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SYNCHRONIZATION OF THE RÖSSLER-LORENZ SYSTEMS WITH FRACTIONAL BROWNIAN MOTION*

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Abstract The current paper is devoted to the dynamics of stochastic chaotic systems with fractional Brownian motion with $H \in (\frac{1}{2}, 1)$. The existence and uniqueness of the so-called stochastic Rössler-Lorenz system driven by fractional Brownian motion is established. Moreover, the stochastic synchronization of stochastic Rössler-Lorenz system is proved, and some numerical simulations are provided to verify the theoretical results.

Keywords Fractional Brownian motion, Rössler-Lorenz system, synchronization.

MSC(2010) 60H15, 37L55.

1. Introduction

In the present paper, we investigate the so-called Rössler-Lorenz system driven by fractional Brownian motion (fBm)

$$\begin{cases} dx(t) = [\sigma(y-x) + \alpha_3(t)z(y-z)]dt + g_1(x,y,z)dB_t^{H_1}, \\ dy(t) = [rx - y - \alpha_1(t)xz + x + ay]dt + g_2(x,y,z)dB_t^{H_2}, \\ dz(t) = [\alpha_2(t)xy - \beta z + b + \alpha_3(t)x(z-c)]dt + g_3(x,y,z)dB_t^{H_3}, \end{cases} \tag{1.1}$$

where $B_t^{H_i} (i = 1, 2, 3)$ is fractional Brownian motion in the defined complete probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. In case of $H_i = \frac{1}{2}$, and the noise intensity $g_i(x,y,z)$ is replaced by $\sum_{j=1}^{3} u_{ij}(x,y,z)$, then the stochastic system (1.1) is simplified to the following stochastic so-called Rössler-Lorenz system

$$\begin{cases} dx(t) = [\sigma(y-x) + \alpha_3(t)z(y-z)]dt + \sum\limits_{i=1}^{3} u_{1i}(x,y,z)dB_i(t), \\ dy(t) = [rx - y - \alpha_1(t)xz + x + ay]dt + \sum\limits_{i=1}^{3} u_{2i}(x,y,z)dB_i(t), \\ dz(t) = [\alpha_2(t)xy - \beta z + b + \alpha_3(t)x(z-c)]dt + \sum\limits_{i=1}^{3} u_{3i}(x,y,z)dB_i(t). \end{cases} \tag{1.2}$$

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where $B_i(t)(i = 1, 2, 3)$ is a Brownian motion. The system (1.2) is earliest derived and developed by Jiang and Yin [9], and they proved the well-posedness as well as p-moment stability for the system, for further details refer to [9]. In particular, if we set $u_{1i}(x, y, z) = 0, u_{2i}(x, y, z) = 0, , u_{3i}(x, y, z) = 0, i = 1, 2, 3,\ \alpha_3(t) = 0, b = 0, a = -1$ and $\alpha_1(t) = \alpha_2(t) = 1$, then the system (1.2) reduces to the classical Lorenz system.

Since fractional Brownian motion is neither Markovian nor semi-martingale, the classical Itô formula and the ergodicty theory are not applicable to stochastic differential equations driven by fractional Brownian motion, we refer it to [15] for details. Previously M. Harier [5,6] proposed a framework for solving stochastic differential equations with fBm, and he proved that the solution of stochastic differential equation with fBm is quasi-Markovian solutions along with the existence of the invariant measure. Zeng et.al. [21] together with Xu et.al. [19] proved the stationary distribution of stochastic Lorenz equation and Robinovich systems with fBm following the ideas of Harier [5,6] and [21], respectively. The aim of this work is to prove the stationary distribution of the stochastic Rössler-Lorenz system (1.2) driven by fBm.

First of all, we denote $u(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ and

$$A = \begin{pmatrix} \sigma & -\sigma & 0 \\ -(r+1) & 1-a & 0 \\ c\alpha_3(t) & 0 & \beta \end{pmatrix}, \quad B(u(t), t) = \begin{pmatrix} -\alpha_3(t)yz + \alpha_3(t)z^2 \\ \alpha_1(t)xz \\ -\alpha_2(t)xy - \alpha_3(t)xz \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix},$$

and

$$G(u(t)) = \begin{pmatrix} g_1(u(t)) & 0 & 0 \\ 0 & g_2(u(t)) & 0 \\ 0 & 0 & g_3(u(t)) \end{pmatrix}, \quad B_t^H = \begin{pmatrix} B_t^{H_1} \\ B_t^{H_2} \\ B_t^{H_3} \end{pmatrix},$$

where $g_i(u(t))(i = 1, 2, 3)$ is a linear function and $g_1(u(t)) = k_1 x(t), g_2(u(t)) = k_2 y(t), g_3(u(t)) = k_3 z(t)$, $k_i(i = 1, 2, 3)$ are all constants. Denote by $|u(t)| = (x^2(t)+ y^2(t) + z^2(t))^{1/2}$ the vector norm, and $\langle \cdot, \cdot \rangle$ represents the inner product. Then the stochastic system (1.1) can be rewritten as the stochastic evolution equation

$$\begin{cases} du(t) = -[Au(t) + B(u(t), t) - D]dt + G(u(t))dB_t^H, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

It is necessary to point out that there exists a typos in [9] when the equation (1.2) is transformed into the abstract evolution equation (1.3), $a_{31} = -c\alpha_3(t)$ should be $a_{31} = c\alpha_3(t)$, but there is no any impact on their arguments.

In the sequel,we suppose the following assumptions hold

(A1) $\beta > 0, \sigma > 0, r + a < 0$ and $1 - a + \sigma > 0$;

(A2) $\alpha_1(t), \alpha_2(t), \alpha_3(t)$ are uniformly bounded for $t \geq 0$, and $\alpha_1(t) = \alpha_2(t)+\alpha_3(t)$.

In fact, the assumption (A1) guarantees that $\langle Au, u \rangle \geq \lambda |u|^2$, where $\lambda > 0$ is a constant.

