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STATIONARY DISTRIBUTION OF A LOTKA-VOLTERRA MODEL WITH STOCHASTIC PERTURBATIONS AND DISTRIBUTED DELAY

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Abstract This paper devotes to the existence of a stationary distribution for a one-prey and two-predator Lotka-Volterra model with stochastic nonlinear perturbations and distributed delay. The studied autonomous system is first proved having a unique global and positive solution. Then, through constructing appropriate Lyapunov function and using Itô formula, sufficient conditions guaranteeing the existence of a stationary distribution of the stochastic system are obtained. Some numerical simulations are also provided in the end to illustrate the main results.

Keywords Lotka-Volterra model, Itô formula, global solution, stationary distribution, distributed delay.

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1. Introduction

In recent years, the dynamic relationship of predator-prey for species has been extensively studied due to its universality and importance in mathematical biology and ecology (see [2, 4, 6, 13, 23, 33]). Meanwhile, it is also a common ecological phenomenon that two predators capture the same prey. Farkas and Freedman in [9] discussed persistence, extinction and global attractivity properties of the two-predator and one-prey system. Dubey and Upadhyay [8] analyzed the dynamics of a two-predator and one-prey system with a ratio-dependent growth rate. Alebraheem and Hasan [1] considered the existence of limit cycle of the three-species food chain model, and revealed different dynamics of the persistence and extinction of predators through numerical simulations. Particularly, in [11], Llibre and Xiao studied the global dynamics of the following 3-dimensional Lotka-Volterra models with two predators competing for a single prey species in a constant and uniform environment

$$\begin{cases} dx(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y_1(t) - a_{13}y_2(t)]dt, \\ dy_1(t) = y_1(t)[-r_2 + a_{21}x(t)]dt, \\ dy_2(t) = y_2(t)[-r_3 + a_{31}x(t)]dt, \end{cases} \tag{1.1}$$

where $x(t)$ is the prey population density, $y_1(t)$ and $y_2(t)$ are the predator population density at time t, respectively, r_1 stands for the intrinsic growth rate of species x,

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$\frac{1}{a_{11}}$ is the carrying capacity of the prey, a_{12} and a_{13} are effects of the $i-$th predation on the prey, respectively, r_2, r_3 denote the mortality rate of predators, a_{21} and a_{31} are the efficiency and propagation rates of the $i-$th predator in the presence of prey. Here all these parameters are positive constants. They characterized the qualitative behavior of the system (1.1) in two cases, one is the resource for prey is limited, the other is the resource for prey is unlimited. The obtained results there showed that there are only two coexistence styles for all three species: periodic oscillation or steady states, which depends on the resource for prey.

At the same time, the models with time delay reflect that the states before time t have frequently a heavy influence on the state at time t. Indeed, as pointed out by Kuang in [16], neglecting time delays means ignoring reality. So it is quite essential to take into account in biological models. It is especially of great interest to consider population models with distributed delays due to the complexity of the natural environment, see [5, 14, 18, 21, 38], for instance.

On the other hand, ecological systems are often influenced by various environmental noises, which inevitably affect the dynamics of populations. However, deterministic systems have many limitations in terms of describing population dynamic behavior. In the past years many scholars incorporated the white noise into the models to study the richer and more complex dynamic behaviors of the resulting stochastic differential systems, see [12,24,30,35]. Among them, most works only considered linear stochastic perturbations in the models, few are on nonlinear perturbations, see [19–21]. Moreover, it is well known that stochastic perturbations may destroy the stability of the equilibria existing in the deterministic systems and lead to a stochastic weak stability named stationary distribution. Recently, existence of stationary distributions, as an essential and important issue of stochastic systems, has also been widely investigated by many mathematicians, see [14, 34, 36, 37] and the references therein.

Motivated by the above works, in this paper, we mainly attempt to discuss the existence of a stationary distribution of a three-species Lotka-Volterra system with nonlinear stochastic perturbations and time delay. We assume that the intrinsic growth rate of the prey $x(t)$ and the mortality rate of predators $y_1(t)$, $y_2(t)$ in Eq. (1.1) are affected by white noise respectively in the form

$$\begin{aligned}&r_1 \to r_1 + (\sigma_{11} + \sigma_{12}x(t))\dot{B}_1(t), \quad -r_2 \to -r_2 + (\sigma_{21} + \sigma_{22}y_1(t))\dot{B}_2(t),\\ &-r_3 \to -r_3 + (\sigma_{31} + \sigma_{32}y_2(t))\dot{B}_3(t).\end{aligned}$$

Here $B_i(t)$ $(i = 1, 2, 3)$ are independent standard Brownian motions defined over the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual condition, i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. And $\sigma_{ij}^2 > 0$ $(i = 1, 2, 3, j = 1, 2)$ denote the intensity of the white noise. Precisely, we consider in this paper the stochastic differential system with distributed delay expressed as

$$\begin{cases} dx(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y_1(t) - a_{13}y_2(t)]dt + [\sigma_{11} + \sigma_{12}x(t)]x(t)dB_1(t),\\ dy_1(t) = y_1(t)\left[-r_2 + a_{21}\int_{-\infty}^{t} F(t-s)x(s)ds\right]dt + [\sigma_{21} + \sigma_{22}y_1(t)]y_1(t)dB_2(t),\\ dy_2(t) = y_2(t)\left[-r_3 + a_{31}\int_{-\infty}^{t} F(t-s)x(s)ds\right]dt + [\sigma_{31} + \sigma_{32}y_2(t)]y_2(t)dB_3(t). \end{cases} \tag{1.2}$$

with initial conditions $x(\theta) = \varphi(\theta) \in C_b((-\infty, 0), \mathbb{R}_+)$ and $(y_1(0), y_2(0)) = \left(y_1^0, y_2^0\right) \in$

