilon^{\frac{3}{2}} + Ct^4\varepsilon^2.$$

Set

$$h_\varepsilon(t) = \frac{1}{2}t^2\|u_\varepsilon\|^2 - \frac{1}{6}t^6\int_{\mathbb{R}^3}|u_\varepsilon|^6 dx - t^5 m\int_{\mathbb{R}^3}|u_\varepsilon|^5 dx + Ct^2\varepsilon + Ct^3\varepsilon^{\frac{3}{2}} + Ct^4\varepsilon^2.$$

Since $\lim\limits_{t\to+\infty} h_\varepsilon(t) = -\infty$ and $h_\varepsilon(0) = 0$, there exist $t_1, t_2 > 0$ such that

$$0 < t_1 \le t_\varepsilon \le t_2 < \infty \tag{2.17}$$

and

$$h_\varepsilon(t_\varepsilon) = \sup_{t\geq 0} h_\varepsilon(t), \quad h'_\varepsilon(t)|_{t=t_\varepsilon} = 0.$$

Note that

$$\int_{\mathbb{R}^3} |u_\varepsilon|^5 dx = C\varepsilon^{\frac{1}{2}} + O(\varepsilon^{\frac{5}{2}}),$$

it follows from (2.16) and (2.17) that

$$\begin{aligned}\sup_{t\geq 0} h_\varepsilon(t) &\leq \sup_{t\geq 0}\left\{\frac{1}{2}t^2 S^{\frac{3}{2}} - \frac{1}{6}t^6 S^{\frac{3}{2}}\right\} + C_1\varepsilon - C_2\varepsilon^{\frac{1}{2}} \\ &\leq \frac{1}{3}S^{\frac{3}{2}} + C_1\varepsilon - C_2\varepsilon^{\frac{1}{2}}\end{aligned}$$

where $C_1, C_2 > 0$ (independent of $\varepsilon, \lambda$). Let $\varepsilon = \lambda^2, 0 < \lambda < \Lambda_1 = \frac{C_2}{C_1+D}$, then we have that

$$\begin{aligned}C_1\varepsilon - C_2\varepsilon^{\frac{1}{2}} &= C_1\lambda^2 - C_2\lambda \\ &= \lambda^2(C_1 - C_2\lambda^{-1}) \\ &< -D\lambda^2\end{aligned}$$

which implies that $\sup_{t\geq 0} h_\varepsilon(t) < \frac{1}{3}S^{\frac{3}{2}} - D\lambda^2$ for all $\lambda \in (0, \Lambda_1)$. The proof is complete. □

**Theorem 2.2.** *Under the conditions of Theorem 1.1, system* (1.1) *has another positive solution* $(u^*, \phi_{u^*})$ *with* $I_\lambda(u^*) > 0$.

**Proof.** Let $\lambda^* = \min\{\Lambda_0, \Lambda_1, (\frac{S^{\frac{2}{3}}}{3D})^{\frac{1}{2}}\}$. By Lemma 2.4, we can choose a sufficiently large $T_0 > 0$ such that $I_\lambda(u_* + T_0 u_\varepsilon) < 0$, with the fact that $I_\lambda(u_*) < 0$. Then we apply the mountain-pass Lemma (see [4]) to obtain that there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$I_\lambda(u_n) \to c > 0 \text{ and } I'_\lambda(u_n) \to 0,$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) \mid \gamma(0) = u_*, \gamma(1) = u_* + T_0 u_\varepsilon\}.$$

By Lemma 2.3, there exists a convergent subsequence $\{u_n\}$ (still denoted by $\{u_n\}$) and $u^* \in H^1(\mathbb{R}^3)$ such that $u_n \to u^*$ in $H^1(\mathbb{R}^3)$, thus $u^*$ is a solution of equation (2.1). Since $I_\lambda(|u|) \leq I_\lambda(u)$, by Theorem 10 in [5], we can get $u^* \geq 0$ and $u_* \not\equiv 0$. By using the strong maximum principle, we have $u^* > 0$ in $\mathbb{R}^3$. Thus, $(u^*, \phi_{u^*})$ is a positive solution of system (1.1). The proof is complete. □

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