We also investigate the synchronization phenomenon of stochastic Rössler-Lorenz systems with fBm, and consider the drive system

$$du(t) = -[Au(t) + B(u(t), t) - D]dt, \tag{1.4}$$

as well as its response system

$$dv(t) = [-Av(t) - B(v(t), t) + D + \Gamma(u(t), v(t))]dt + G(u(t), v(t))dB_t^H, \tag{1.5}$$

where $\Gamma(u(t), v(t)), G(u(t), v(t))$ are both control input vectors, which operate as feedback controllers, and the error system can be obtained by the following equation

$$\begin{aligned} de(t) =& [-Ae(t) - B(v(t), t) + B(u(t), t) + \Gamma(u(t), v(t))]dt \\ &+ G(u(t), v(t))dB_t^H, \end{aligned} \tag{1.6}$$

where $e(t) = v(t) - u(t)$.

This thesis documents several key contributions and novelty. The first novelty of the present paper is to establish the stationary distribution of stochastic Rössler-Lorenz system driven by fBm with H $(1/2 < H < 1)$. Then different kinds of controllers are introduced to realize finite-time synchronization between the drive system and its corresponding response system. Besides, we analyze the efficiency and convenience of the two different controllers, and realize the synchronization at exponential speed. Moreover, we also present the phenomenon that the finite-time synchronization can be achieved with appropriate controller.

The rest of the paper is arranged as follows. In section 2, the existence of invariant measure of stochastic systems with fBm (1.3) is established by applying the framework of Harier [5, 6] and combining Itô formula for fractional Brownian motion and Lyapunov functions as well. Some examples and numerical simulations are provided at the end of this section. In section 3, we introduce two kinds of different controllers to realize finite-time synchronization between the drive system and the response system, then analyze the efficiency and convenience of the two different controllers, respectively. A series of numerical simulations are provided to illustrate the theoretical results.

2. Stationary distribution of stochastic Rössler-Lorenz system with fBm

In this section, we firstly introduce the Itô formula [14] for fractional Brownian motion, which is the critical tool to prove the stationary distribution of stochastic Rössler-Lorenz system with fBm.

Lemma 2.1 ([14], **Itô formula for fBm when** $H \in [1/2, 1)$**).** *Assume that* $X_t = \sum_{i=1}^m \sigma_i B_t^{H_i}$*, where* σ_i *are constants,* $H_1 = \frac{1}{2}$ *and* $H_i \in (\frac{1}{2}, 1)$ *for* $i \geq 2$*, then* B^H *is fBm with Hurst parameter* H*. Let* $F \in C^2(\mathbb{R})$*, then for any* $t > 0$*,*

$$F(X_t) = F(0) + \sigma_1 \int_0^t F'(X_s)dW_s + \sum_{i=2}^m \sigma_i \int_0^t F'(X_s)dB_s^{H_i} + \frac{\sigma_1^2}{2} \int_0^t F''(X_s)ds,$$

where W_t *is a Brownian motion.*

2.1. Well-Posedness of stochastic Rössler-Lorenz system with fBm

Since

$$G(u(t)) = \begin{pmatrix} k_1x(t) & 0 & 0 \\ 0 & k_2y(t) & 0 \\ 0 & 0 & k_3z(t) \end{pmatrix},$$

and k_1, k_2, k_3 are constants.

Theorem 2.1. *Stochastic Rössler-Lorenz system* (1.3) *admits a unique global solution with the form*

$$u(t) = u_0 - \int_0^t [Au(s) + B(u(s), s) - D]ds + \int_0^t G(u(s))dB_s^H.$$

Proof. Denote $F(u(t)) = Au(t) + B(u(t), t) - D$, and define

$$F_N(u(t)) = \begin{cases} F(u(t)), & |u(t)| \leq N, \\ F\big(\frac{u(t)}{|u(t)|}N\big), & |u(t)| > N. \end{cases}$$

Then $F_N(u(t))$ is Lipschitz continuous, and we have

$$\begin{aligned} |G(u(t)) - G(v(t))| &= \begin{vmatrix} k_1(x_1(t) - x_2(t)) & 0 & \\ 0 & k_2(y_1(t) - y_2(t)) & 0 \\ 0 & 0 & k_3(z_1(t) - z_2(t)) \end{vmatrix} \\ &\leq \max_{i=1,2,3}\{|k_i|\}|u(t) - v(t)|. \end{aligned}$$

Therefore, we can obtain the truncated system

$$\begin{cases} du^N(t) = -F_N(u^N(t))dt + G(u^N(t))dB_t^H, \\ u^N(0) = u_0, \end{cases} \tag{2.1}$$

and it has a unique solution in the form of

$$u^N(t) = u_0 - \int_0^t F_N(u^N(s))ds + \int_0^t G(u^N(s))dB_s^H.$$

Define the stopping time $\{\tau_N\}$ by $\tau_N := \inf\{t > 0 : |u(t)| \geq N\}$, then refer to [19], Borel-Cantelli lemma guarantees that $u^N(t)$ converges to $u(t)$ as $N \to \infty$ and Theorem 2.1 holds. □

In order to visualize theoretical results, we use truncated EM method proposed by Mao [13] in all numerical simulations in this paper, and $u_0 = (1, 0.1, 1)^T$ as well as all coefficients in the system are chosen to satisfy the demand for the assumptions **(A1)** and **(A2)**.

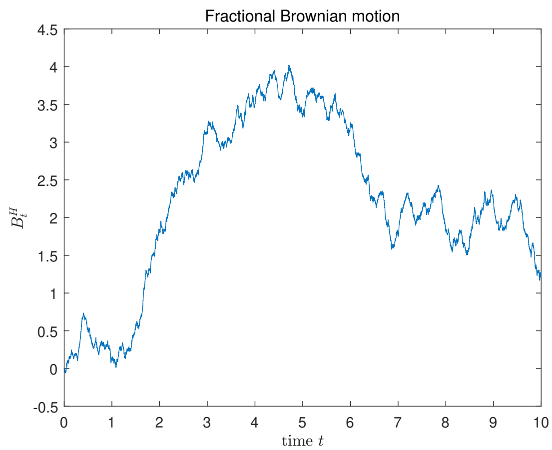

Figure 1. Trajectory of fBm

Example 2.1. Set the Hurst parameter $H = 0.7$, the step length $dt = 10^{-4}$, then we have the trajectory of fractional Brownian motion as Fig.1.