$\mathbb{R}^2_+$, where the parameters a_{ij} are the same as in (1.1), the kernel $F : [0, \infty) \to [0, \infty)$ is a L^1-function normalized as $\int_0^\infty F(s)ds = 1$. MacDonald in [25] initially pointed out it is reasonable to take the Gamma distribution

$$F(s) = \frac{s^n \alpha^{n+1} e^{-\alpha s}}{n!}, \ s \in (0, \infty),$$

as a kernel function to represent the distribution delay. Here n is a nonnegative integer, $\alpha > 0$ denotes the rate of decay of effects of past memories and is also regarded as exponentially fading memory. In this paper, we take for brevity the weak kernel for $n = 0$, i.e., $F(s) = \alpha e^{-\alpha s}$. Note that the weak kernel and the strong kernel have been extensively used in biological systems, such as the population systems discussed in [7,18,21,29] and epidemiology in [3]. Such an infinite distributed delay means that the variation rate of the predators y_i or, say, the prodation rate at time t depends on the scale of the prey x at every moment in the past or whole history with weight (kernel) function $F(t)$. Up to now, the population models of three-species with finite and infinite delays have been much studied in literature, see [10,17,31]. However very few considered the three-species systems with infinite distributed delay.

To deal with the term of distribution delay in (1.2), we set

$$z(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} x(s) ds,$$

then by the linear chain technique, the model (1.2) is transformed into the following equivalent system

$$\begin{cases} dx(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y_1(t) - a_{13}y_2(t)]dt + [\sigma_{11} + \sigma_{12}x(t)]x(t)dB_1(t), \\ dy_1(t) = y_1(t)[-r_2 + a_{21}z(t)]dt + [\sigma_{21} + \sigma_{22}y_1(t)]y_1(t)dB_2(t), \\ dy_2(t) = y_2(t)[-r_3 + a_{31}z(t)]dt + [\sigma_{31} + \sigma_{32}y_2(t)]y_2(t)dB_3(t), \\ dz(t) = \alpha[x(t) - z(t)]dt. \end{cases} \tag{1.3}$$

As a result, in what follows we turn to study equivalently the degenerate system (1.3). That is, we shall discuss the existence of a stationary distribution for system (1.3). First, in the next section, by applying the Itô formula and using Lyapunov analysis method, we prove that there exists a global positive solution of the system (1.3) for any given initial (positive) value. Following that, in Section 3 we investigate and obtain the sufficient conditions for the existence of stationary distribution of the system (1.3) by employing the theory of stationary distributions developed by [15,32]. Finally, in Section 4, we provide some concrete numbers to the coefficients in (1.3) and do the corresponding numerical simulations to illustrate the obtained conclusions.

Let's mention here that, to the best of our knowledge, up to now there are very few similar results about the dynamical behaviors of three species stochastic Lotka-Volterra model with distributed delay. On the other hand, because the Fokker-Planck equation corresponding to system (1.3) is degenerate, as it can be seen later, it becomes much more difficult for us to construct an appropriate Lyapunov function in 4-dimensional space. Obviously, the work in this article extends and develops directly the existing results in [5,11,14,18,19] as well as the above mentioned results on this topic.

We now end this section by introducing a differential operator to be used throughout this paper. Consider a l-dimensional stochastic equation

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t), t \geq t_0, \tag{1.4}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^l$, where $f \in L^1(\mathbb{R}^l, \mathbb{R}_+)$, $g \in L^2(\mathbb{R}^{l\times m}, \mathbb{R}_+)$, and $B(t)$ is an m-dimensional standard Brownian motion. Denote by $C^{2,1}(\mathbb{R}^l \times \mathbb{R}_+; \mathbb{R}_+)$ the family of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^l \times \mathbb{R}_+$ which are twice continuously differentiable in x and once continuously differentiable in t. We define the differential operator $\mathscr{L}$ associated with (1.4) by

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{l} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l}\left[g^T(x,t)g(x,t)\right]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If $\mathscr{L}$ acts on a function $V \in C^{2,1}(\mathbb{R}_+ \times \mathbb{R}^l; \mathbb{R}_+)$, then

$$\mathscr{L}V(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\mathrm{tr}\Big(g^T(x,t)V_{xx}(x,t)g(x,t)\Big), \tag{1.5}$$

in which $V_t(x,t) = \frac{\partial V(x,t)}{\partial t}$, $V_x(x,t) = \Big(\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_l}\Big)$, $V_{xx}(x,t) = \Big(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\Big)_{l\times l}$, and $g^T(x,t)$ denotes the transpose of $g(x,t)$. Then from Itô formula it follows easily that, for $x(t) \in \mathbb{R}^l$,

$$dV(x(t),t) = \mathscr{L}V(x(t),t)dt + V_x\big(x(t),t\big)g\big(x(t),t\big)dB(t).$$

2. Existence of unique global positive solutions

In this section we focus on the existence and uniqueness of solutions of the system (1.3). From the view point of biology, the population density $x(t), y_1(t), y_2(t), z(t)$ should be nonnegative. Hence, we shall prove here that, for any initial values $X(0) = (x(0), y_1(0), y_2(0), z(0)) \in \mathbb{R}^4_+$, which is given by $\Big(\varphi(0), y_1^0, y_2^0, \int_{-\infty}^0 \alpha e^{\alpha s}\varphi(s)ds\Big)$ corresponding to the initial values $\big(\varphi(\cdot), y_1^0, y_2^0\big)$ of (1.2), there exists a unique global and positive solution $X(t) = (x(t), y_1(t), y_2(t), z(t))$ for system (1.3). Namely,

Theorem 2.1. *For any initial value $X(0) \in \mathbb{R}^4_+$, the system (1.3) has a unique positive solution $X(t)$ almost surely (which is a Markov process).*

Proof. Since all the nonlinear terms in system (1.3) satisfy the locally Lipschitz condition, by the existence and uniqueness theorem of solutions for stochastic differential equations, there exists a unique maximal local solution $X(t)$ $\big(t \in [0, \tau_e)\big)$ for any initial value $X(0) \in \mathbb{R}^4_+$, where τ_e is the explosion time. In order to prove that system (1.3) has a global solution, it is sufficient to show $\tau_e = +\infty$ a.s..