Set the coefficients $\sigma = 2, b = 1, r = 0, c = 0, a = -18, \beta = 8(\cos t + 1)$, and

$$\alpha_1(t) = \frac{2}{2t+2} - 2\sin t + 3 + \frac{1}{\cos 3t + 2}, \quad \alpha_2(t) = -2\sin t + 3 + \frac{1}{\cos 3t + 2}, \quad \alpha_3(t) = \frac{2}{2t+2},$$

the numerical solutions with $k_i = 1, i = 1, 2, 3$ and $k_i = 0.1, i = 1, 2, 3$ are presented in Fig.2 and Fig.3, respectively.

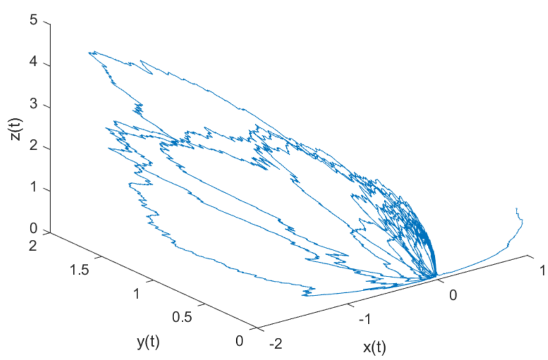

Figure 2. Sample path of Example 2.1 with $k_i = 1$, $i = 1, 2, 3$

Example 2.2. Let $B(u(t), t)$ be the same as in Example 2.1, and take $\sigma = 2, a = -7, r = 0, b = 1, c = 0, \beta = 2$, the numerical solution is shown in Fig.4. Moreover, we set the coefficient matrix A has the same parameters as in Example 2.1, and

$$\alpha_1(t) = \frac{1}{\sin t + 2} + \frac{5}{\sin t^2 + 2}, \quad \alpha_2(t) = \frac{5}{\sin t^2 + 2}, \quad \alpha_3(t) = \frac{1}{\sin t + 2}.$$

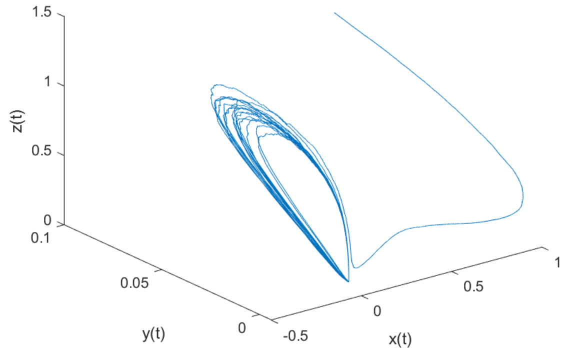

Figure 3. Sample path of Example 2.1 with $k_i = 0.1$, $i = 1, 2, 3$

The corresponding numerical simulation is presented in Fig.5.

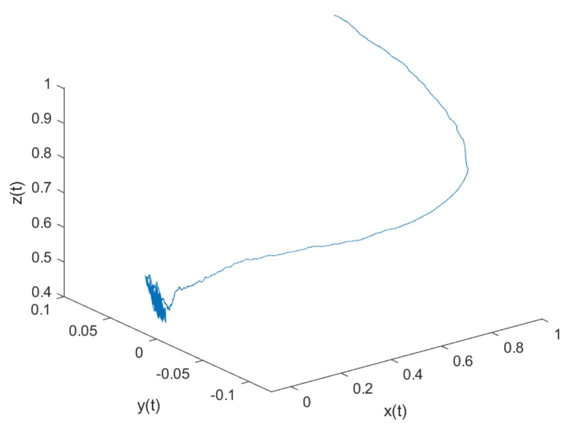

Figure 4. Sample path of Example 2.2 with $k_i = 1$, $i = 1, 2, 3$

2.2. Invariant measure when $H \in (\frac{1}{2}, 1)$

In this section, we introduce the stationary distribution of stochastic Rössler-Lorenz system (1.3) on the basis of the idea [21]. Define the two-side fBm with the Hurst parameter $H \in (0, 1)$ as

$$B_t^H = \alpha_H \int_{-\infty}^{0} (-r)^{H-1/2}(dB_{t+r} - dB_r).$$

Denote $H_0 = \min\{H_1, H_2, H_3\}$, $\mathbb{R}^- = (-\infty, 0]$, $C_0^\infty(\mathbb{R}^-; \mathbb{R}^3)$ be the set of smooth compactly supported functions f and $f(0) = 0$, and $W_{\lambda,\gamma}$ be the completion

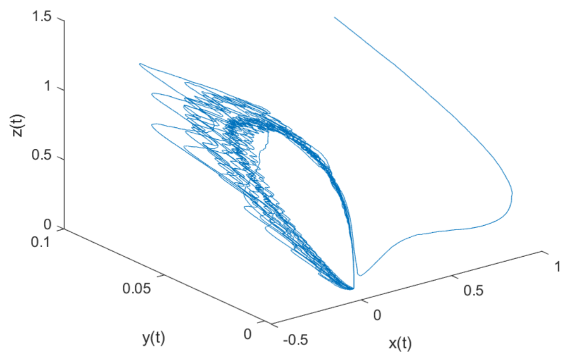

Figure 5. Sample path of Example 2.2 with $k_i = 0.1$, $i = 1, 2, 3$

of $C_0^\infty(\mathbb{R}^-;\mathbb{R}^3)$ with the norm:

$$||\omega||_{\lambda,\gamma} = \sup_{s,t\in\mathbb{R}^-,s\neq t} \frac{|\omega(t)-\omega(s)|}{|t-s|^\gamma(1+|t|+|s|)^\lambda},$$

where $0 < \lambda, \gamma < 1$. Obviously, the norm $||\cdot||_{\lambda,\gamma}$ is equivalent to the Hölder norm $||\cdot||_\gamma$ on $[0,T]$.

Let $\widetilde{W}_{\lambda,\gamma}$ be the function space on $\mathbb{R}^+ = [0,+\infty)$. Furthermore, if $\gamma \in (\frac{1}{2}, H_0)$ and $\gamma + \lambda \in (H_0, 1)$, it can be found that there is a Borel probability measure P_{H_0} on $W_{\lambda,\gamma} \times \widetilde{W}_{\lambda,\gamma}$ such that the cannonical process associated with P_{H_0} is a two-side fractional Brownian motion. Let $\Pi : W_{\lambda,\gamma} \to \widetilde{W}_{\lambda,\gamma}$ be linear and bounded,

$$\Pi\omega(t) = \beta_{H_0} \int_0^\infty \frac{1}{r} h(\frac{t}{r})\omega(-r)dr,$$

where $\beta_{H_0} = (H_0 - \frac{1}{2})\alpha_{H_0}\alpha_{1-H_0}$ and

$$h(r) = r^{H_0-\frac{1}{2}} + (H_0 - \frac{3}{2})r \int_0^1 \frac{(r+u)^{H_0-\frac{5}{2}}}{(1-u_0)^{H_0-\frac{1}{2}}}du.$$

Let $K_h : \omega \to \omega + h$ be a shift map in $\widetilde{W}_{\lambda,\gamma}$, and the transition kernel $\mathcal{K}(\omega,\cdot) = (K_{\Pi\omega} \circ I^{H_0-\frac{1}{2}}) * W$, where $I^{H_0-\frac{1}{2}}$ can be defined as [16]:

$$I^{H_0-\frac{1}{2}} f(t) = \frac{1}{\Gamma(H_0-\frac{1}{2})} \int_0^t (t-s)^{H_0-\frac{2}{3}} f(s)ds.$$

Define the one-side Wiener shift $\theta_t : W_{\lambda,\gamma} \to W_{\lambda,\gamma}$ by

$$\theta_t\omega(s) = \omega(s-t) - \omega(-t), \quad s \in \mathbb{R}^-, t \in \mathbb{R}^+,$$

and $\beta_t : W_{\lambda,\gamma} \times \widetilde{W}_{\lambda,\gamma} \to W_{\lambda,\gamma}$, that is

$$\beta_t(\omega,\tilde{\omega}) = \begin{cases} \tilde{\omega}(s+t) - \tilde{\omega}(t), & t > -s, \\ \omega(s+t) - \tilde{\omega}(t), & t \leq -s, \end{cases}$$

where $\omega \in W_{\lambda,\gamma}, t \in \mathbb{R}^+$, and the transition operator P_t is defined by $P_t(\omega, \cdot) = \beta_t^*(\omega, \cdot)\mathcal{K}_t(\omega, \cdot)$. Hence, the quadruple $(W_{\lambda,\gamma}, \{P_t\}_{t\geq 0}, \mathbb{P}_{H_0}, \{\theta_t\}_{t\geq 0})$ is a stationary process.

For $u_0 = (x_0, y_0, z_0)^T$ and $\xi \in C_0([0,T], \mathbb{R}^3)$, denote that

$$\phi_T(u(t),\xi)(t)=u_0-\int_0^t(A\phi_T(u,\xi)(s)+B(\phi_T(u,\xi)(s))-D)ds+\int_0^t G(\phi_T(u,\xi)(s))dB_s^H.$$

Theorem 2.1 yields that there has an exact $\phi_T(u(t), \xi)$, which implies that there exists a stationary process $(W_{\lambda,\gamma}, \{P_t\}_{t\geq 0}, \mathbb{P}_{H_0}, \{\theta_t\}_{t\geq 0})$. Thus we can define $\Xi : \mathbb{R}^+ \times \mathbb{R}^3 \times W_{\lambda,\gamma} \to \mathbb{R}^3$ by

$$\Xi_t(u(t), \omega) = \phi_T(u(t), G_t\omega)(t),$$

where G_T is a continuous shift operator expressed as

$$(G_T h)(t) = h(t-T) - h(-T), \quad G_T : W_{\lambda,\gamma} \to \widetilde{W}_{\lambda,\gamma}.$$

It follows from Lemma 2.12 in [6] that for given $t \geq 0$, the Feller semigroup $Q_t(u(t), \omega; \cdot)$ can be defined on $\mathbb{R}^3 \times W_{\lambda,\gamma}$ by

$$Q_t(u(t), \omega; Q_1 \times Q_2) = \int_{Q_2} \delta_{\Xi_t(u(t),\omega')}(Q_1)P_t(\omega, d\omega'),$$

where δ_x located at x represents the delta measure.

Theorem 2.2. *Assume that* **(A1)** *and* **(A2)** *hold. Then the stochastic dynamical system generated by the solution of stochastic Rössler-Lorenz system* (1.3) *admits a unique invariant measure.*

Proof. Since ϕ_T is the pathwise integral, it follows from [6] $\{\Xi_t\}_{t\geq 0}$ generates a continuous stochastic dynamical system. For the sake of proving the existence of the invariant measure, we consider the following deterministic Rössler-Lorenz system

$$\begin{cases} d\hat{u}(t) = -[A\hat{u}(t) + B(\hat{u}(t), t) - D]dt, \\ \hat{u}(0) = u_0, \end{cases} \tag{2.2}$$

where $\hat{u}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t)) \in \mathbb{R}^3$. Let $V_1 = \hat{x}^2(t) + \hat{y}^2(t) + \hat{z}^2(t)$, then direct calculation implies that

$$\frac{d|\hat{u}(t)|^2}{dt} \leq \frac{dV_1}{dt} = 2\langle \hat{u}(t), d\hat{u}(t)\rangle \leq -\lambda|\hat{u}(t)|^2 + \langle D, \hat{u}(t)\rangle \leq -\frac{\lambda}{2}|\hat{u}(t)|^2 + \frac{d^2}{2\lambda}.$$

Multiply the above equation by $e^{\frac{\lambda}{2}t}$ can be derived

$$e^{\frac{\lambda}{2}t}d|\hat{u}(t)|^2 + \frac{\lambda}{2}|\hat{u}(t)|^2 e^{\frac{\lambda}{2}t}dt \leq \frac{d^2}{2\lambda}e^{\frac{\lambda}{2}t}dt, \quad de^{\frac{\lambda}{2}t}|\hat{u}(t)|^2 \leq \frac{d^2}{2\lambda}e^{\frac{\lambda}{2}t}dt.$$

Integrating the above equation from 0 to t, we can obtain

$$e^{\frac{\lambda}{2}t}|\hat{u}(t)|^2 \leq |\hat{u}(0)|^2 + \int_0^t \frac{d^2}{2\lambda}e^{\frac{\lambda}{2}s}ds, \quad |\hat{u}(t)|^2 \leq |\hat{u}(0)|^2 e^{-\frac{\lambda}{2}t} + \frac{d^2}{\lambda^2}.$$