Let $k_0 > 0$ be sufficiently large so that every component of $X(0)$ lies within the interval $\Big[\frac{1}{k_0}, k_0\Big]$. For any integer $k \geq k_0$, we define the stopping time

$$\begin{aligned}\tau_k = \inf\Bigg\{& t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{k}, k\right) \text{ or } y_1(t) \notin \left(\frac{1}{k}, k\right) \text{ or } y_2(t) \notin \left(\frac{1}{k}, k\right) \\ & \text{or } z(t) \notin \left(\frac{1}{k}, k\right)\Bigg\},\end{aligned}$$

here we set $\inf\emptyset = +\infty$. Then it is easily known that τ_k is increasing as $k \to +\infty$ and clearly $\tau_\infty := \lim_{k\to+\infty} \tau_k \le \tau_e$, *a.s.*. Therefore, we only need to verify $\tau_\infty = \infty$ *a.s.* in the sequel. If it is false, then there exist two constants $T > 0$ and $\gamma \in (0,1)$ such that

$$P\{\tau_\infty \le T\} > \gamma.$$

Thus there exists an integer $k_1 \ge k_0$ satisfying

$$P\{\tau_k \le T\} \ge \gamma, \quad k \ge k_1. \tag{2.1}$$

Next we derive a contradiction via the method of Lyapunov function. To do so, we define a C^2-function $V_1 : \mathbb{R}_+^4 \to \mathbb{R}_+$ as

$$V_1(x, y_1, y_2, z) = 2\sqrt{x} - \ln x + 2\sqrt{y_1} - \ln y_1 + 2\sqrt{y_2} - \ln y_2 + \frac{a_{21} + a_{31}}{\alpha} z^2.$$

Then the nonnegativity of this function follows immediately from the inequality $2\sqrt{u} - \ln u \ge 0$, for $u > 0$. Applying Itô formula to $V_1(x, y_1, y_2, z)$ we obtain that

$$\begin{aligned} dV_1(x, y_1, y_2, z) =& \mathscr{L}V_1(x, y_1, y_2, z)dt + \left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)(\sigma_{11} + \sigma_{12}x)x dB_1(t) \\ &+ \left(\frac{1}{\sqrt{y_1}} - \frac{1}{y_1}\right)(\sigma_{21} + \sigma_{22}y_1)y_1 dB_2(t) \\ &+ \left(\frac{1}{\sqrt{y_2}} - \frac{1}{y_2}\right)(\sigma_{31} + \sigma_{32}y_2)y_2 dB_3(t), \end{aligned}$$

where (by (1.5))

$$\begin{aligned} &\mathscr{L}V_1(x, y_1, y_2, z) \\ =& \left(x^{\frac{1}{2}} - 1\right)(r_1 - a_{11}x - a_{12}y_1 - a_{13}y_2) - \frac{1}{4}x^{\frac{1}{2}}(\sigma_{11} + \sigma_{12}x)^2 \\ &+ \frac{1}{2}(\sigma_{11} + \sigma_{12}x)^2 + 2(a_{21} + a_{31})z(x - z) \\ &+ \left(y_1^{\frac{1}{2}} - 1\right)(-r_2 + a_{21}z) - \frac{1}{4}y_1^{\frac{1}{2}}(\sigma_{21} + \sigma_{22}y_1)^2 + \frac{1}{2}(\sigma_{21} + \sigma_{22}y_1)^2 \\ &+ \left(y_2^{\frac{1}{2}} - 1\right)(-r_3 + a_{31}z) - \frac{1}{4}y_2^{\frac{1}{2}}(\sigma_{31} + \sigma_{32}y_2)^2 + \frac{1}{2}(\sigma_{31} + \sigma_{32}y_2)^2 \\ \le& -\frac{\sigma_{12}^2}{4}x^{\frac{5}{2}} + (a_{21} + a_{31} + \sigma_{12}^2)x^2 + a_{11}x + r_1 x^{\frac{1}{2}} - r_1 + \sigma_{11}^2 \\ &- \frac{\sigma_{22}^2}{4}y_1^{\frac{5}{2}} + \sigma_{22}^2 y_1^2 + a_{12}y_1 + a_{21}zy_1^{\frac{1}{2}} + r_2 + \sigma_{21}^2 + r_3 + \sigma_{31}^2 \\ &- \frac{\sigma_{32}^2}{4}y_2^{\frac{5}{2}} + \sigma_{32}^2 y_2^2 + a_{13}y_2 + a_{31}zy_2^{\frac{1}{2}} - (a_{21} + a_{31})z^2. \end{aligned}$$

Obviously, there exists a constant $K_1 > 0$ satisfying $\mathscr{L}V_1(x, y_1, y_2, z) \le K_1$. So

$$\begin{aligned} &dV_1(x, y_1, y_2, z) \\ \le& K_1 dt + \left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)(\sigma_{11} + \sigma_{12}x)x dB_1(t) + \left(\frac{1}{\sqrt{y_1}} - \frac{1}{y_1}\right)(\sigma_{21} + \sigma_{22}y_1)y_1 dB_2(t) \\ &+ \left(\frac{1}{\sqrt{y_2}} - \frac{1}{y_2}\right)(\sigma_{31} + \sigma_{32}y_2)y_2 dB_3(t). \end{aligned}$$