Hence, there exists a positive constant C such that for $p \geq 1$

$$|\hat{u}(t)|^p \leq C|\hat{u}(0)|^p e^{-\frac{p\lambda}{2}t} + C.$$

Let $\tilde{u}(t) = u(t) - \hat{u}(t)$, then it follows that

$$\begin{aligned}\tilde{u}(t) = &\int_0^t [-A\tilde{u}(s) - B(\tilde{u}(s) + \hat{u}(s), s) + B(\hat{u}(s), s)]ds + \int_0^t G(\hat{u}(s) + \tilde{u}(s))dB_s^H \\ =: &I_1 + I_2.\end{aligned}$$

For any $\alpha \in (1 - H_0, 1/2)$, direct calculation shows that

$$|I_1(t)| \leq \int_0^t |A\tilde{u}(s)|ds + \int_0^t |B(\hat{u}(s) + \tilde{u}(s), s) - B(\hat{u}(s), s)|ds \leq C\int_0^t \frac{\tilde{u}(s)}{|t-s|^\alpha}ds,$$

and

$$\begin{aligned}&\int_0^t \frac{|I_1(t) - I_1(s)|}{|t-s|^{1+\alpha}}ds \\ =&\int_0^t \frac{|\int_s^t[-A(\hat{u}(\tau) + \tilde{u}(\tau)) - B(\hat{u}(\tau) + \tilde{u}(\tau), \tau) + A\hat{u}(\tau) + B(\hat{u}(\tau), \tau)]d\tau|}{|t-s|^{1+\alpha}}ds \\ \leq &C\int_0^t \frac{\tilde{u}(s)}{|t-s|^\alpha}ds.\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}|I_2(t)| &\leq C||B^{H_0}||_\alpha \int_0^t |G(\hat{u}(s) + \tilde{u}(s))|ds \\ &\leq C||B^{H_0}||_\alpha\Big(1 + |u_0| + \int_0^t\int_0^s \frac{|\tilde{u}(s) - \tilde{u}(\tau)|}{|s-\tau|^{1+\alpha}}d\tau ds\Big),\end{aligned}$$

and

$$\begin{aligned}\int_0^t \frac{|I_2(t) - I_2(s)|}{|t-s|^{1+\alpha}}ds &\leq C||B^{H_0}||_\alpha(1 + |u_0| + \int_0^t \frac{\int_0^s \frac{|\tilde{u}(s) - \tilde{u}(\tau)|}{|s-\tau|^{1+\alpha}}d\tau}{|t-s|^\alpha}ds) \\ &\leq C||B^{H_0}||_\alpha\Big(1 + |u_0| + \int_0^t\int_0^s \frac{|\tilde{u}(s) - \tilde{u}(\tau)|}{|s-\tau|^{1+\alpha}|t-s|^\alpha}d\tau ds\Big).\end{aligned}$$

Thus, combining the above arguments, we arrive at

$$\begin{aligned}&|\tilde{u}(t)| + \int_0^t \frac{|\tilde{u}(t) - \tilde{u}(s)|}{|t-s|^{1+\alpha}}ds \\ \leq &|I_1(t)| + |I_2(t)| + \int_0^t \frac{|I_1(t) - I_1(s)|}{|t-s|^{1+\alpha}}ds + \int_0^t \frac{|I_2(t) - I_2(s)|}{|t-s|^{1+\alpha}}ds \\ \leq &2C\int_0^t \frac{\tilde{u}(s)}{|t-s|^\alpha}ds + 2C||B^{H_0}||_\alpha\Big(1 + |u_0| + \int_0^t\int_0^s \frac{|\tilde{u}(s) - \tilde{u}(\tau)|}{|s-\tau|^{1+\alpha}|t-s|^\alpha}d\tau ds\Big) \\ \leq &2C||B^{H_0}||_\alpha\Big(1 + |u_0| + t^\alpha\int_0^t \frac{|\tilde{u}(s)| + \int_0^s \frac{|\tilde{u}(s) - \tilde{u}(\tau)|}{|s-\tau|^{1+\alpha}}d\tau}{|t-s|^\alpha s^\alpha}ds\Big)\end{aligned}$$

$$\leq 2C||B^{H_0}||_\alpha(1+|u_0|)\exp(2C||B^{H_0}||_\alpha^{1/(1-\alpha)}t).$$

Therefore, we have

$$|u(t)|^p \leq C|u_0|e^{-\frac{p\lambda}{2}t} + Ce^{2C||B^{H_0}||_\alpha^{1/(1-\alpha)}t}. \tag{2.3}$$

It follows from Fernique¡¯s theorem [4], (2.3) and $||B^{H_0}||_\alpha < \infty$ that

$$\int |u(t)|^p (Q_t\mu)(dx, d\omega) \leq Ce^{-\frac{p\lambda}{2}t} \int |u(t)|^p \mu(dx, d\omega) + C, \quad \forall \mu.$$

Since $V(u(t)) = |u(t)|^p$ is a Lyapunov function, it follows from [6] that there exists an invariant measure.

Next, we will prove the uniqueness of invariant measure. Assume that $\varphi : C([1,+\infty),\ \mathbb{R}^3) \to \mathbb{R}$ is a measurable function, the derivative of function $D\varphi$ is given by

$$D\varphi(u(t),\omega) = \int_{C([1,+\infty),\mathbb{R}^3)} \varphi(z)R_1^* D\delta_{u,\omega}(dz) = \mathbb{E}_\omega \varphi_T(\phi_T(u(t),\tilde{\omega})),$$

where $\mathbb{E}_\omega$ is the expectation over $\tilde{\omega}$ with respect to the probability measure $\mathcal{K}(\omega,\cdot)$. it derives that

$$D\varphi_T(u(t),\omega) = \mathbb{E}_\omega \varphi_T(\phi_T(u(t),\tilde{\omega})).$$

Taking the transformations $\phi_s = su(t) + (t-s)z$ and $\zeta = u(t) - v(t)$, one obtain

$$D\varphi_T(v(t),\omega) - D\varphi_T(u(t),\omega) = \mathbb{E}_\omega \int_0^1 \langle D\varphi_T(\phi_T(\phi_s,\tilde{\omega})), D_x\phi_T(\phi_s,\tilde{\omega})\zeta\rangle.$$

Due to the boundedness of $D\varphi_T$ and $D_x\phi_T$ [8], we can find a jointly continuous function f satisfying

$$||D\phi_T(v(t),\omega) - D\phi_T(u(t),\omega)||_{TV} \leq f(u(t),v(t),\omega).$$

Thus, it follows from [6] that

$$||R_1^* D\delta_{(v(t),\omega)} - R_1^* D\delta_{(u(t),\omega)}||_{TV} \leq f(u(t),v(t),\omega).$$

Applying the Bismut-Elworthy-Li formula [3] and Theorem 3.10 in [6], we deduce that the stochastic dynamical systems Ξ is strong Feller and topological irreducible at $t = 1$. Moreover, the stochastic system (1.3) is quasi-Markovian. Synthesizing above all outcomes, we finish the proof of Theorem 2.2. □

2.3. Numerical simulation

For convenience, we take the same coefficients as Example 2.1 in section 2, and let $k_i = 0.1, i = 1,2,3$. Denote the invariant measure by η, and since the solution to equation (1.3) is 3-dimensional, we intend to visualize the invariant measure by projection and the approximation of empirical density. The statistical distribution is given under 100 samples. For visualization, we give the distribution maps of the empirical density in XY, XZ and YZ spaces as following, which are shown as Fig.7-Fig.9.

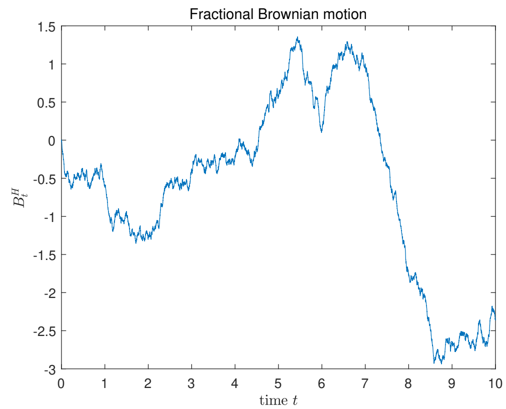

Figure 6. Trajectory of fBm

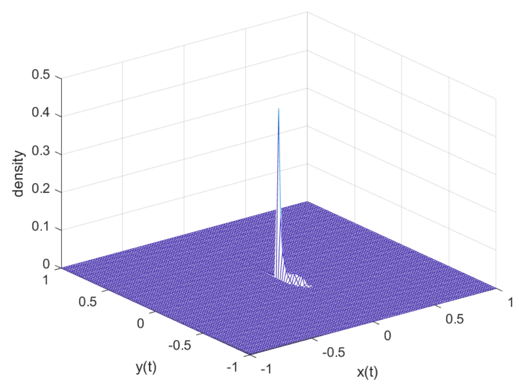

Figure 7. η in XY space

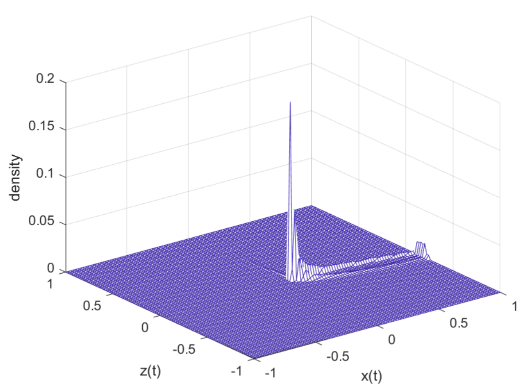

Figure 8. η in XZ space

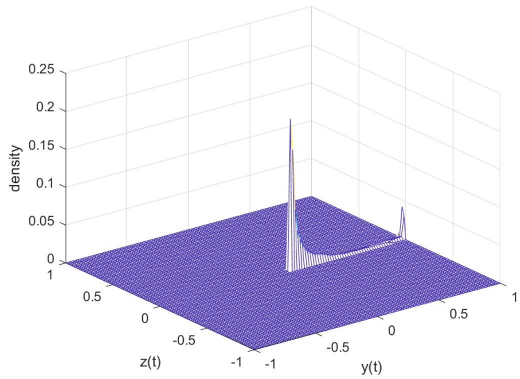

Figure 9. η in YZ space

3. Synchronization of stochastic Rössler-Lorenz system with fBm

In this section, we will consider the synchronization of stochastic Rössler-Lorenz system with fBm. To the end, we introduce the definition of stochastic synchronization from [22].

Definition 3.1 ([22])**.** The drive system (1.4) and response system (1.5) driven by fractional Brownian motion are said to be synchronous in the mean square if the error system (1.6) with fractional Brownian motion is asymptotic stable in the mean square, that is, for any initial condition, the mild solution $e(t,\cdot)$ of the error system (1.6) satisfies the following estimate

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\|e(t,\cdot)\|^2)<0, \tag{3.1}$$

or there exists the two constants K and $\gamma>0$ such that

$$\mathbb{E}\|e(t,\cdot)\|^2\le Ke^{-\gamma t}. \tag{3.2}$$

3.1. Exponential asymptotic stability of error system

Consider the following drive system (1.4)

$$du(t)=[-Au(t)-B(u(t),t)+D]dt,$$

and its corresponding response system (1.5)

$$dv(t)=[-Av(t)-B(v(t),t)+D+\Gamma(u(t),v(t))]dt+G(u(t),v(t))dB_t^H.$$

In order to prove the synchronization by constructing the input vector $\Gamma(t)$ as a controller, we consider the error system

$$de(t)=[-Ae(t)-B(v(t),t)+B(u(t),t)+\Gamma(u(t),v(t))]dt+G(u(t),v(t))dB_t^H,$$

where $e(t) = (e_1(t), e_2(t), e_3(t))^T$, $\Gamma(u(t), v(t)) = C(t) = (c_1(t), c_2(t), c_3(t))^T$ is an input vector, and

$$G(u(t), v(t)) = \begin{pmatrix} x_2(t) - x_1(t) & 0 & 0 \\ 0 & y_2(t) - y_1(t) & 0 \\ 0 & 0 & z_2(t) - z_1(t) \end{pmatrix}.$$

To measure the effects caused by the error and noise, we take

$$V(t) \triangleq V(e_1(t), e_2(t), e_3(t)) = \frac{1}{2}(|e_1(t)|^2 + |e_2(t)|^2 + |e_3(t)|^2),$$

and $\delta^2(t) = (x_2(t) - x_1(t))^2 + (y_2(t) - y_1(t))^2 + (z_2(t) - z_1(t))^2$.