Integrating from 0 to $\tau_k \wedge T$ on both sides of this inequality and then taking the expectations lead to

$$\begin{aligned}&\mathbb{E}\left[V_1(x(\tau_k \wedge T), y_1(\tau_k \wedge T), y_2(\tau_k \wedge T), z(\tau_k \wedge T))\right]\\ \leq& V_1(x(0), y_1(0), y_2(0), z(0)) + K_1 T.\end{aligned}$$

Note that for every $\omega \in \{\tau_k \leq T\}$, there is at least one of $x(\tau_k, \omega)$, $y_1(\tau_k, \omega)$, $y_2(\tau_k, \omega)$ and $z(\tau_k, \omega)$ equal to k or $\frac{1}{k}$. Hence,

$$V_1(x(\tau_k, \omega), y_1(\tau_k, \omega), y_2(\tau_k, \omega), z(\tau_k, \omega)) \geq \min\left\{2\sqrt{k} - \ln k, 2\sqrt{\frac{1}{k}} - \ln\frac{1}{k}\right\}.$$

Consequently, due to (2.1),

$$\begin{aligned}&V_1(x(0), y_1(0), y_2(0), z(0)) + K_1 T\\ \geq& \mathbb{E}\Big[1_{\{\tau_k \leq T\}}(\omega) V_1\Big(x(\tau_k, \omega), y_1(\tau_k, \omega), y_2(\tau_k, \omega), z(\tau_k, \omega)\Big)\Big]\\ \geq& \gamma \min\left\{2\sqrt{k} - \ln k, 2\sqrt{\frac{1}{k}} - \ln\frac{1}{k}\right\},\end{aligned}$$

where $1_{\{\tau_k \leq T\}}(\omega)$ is the indicator function of $\{\tau_k \leq T\}$. Then letting $k \to +\infty$ results in the following contradiction

$$+\infty > V_1(x(0), y_1(0), y_2(0), z(0)) + K_1 T = +\infty.$$

Therefore we infer that $\tau_e = \infty$ a.s. and the proof is completed. □

3. Existence of the stationary distribution

Based on the discussion of the previous section, we investigate in this part the existence of a stationary distribution of the solution for system (1.3). Let $P_{x_0,t}(\cdot)$ denote the probability measure induced by $x(t)$ with initial value $x(0) = x_0$, that is

$$P_{x_0,t}(A) = \mathbb{P}(x(t) \in A), \quad A \in \mathscr{B}(\mathbb{R}^n_+),$$

where $\mathscr{B}(\mathbb{R}^n_+)$ is the σ-algebra of all the Borel sets $A \subseteq \mathbb{R}^n_+$. If there is a probability measure $\mu(\cdot)$ on the measurable space $(\mathbb{R}^n_+, \mathscr{B}(\mathbb{R}^n_+))$ such that

$$P_{x_0,t}(\cdot) \to \mu(\cdot) \ \text{ in distribution for any } x_0 \in \mathbb{R}^n_+,$$

we then say that the SDE model (1.3) has a stationary distribution $\mu(\cdot)$ (see e.g. [4, 15, 26, 27]).

Before establishing the main result of this section, here we still need to state a lemma which comes from [15] and will paly a crucial role in our later discussion.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^l$ satisfying the following stochastic integral equation (which can be interpreted as an integral form of eq.(1.4))

$$X(t) = X(t_0) + \int_{t_0}^{t} f(X(s), s)ds + \sum_{i=1}^{k} \int_{t_0}^{t} g_i(X(s), s)dB_i(s), \tag{3.1}$$

where $f(\cdot, \cdot)$ and $g_i(\cdot, \cdot) : \mathbb{R}^l \times \mathbb{R}_+ \to \mathbb{R}^l$ are vector functions.

Lemma 3.1. *Suppose that the coefficients of* (3.1) *are independent of* t *and satisfy that*

(i) For any $R > 0$*, there is a constant* $K > 0$ *such that*

$$|f(x,s) - f(y,s)| + \sum_{r=1}^{k} |g_i(x,s) - g_i(y,s)| \le K|x-y|, \tag{3.2}$$

$$|f(x,s)| + \sum_{r=1}^{k} |g_i(x,s)| \le K(1+|x|), \tag{3.3}$$

for $x, y \in U_R := \{u \in \mathbb{R}^l, |u| < R\}$.

(ii) There exists a nonnegative C^2*-function* $V(x)$ *in* $\mathbb{R}^l_+$ *such that*

$$\mathscr{L}V(x) \le -1 \text{ outside some compact set.}$$

Then the system (3.1) *exists a solution which is a stationary distribution.*

Remark 3.1. According to Remark 5 in Xu [32], for the equation (3.1) the conditions (3.2) and (3.3) of Lemma 3.1 can be replaced by the global existence of the solutions, which was just well proved in Theorem 2.1.

Now we are in the position to present the main result of this part.

Theorem 3.1. *Assume that*

$$r_2 + \frac{\sigma_{21}^2}{4} - \frac{(2a_{12} + a_{21})^2}{8\sigma_{21}\sigma_{22}} > 0, \tag{3.4}$$

$$r_3 + \frac{\sigma_{31}^2}{4} - \frac{(2a_{13} + a_{31})^2}{8\sigma_{31}\sigma_{32}} > 0, \tag{3.5}$$

$$\frac{2a_{11} - 2\sigma_{12}^2 - a_{21} - a_{31} - \sqrt{k^*}}{\sigma_{12}^2 - r_1} > 0 \tag{3.6}$$

and

$$\eta := r_1 - \sigma_{11}^2 - \frac{1}{4} - \left(\frac{4}{5}\right)^{\frac{5}{4}} \frac{\sigma_{12}^2(2a_{11} - 2\sigma_{12}^2 - a_{21} - a_{31} - \sqrt{k^*})}{\sigma_{12}^2 - r_1} > 0, \tag{3.7}$$

where $k^* := 2a_{11} - r_1 - 3\sigma_{12}^2 - \frac{3(a_{21}+a_{31})}{2} > 0$*. Then for any initial value* $X(0) \in \mathbb{R}^4_+$*, there exists a stationary distribution* $\mu(\cdot)$ *for the system* (1.3).