Theorem 3.1. *Let*

$$\begin{aligned}\Gamma(u(t), v(t)) =&(\alpha_2(t)(y_2(t)z_1(t) - y_1(t)z_2(t)), -\frac{\delta^2(t)sgn(y_2(t) - y_1(t))}{|y_2(t) - y_1(t)|}, \\ &\alpha_3(t)(x_1(t)y_2(t) - x_2(t)y_1(t)) + \alpha_3(t)(x_2(t)z_1(t) - x_1(t)z_2(t)))^T,\end{aligned}$$

the zero solution to (1.6) *is exponential asymptotic stable if both (A1) and (A2) hold, in other words, it can be said that the error dynamics converges to zero at exponential rate.*

Proof. The assumption **(A1)** implies that there is a positive constant $\lambda > 0$ such that

$$\langle Au, u\rangle \geq \lambda |u|^2.$$

According to Lemma 2.1 and Itô formula, there holds

$$\begin{aligned}
&\mathbb{E}[V(t)] \\
=&\frac{1}{2}\mathbb{E}|e_0|^2 - \mathbb{E}\int_0^t \langle Ae(s), e(s)\rangle ds - \mathbb{E}\int_0^t \langle B(v(s), s) - B(u(s), s), e(s)\rangle ds \\
&+ \frac{1}{2}\mathbb{E}\int_0^t \delta^2(s)ds + \frac{1}{2}\mathbb{E}\Big[\sum_{i=1}^3 \int_0^t |e_i(s)|^2 dB_s^{H_i}\Big] + \mathbb{E}\int_0^t \langle \Gamma(u(s), v(s)), e(s)\rangle ds \\
\leq&\mathbb{E}\int_0^t \Big[-\lambda|e(s)|^2 + \alpha_3(s)\Big(x_2(s)y_2(s)z_2(s) - x_1(s)y_2(s)z_2(s) - x_2(s)z_2^2(s) \\
&- x_1(s)z_2^2(s) - x_2(s)y_1(s)z_1(s) + x_2(s)z_1^2(s) + x_1(s)y_1(s)z_1(s) - x_1(s)z_1^2(s)\Big) \\
&- \alpha_1(s)\Big(x_2(s)y_2(s)z_2(s) - x_2(s)y_1(s)z_2(s) - x_1(s)y_2(s)z_1(s) + x_1(s)y_1(s)z_1(s)\Big) \\
&+ \alpha_2(s)\Big(x_2(s)y_2(s)z_2(s) - x_2(s)y_2(s)z_1(s) - x_1(s)y_1(s)z_2(s) + x_1(s)y_1(s)z_1(s)\Big) \\
&+ \alpha_3(s)\Big(x_2(s)z_2^2(s) - x_2(s)z_1(s)z_2(s) - x_1(s)z_1(s)z_2(s) + x_1(s)z_1^2(s)\Big) \\
&+ \langle \Gamma(u(s), v(s)), e(s)\rangle + \frac{1}{2}\delta^2(s)\Big] ds + \frac{1}{2}\mathbb{E}|e_0|^2 \\
=&\frac{1}{2}\mathbb{E}|e_0|^2 + \mathbb{E}\int_0^t \Big[-\lambda|e(s)|^2 - \alpha_3(s)\Big(x_1(s)y_2(s)z_2(s) + x_2(s)y_1(s)z_1(s)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_3(s)\Big(x_1(s)z_2^2(s)+x_2(s)z_1^2(s)\Big)+\alpha_1(s)\Big(x_2(s)y_1(s)z_2(s)+x_1(s)y_2(s)z_1(s)\Big)\\
&-\alpha_2(s)\Big(x_2(s)y_2(s)z_1(s)+x_1(s)y_1(s)z_2(s)\Big)-\alpha_3(s)\Big(x_2(s)z_2^2(s)+x_1(s)z_1^2(s)\Big)\\
&+\alpha_2(s)\Big(x_2(s)y_2(s)z_1(s)-x_2(s)y_1(s)z_2(s)-x_1(s)y_2(s)z_1(s)+x_1(s)y_1(s)z_2(s)\Big)\\
&+\alpha_3(s)\Big(x_1(s)y_2(s)z_2(s)-x_2(s)y_1(s)z_2(s)-x_1(s)y_2(s)z_1(s)+x_2(s)y_1(s)z_1(s)\Big)\\
&+\alpha_3(s)\Big(x_2(s)z_1(s)z_2(s)-x_1(s)z_2^2(s)-x_2(s)z_1^2(s)+x_1(s)z_1(s)z_2(s)\Big)-\frac{1}{2}\delta^2(s)\Big]ds\\
\le&\frac{1}{2}\mathbb{E}|e_0|^2-\lambda\mathbb{E}\int_0^t[|e(s)|^2]ds.
\end{aligned}
$$

Applying the Gronwall inequality, we deduce that there exists a positive constant $C_1>0$ such that

$$\mathbb{E}|e(t)|^2\le C_1\mathbb{E}|e_0|^2e^{-\lambda t}, \tag{3.3}$$

which implies $\lim\limits_{t\to\infty}\sup\frac{1}{t}log(\mathbb{E}|e(t)|^2)\le-\lambda<0$. Therefore, both the drive system and response system are synchronous in the mean square, and the error system (1.6) is exponential asymptotic stable. □

Theorem 3.2. *Assume that* **(A1)** *and* **(A2)** *hold, and in addition* $\beta>\frac{1}{2}$, $1-a+\sigma>1+\sqrt{(1-a+\sigma)^2+4\sigma(a+r)}$. *Let*

$$
\begin{aligned}
\tilde{\Gamma}(u(t),v(t))=&\Big(\alpha_2(t)(y_2(t)z_1(t)-y_1(t)z_2(t)),0,\\
&\alpha_3(t)(x_1(t)y_2(t)-x_2(t)y_1(t))+\alpha_3(t)(x_2(t)z_1(t)-x_1(t)z_2(t))\Big)^T.
\end{aligned}
$$