Proof. As mentioned above, we prove this result by applying Lemma 3.1, namely, we only need to construct a nonnegative C^2-function $V(x, y_1, y_2, z)$ and a closed set $U \subset \mathbb{R}^4_+$ such that

$$\mathscr{L}V(x, y_1, y_2, z) \le -1, \text{ for any } (x, y_1, y_2, z) \in \mathbb{R}^4_+ \setminus U.$$

To do this, we first define a C^2-function $V_{21} : \mathbb{R}^4_+ \to \mathbb{R}$ as

$$V_{21}(x, y_1, y_2, z) = 2k_1 x^{\frac{1}{2}} + x - \ln x + 2y_1^{\frac{1}{2}} + 2y_2^{\frac{1}{2}} + \frac{k_2}{\alpha} z^2 + \frac{k^*}{\alpha} z - \frac{1}{4\alpha} \ln z,$$

where k_1 and k_2 will be determined later. Applying Itô formula to V_{21} yields that

$\mathscr{L}V_{21}(x, y_1, y_2, z)$

$$
\begin{aligned}
&= \left(k_1 x^{\frac{1}{2}} + x - 1\right)(r_1 - a_{11}x - a_{12}y_1 - a_{13}y_2) + \left(-\frac{k_1}{4}x^{\frac{1}{2}} + \frac{1}{2}\right)(\sigma_{11} + \sigma_{12}x)^2 \\
&\quad + y_1^{\frac{1}{2}}(-r_2 + a_{21}z) - \frac{1}{4}y_1^{\frac{1}{2}}(\sigma_{21} + \sigma_{22}y_1)^2 + y_2^{\frac{1}{2}}(-r_3 + a_{31}z) - \frac{1}{4}y_2^{\frac{1}{2}}(\sigma_{31} + \sigma_{32}y_2)^2 \\
&\quad + 2k_2 z(x - z) + k^*(x - z) - \frac{1}{4z}(x - z) \\
&\le k_1 r_1 x^{\frac{1}{2}} + r_1 x - a_{11}x^2 - r_1 + a_{11}x + a_{12}y_1 + a_{13}y_2 - \frac{k_1\sigma_{12}^2}{4}x^{\frac{5}{2}} + \sigma_{12}^2 x^2 + \sigma_{11}^2 \\
&\quad - r_2 y_1^{\frac{1}{2}} + a_{21}z y_1^{\frac{1}{2}} - \frac{\sigma_{22}^2}{4}y_1^{\frac{5}{2}} - \frac{1}{4}y_1^{\frac{1}{2}}(\sigma_{21}^2 + 2\sigma_{21}\sigma_{22}y_1) - r_3 y_2^{\frac{1}{2}} + a_{31}z y_2^{\frac{1}{2}} - \frac{\sigma_{32}^2}{4}y_2^{\frac{5}{2}} \\
&\quad - \frac{1}{4}y_2^{\frac{1}{2}}(\sigma_{31}^2 + 2\sigma_{31}\sigma_{32}y_2) + k_2 x^2 - k_2 z^2 + k^* x - k^* z - \frac{x}{4z} + \frac{1}{4} \\
&\le -\frac{k_1\sigma_{12}^2}{4}x^{\frac{5}{2}} - (a_{11} - \sigma_{12}^2 - k_2)x^2 + (r_1 + a_{11} + k^*)x + k_1 r_1 x^{\frac{1}{2}} - \sqrt{k^*}x^{\frac{1}{2}} \\
&\quad - \frac{\sigma_{22}^2}{4}y_1^{\frac{5}{2}} - r_2 y_1^{\frac{1}{2}} + \frac{a_{21} + 2a_{12}}{2}y_1 - y_1^{\frac{1}{2}}\left[\frac{\sigma_{21}\sigma_{22}}{2}y_1 + \frac{\sigma_{21}^2}{4}\right] - r_1 + \sigma_{11}^2 + \frac{1}{4} \\
&\quad - \frac{\sigma_{32}^2}{4}y_2^{\frac{5}{2}} - r_3 y_2^{\frac{1}{2}} + \frac{a_{31} + 2a_{13}}{2}y_2 - y_2^{\frac{1}{2}}\left[\frac{\sigma_{31}\sigma_{32}}{2}y_2 + \frac{\sigma_{31}^2}{4}\right] + \left(\frac{a_{21} + a_{31}}{2} - k_2\right)z^2 \\
&= -\frac{k_1\sigma_{12}^2}{4}x^{\frac{5}{2}} + (a_{11} - \sigma_{12}^2 - k_2)\left(-x^2 + \frac{r_1 + a_{11} + k^*}{a_{11} - \sigma_{12}^2 - k_2}x\right) + \left(k_1 r_1 - \sqrt{k^*}\right)x^{\frac{1}{2}} \\
&\quad - y_1^{\frac{1}{2}}\left[\frac{\sigma_{21}\sigma_{22}}{2}\left(y_1 - \frac{a_{21} + 2a_{12}}{2\sigma_{21}\sigma_{22}}\right)^2 - \frac{(2a_{12} + a_{21})^2}{8\sigma_{21}\sigma_{22}} + \frac{\sigma_{21}^2}{4} + r_2\right] \\
&\quad - y_2^{\frac{1}{2}}\left[\frac{\sigma_{31}\sigma_{32}}{2}\left(y_2 - \frac{a_{31} + 2a_{13}}{2\sigma_{31}\sigma_{32}}\right)^2 - \frac{(2a_{13} + a_{31})^2}{8\sigma_{31}\sigma_{32}} + \frac{\sigma_{31}^2}{4} + r_3\right] \\
&\quad + \left(\frac{a_{21} + a_{31}}{2} - k_2\right)z^2 - r_1 + \sigma_{11}^2 + \frac{1}{4}.
\end{aligned}
$$

Take here $k_2 = \frac{a_{21}+a_{31}}{2}$, then it follows from (3.4) and (3.5) that

$$
\begin{aligned}
&\mathscr{L}V_{21}(x, y_1, y_2, z) \\
&\le -\frac{k_1\sigma_{12}^2}{4}x^{\frac{5}{2}} + (a_{11} - \sigma_{12}^2 - k_2)\left(-x^2 + \frac{r_1 + a_{11} + k^*}{a_{11} - \sigma_{12}^2 - k_2}x\right) + \left(k_1 r_1 - \sqrt{k^*}\right)x^{\frac{1}{2}} \\
&\quad - r_1 + \sigma_{11}^2 + \frac{1}{4} \\
&:= f(x) - r_1 + \sigma_{11}^2 + \frac{1}{4},
\end{aligned}
$$

where

$$
f(x) = -\frac{k_1\sigma_{12}^2}{4}x^{\frac{5}{2}} + (a_{11} - \sigma_{12}^2 - k_2)\left(-x^2 + \frac{r_1 + a_{11} + k^*}{a_{11} - \sigma_{12}^2 - k_2}x\right) + \left(k_1 r_1 - \sqrt{k^*}\right)x^{\frac{1}{2}}. \quad (3.8)
$$

Next we estimate the function $f(x)$. Since it is assumed $k^* = 2a_{11} - r_1 - 3\sigma_{12}^2 - \frac{3(a_{21}+a_{31})}{2}$ (> 0), we get easily that $\frac{r_1+a_{11}+k^*}{a_{11}-\sigma_{12}^2-k_2} = 3$. Thus, making use of the inequality (see [22] Lemma 4.2)

$$
-x^2 + 3x \le 2\sqrt{x}, \quad \text{for} \quad x \ge 0,
$$

we deduce that

$$\begin{aligned}
&(a_{11}-\sigma_{12}^2-k_2)\left(-x^2+\frac{r_1+a_{11}+k^*}{a_{11}-\sigma_{12}^2-k_2}x\right)\\
=&(a_{11}-\sigma_{12}^2-k_2)\left(-x^2+3x\right)\\
\leq&2(a_{11}-\sigma_{12}^2-k_2)x^{\frac{1}{2}},
\end{aligned}$$

from which (3.8) implies

$$f(x)\leq-\frac{k_1\sigma_{12}^2}{4}x^{\frac{5}{2}}+\left(k_1r_1+2(a_{11}-\sigma_{12}^2-k_2)-\sqrt{k^*}\right)x^{\frac{1}{2}}. \tag{3.9}$$

Now we consider the function $g(v):=c_1v^{\frac{1}{2}}-c_2v^{\frac{5}{2}}$ with $c_1,\ c_2>0$, for $v>0$. It is easy to show that $g(v)$ reaches its maximum at $v^*=\left(\frac{c_1}{5c_2}\right)^{\frac{1}{2}}$ on $(0,+\infty)$, i.e. $\max\limits_{v>0}g(v)=\frac{4c_1}{5}\left(\frac{c_1}{5c_2}\right)^{\frac{1}{4}}$. Applying this fact to (3.9) we further obtain that

$$f(x)\leq\left(\frac{4}{5}\right)^{\frac{5}{4}}\left(k_1r_1+2(a_{11}-\sigma_{12}^2-k_2)-\sqrt{k^*}\right)\cdot\left(\frac{k_1r_1+2(a_{11}-\sigma_{12}^2-k_2)-\sqrt{k^*}}{k_1\sigma_{12}^2}\right)^{\frac{1}{4}}.$$

We choose

$$k_1=\frac{2(a_{11}-\sigma_{12}^2-k_2)-\sqrt{k^*}}{\sigma_{12}^2-r_1}=\frac{2a_{11}-2\sigma_{12}^2-a_{21}-a_{31}-\sqrt{k^*}}{\sigma_{12}^2-r_1}>0\quad\text{(due to(3.6))}$$

so that $\left(\frac{k_1r_1+2(a_{11}-\sigma_{12}^2-k_2)-\sqrt{k^*}}{k_1\sigma_{12}^2}\right)^{\frac{1}{4}}=1$, then we have

$$\begin{aligned}
f(x)&\leq\left(\frac{4}{5}\right)^{\frac{5}{4}}\frac{\sigma_{12}^2(2(a_{11}-\sigma_{12}^2-k_2)-\sqrt{k^*})}{\sigma_{12}^2-r_1}\\
&=\left(\frac{4}{5}\right)^{\frac{5}{4}}\frac{\sigma_{12}^2(2a_{11}-2\sigma_{12}^2-a_{21}-a_{31}-\sqrt{k^*})}{\sigma_{12}^2-r_1}.
\end{aligned}$$

Therefore, we return to the estimate of $\mathscr{L}V_{21}$ to obtain that

$$\begin{aligned}
&\mathscr{L}V_{21}(x,y_1,y_2,z)\\
\leq&\left(\frac{4}{5}\right)^{\frac{5}{4}}\frac{\sigma_{12}^2(2a_{11}-2\sigma_{12}^2-a_{21}-a_{31}-\sqrt{k^*})}{\sigma_{12}^2-r_1}-r_1+\sigma_{11}^2+\frac{1}{4}\\
=&-\eta.\quad\text{(by(3.7))}
\end{aligned}\tag{3.10}$$

We then introduce the function

$$V_{22}(x,y_1,y_2,z)=\frac{1}{\theta}(x^\theta+y_1^\theta+y_2^\theta)+\frac{a_{21}+a_{31}}{\alpha}z^2-\ln z,\quad\theta\in(0,1).$$

Also conducting Itô formula to V_{22} we see

$$\begin{aligned}
&\mathscr{L}V_{22}(x,y_1,y_2,z)\\
=&x^\theta(r_1-a_{11}x-a_{12}y_1-a_{13}y_2)+y_1^\theta(-r_2+a_{21}z)+y_2^\theta(-r_3+a_{31}z)
\end{aligned}$$

$$
\begin{aligned}
&+2(a_{21}+a_{31})xz-2(a_{21}+a_{31})z^2-\frac{1}{2}(1-\theta)x^\theta(\sigma_{11}+\sigma_{12}x)^2\\
&-\frac{1}{2}(1-\theta)y_1^\theta(\sigma_{21}+\sigma_{22}y_1)^2-\frac{1}{2}(1-\theta)y_2^\theta(\sigma_{31}+\sigma_{32}y_1)^2-\frac{\alpha x}{z}+\alpha\\
\le&-\frac{(1-\theta)\sigma_{12}^2}{2}x^{\theta+2}-\frac{(1-\theta)\sigma_{22}^2}{2}y_1^{\theta+2}-\frac{(1-\theta)\sigma_{32}^2}{2}y_2^{\theta+2}+r_1x^\theta+a_{21}zy_1^\theta+a_{31}zy_2^\theta\\
&+(a_{21}+a_{31})x^2-(a_{21}+a_{31})z^2-\frac{\alpha x}{z}+\alpha. \qquad (3.11)
\end{aligned}
$$

Now, let

$$\tilde{V}_2(x,y_1,y_2,z)=MV_{21}(x,y_1,y_2)+V_{22}(x,y_1,y_2,z),$$

where $M>0$ is a constant will be determined below. Observe that $\tilde{V}_2(x,y_1,y_2,z)$ is continuous and tends to $+\infty$ as (x,y_1,y_2,z) goes to the boundary of $\mathbb{R}_+^4$. Hence $\tilde{V}_2(x,y_1,y_2,z)$ reaches its minimum in the interior of $\mathbb{R}_+^4$, which is denoted by $\tilde{V}_2(x^*,y_1^*,y_2^*,z^*)$. Hence we can define the nonnegative Lyapunov function $V_2:\mathbb{R}_+^4\to\mathbb{R}_+$ as

$$
\begin{aligned}
V_2(x,y_1,y_2,z)&=\tilde{V}_2(x,y_1,y_2,z)-\tilde{V}_2(x^*,y_1^*,y_2^*,z^*)\\
&=MV_{21}(x,y_1,y_2)+V_{22}(x,y_1,y_2,z)-\tilde{V}_2(x^*,y_1^*,y_2^*,z^*).
\end{aligned}
$$

Combining (3.10) and (3.11), we then find that

$$
\begin{aligned}
&\mathscr{L}V_2(x,y_1,y_2,z)\\
\le&-M\eta-\frac{(1-\theta)\sigma_{12}^2}{2}x^{\theta+2}-\frac{(1-\theta)\sigma_{22}^2}{2}y_1^{\theta+2}-\frac{(1-\theta)\sigma_{32}^2}{2}y_2^{\theta+2}+r_1x^\theta+a_{21}zy_1^\theta\\
&+a_{31}zy_2^\theta+(a_{21}+a_{31})x^2-(a_{21}+a_{31})z^2-\frac{\alpha x}{z}+\alpha\\
\le&-\frac{(1-\theta)\sigma_{12}^2}{4}x^{\theta+2}-\frac{(1-\theta)\sigma_{22}^2}{4}y_1^{\theta+2}-\frac{(1-\theta)\sigma_{32}^2}{4}y_2^{\theta+2}-\frac{a_{21}+a_{31}}{3}z^2\\
&-\frac{\alpha x}{z}-M\eta+A\\
:=&G(x,y_1,y_2,z),
\end{aligned}
$$

where

$$
\begin{aligned}
A=\max_{(x,y_1,y_2,z)\in\mathbb{R}_+^4}\Bigg\{&-\frac{(1-\theta)\sigma_{12}^2}{4}x^{\theta+2}-\frac{(1-\theta)\sigma_{22}^2}{4}y_1^{\theta+2}-\frac{(1-\theta)\sigma_{32}^2}{4}y_2^{\theta+2}\\
&-\frac{2(a_{21}+a_{31})}{3}z^2+(a_{21}+a_{31})x^2+r_1x^\theta+a_{21}zy_1^\theta+a_{31}zy_2^\theta+\alpha\Bigg\}.
\end{aligned}
$$

Clearly $A<+\infty$ and so we are able to take $M>0$ so that $-M\eta+A\le-1$. Thus, from the expression of G it follows that

$$
G(x,y_1,y_2,z)\begin{cases}
=-\infty, & x\to+\infty,\\
=-\infty, & y_1\to+\infty,\\
=-\infty, & y_2\to+\infty,\\
=-\infty, & z\to+\infty,\\
=-\infty, & z\to0^+,\\
\le-M\eta+A\le-1, & x\to0^+,\\
\le-M\eta+A\le-1, & y_1\to0^+,\\
\le-M\eta+A\le-1, & y_2\to0^+.
\end{cases}
$$

As a result, there is a closed set $U \subset \mathbb{R}_+^4$ such that $\mathscr{L}V_2 \leq -1$ on $\mathbb{R}_+^4 \setminus U$. Consequently, by virtue of Lemma 3.1, we conclude that the system (1.3) has a stationary distribution $\mu(\cdot)$. The proof is completed. $\square$

4. Numerical simulations

In this section, we give some numerical simulations to illustrate the obtained results. By using the Eular-Maruyama Method developed in [28], we mainly observe the relations and differences between the solutions of the considered deterministic and stochastic systems by simulating their solutions.

For the system (1.3), let $r_1 = 0.2$, $r_2 = 1$, $r_3 = 1$, $a_{11} = 0.26$, $a_{12} = 0.1$, $a_{13} = 0.1$, $a_{21} = 0.01$, $a_{31} = 0.01$. Then we obtain by direct computation that the corresponding deterministic system of (1.3) has a positive equilibrium point $\bar{X} = (0.6962, 1.10937, 0.1331, 0.9709)$. Next, we explore the effects of the stochastic fluctuation of the environment white noise on population density.

We take $\sigma_{11} = 0.23$, $\sigma_{12} = 0.01$, $\sigma_{21} = 0.05$, $\sigma_{22} = 0.06$, $\sigma_{31} = 0.05$, $\sigma_{32} = 0.06$, we then have $k^* = 0.0997 > 0$, $r_2 + \frac{\sigma_{21}^2}{4} - \frac{(2a_{12}+a_{21})^2}{8\sigma_{21}\sigma_{22}} = 0.8506 > 0$, $r_3 + \frac{\sigma_{31}^2}{4} - \frac{(2a_{13}+a_{31})^2}{8\sigma_{31}\sigma_{32}} = 0.8506 > 0$, $\frac{2(a_{11}-\sigma_{12}^2-k_2)-\sqrt{k^*}}{\sigma_{12}^2-r_1} = 0.1923 > 0$ and

$$\eta = r_1 - \sigma_{11}^2 - \frac{1}{2} - \left(\frac{4}{5}\right)^{\frac{5}{4}} \frac{\sigma_{12}^2(2a_{11} - 2\sigma_{12}^2 - a_{21} - a_{31} - \sqrt{k^*})}{\sigma_{12}^2 - r_1} = 0.0031 > 0.$$

Thus, all conditions of Theorem 3.1 are fulfilled and consequently the system (1.3) admits a stationary distribution, which is well illustrated in Fig.1 below.

The left diagrams of Fig.1 represent the solution $x(t)$, $y_1(t)$, $y_2(t)$ of the stochastic system (1.3) and of the corresponding deterministic system respectively. We can find that after some initial transients the population density fluctuates near the deterministic steady state values $\bar{x} = 0.6962$, $\bar{y_1} = 1.0937$, $\bar{y_2} = 0.1331$, respectively.

The right diagrams are the probability density functions of the prey $x(t)$ and the predators $y_1(t)$, $y_2(t)$, respectively. It is clear that they are distributed normally around the mean values 0.6962, 1.0937 and 0.1331 separately. This indicates that the system may still remains some stability if the intensity of the white noises are relatively small.

5. Conclusion

In this work we study a one-prey and two-predator Lotka-Volterra model with stochastic nonlinear perturbations and distributed delay. We first establish the result of existence and uniqueness of global positive solutions for this system. Based on this result we then explore the existence of stationary distributions for the considered stochastic system and achieve successfully some sufficient conditions on it (Theorem 3.1), which is the main result of this note. Meanwhile, we also present some numerical simulations to support the obtained results.

The obtained theoretical result here and the numerical simulations reveal that the dynamics of the system depends apparently the vital rates appearing in it. In particular, white noises may affect directly the dynamical behaviors of this stochastic Lotka-Volterra model. Actually, as manifested exactly by the result of Theorem

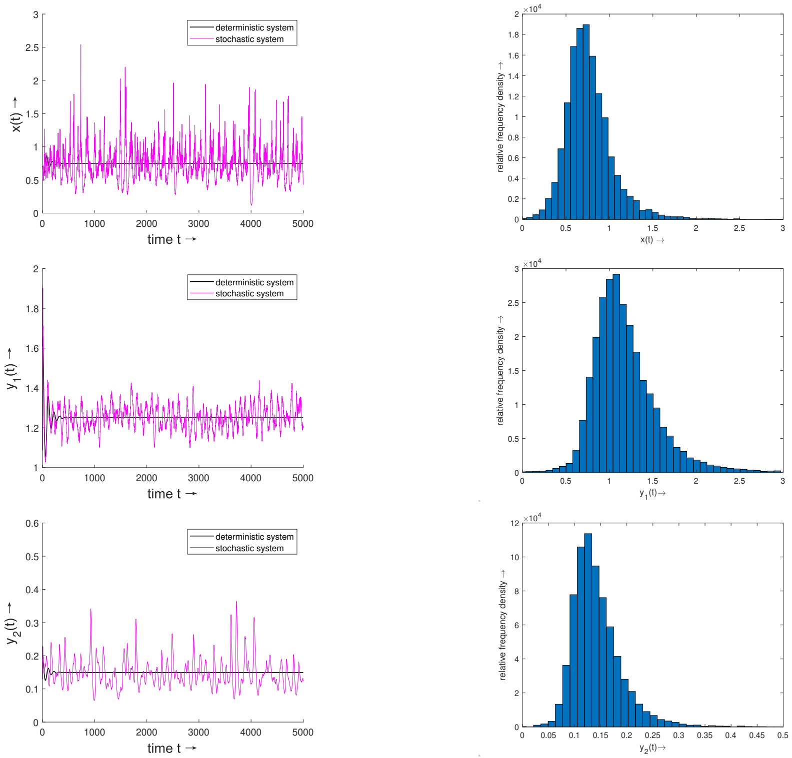

Figure 1. The left is the solution of the stochastic system (1.3) and the corresponding deterministic system. The right is the density function of $x(t)$, $y_1(t)$ and $y_2(t)$ for the system (1.3) obtained by 10,000 simulations running at $t = 200$.

3.1 and the numerical simulations above, if the intensity of white noise in the system is relatively small, the population of the species can maintain a certain stability (stochastic weak stability), which is conducive to the survival and development of the species. If, however, the intensity of white noise is relatively large, its impact on species for the system is somewhat heavy and even harmful, and some may possibly lead to species extinction, for which we are going to further study in the next work.

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