Then, both the drive system (1.4) *and response system* (1.5) *achieve synchronous in the mean square.*

Proof. The proof is similar to one of Theorem 3.1. Indeed, under the assumptions in Theorem 3.2, we can obtain that $\langle Au,u\rangle\ge(\frac{1}{2}+\epsilon)|u|^2$, where $\epsilon>0$ is a positive constant. In detail

$$\det|\lambda I-A|=(\lambda-\beta)[\lambda^2-(1-a+\sigma)\lambda-\sigma(r+\alpha)]. \tag{3.4}$$

Let $\det|\lambda I-A|=0$, with given $\beta>\frac{1}{2}$ and $1-a+\sigma>1+\sqrt{(1-a+\sigma)^2+4\sigma(a+r)}$, we obtain that $\lambda_{min}>\frac{1}{2}$, which implies that there exists a positive constant $\epsilon>0$ such that

$$\langle Au,u\rangle\ge(\frac{1}{2}+\epsilon)|u|^2. \tag{3.5}$$

By straightforward calculations we get the following estimates:

$$
\begin{aligned}
&\mathbb{E}\langle -B(v(t),t)+B(u(t),t),e(t)\rangle+\mathbb{E}\langle\tilde{\Gamma}(u(t),v(t)),e(t)\rangle\\
=&\mathbb{E}\Big[-\lambda|e(t)|^2+\alpha_3(t)\Big(x_2(t)y_2(t)z_2(t)-x_1(t)y_2(t)z_2(t)-x_2(t)z_2^2(t)\\
&-x_1(t)z_2^2(t)-x_2(t)y_1(t)z_1(t)+x_2(t)z_1^2(t)+x_1(t)y_1(t)z_1(t)-x_1(t)z_1^2(t)\Big)\\
&-\alpha_1(t)\Big(x_2(t)y_2(t)z_2(t)-x_2(t)y_1(t)z_2(t)-x_1(t)y_2(t)z_1(t)+x_1(t)y_1(t)z_1(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_2(t)\Big(x_2(t)y_2(t)z_2(t) - x_2(t)y_2(t)z_1(t) - x_1(t)y_1(t)z_2(t) + x_1(t)y_1(t)z_1(t)\Big)\\
&+ \alpha_3(t)\Big(x_2(t)z_2^2(t) - x_2(t)z_1(t)z_2(t) - x_1(t)z_1(t)z_2(t) + x_1(t)z_1^2(t)\Big)\\
&+ \alpha_2(t)\Big(x_2(t)y_2(t)z_1(t) - x_2(t)y_1(t)z_2(t) - x_1(t)y_2(t)z_1(t) + x_1(t)y_1(t)z_2(t)\Big)\\
&+ \alpha_3(t)\Big(x_1(t)y_2(t)z_2(t) - x_2(t)y_1(t)z_2(t) - x_1(t)y_2(t)z_1(t) + x_2(t)y_1(t)z_1(t)\Big)\\
&+ \alpha_3(t)\Big(x_2(t)z_1(t)z_2(t) - x_1(t)z_2^2(t) - x_2(t)z_1^2(t) + x_1(t)z_1(t)z_2(t)\Big)\Big]\\
=&0, \qquad (3.6)
\end{aligned}
$$

and

$$
-\mathbb{E}\langle Ae(t), e(t)\rangle + \frac{1}{2}\mathbb{E}\delta^2(t) \le -\epsilon\mathbb{E}|e(t)|^2. \qquad (3.7)
$$

Applying the Gronwall inequality and (3.6), (3.7), we drive that there exists a constant $C_2 > 0$ such that

$$
\mathbb{E}|e(t)|^2 \le C_2\mathbb{E}|e_0|^2 e^{-\epsilon t}, \qquad (3.8)
$$

thus, the proof of Theorem 3.2 is complete. $\square$

Furthermore, we can prove the error dynamical system will converge to zero in finite time and the finite-time synchronization can be achieved with appropriate controller $\bar{\Gamma}(u(t), v(t))$.

Theorem 3.3. *Let*

$$
\begin{aligned}
\bar{\Gamma}(u(t), v(t)) =&(\alpha_2(t)(y_2(t)z_1(t) - y_1(t)z_2(t)) - |x_2(t) - x_1(t)|^\gamma sgn(x_2(t) - x_1(t)),\\
&- \frac{\delta^2(t)sgn(y_2(t) - y_1(t))}{|y_2(t) - y_1(t)|} - |y_2(t) - y_1(t)|^\gamma sgn(y_2(t) - y_1(t)),\\
&\alpha_3(t)(x_1(t)y_2(t) - x_2(t)y_1(t)) + \alpha_3(t)(x_2(t)z_1(t) - x_1(t)z_2(t))\\
&- |z_2(t) - z_1(t)|^\gamma sgn(z_2(t) - z_1(t)))^T,
\end{aligned}
$$

$\gamma \in (0, 1)$*, if (A1) and (A2) hold, then both the drive system and response system achieve synchronization in finite time.*

Proof. Analogously, via Itô formula we obtain

$$
\begin{aligned}
&\mathbb{E}[\dot{V}(t)]\\
=&-\mathbb{E}\langle Ae(t), e(t)\rangle - \mathbb{E}\langle B(v(t), t) - B(u(t), t), e(t)\rangle + \mathbb{E}\langle\bar{\Gamma}(u(t), v(t)), e(t)\rangle\\
&+ \frac{1}{2}\mathbb{E}\delta^2(t) + \frac{1}{2}\mathbb{E}\Big[\sum_{i=1}^{3}|e_i(s)|^2\dot{B}_t^{H_i}\Big]\\
\le&\mathbb{E}\Big[-\lambda|e(t)|^2 - \alpha_3(t)\Big(x_1(t)y_2(t)z_2(t) + x_2(t)y_1(t)z_1(t)\Big)\\
&+ \alpha_3(t)\Big(x_1(t)z_2^2(t) + x_2(t)z_1^2(t)\Big) + \alpha_1(t)\Big(x_2(t)y_1(t)z_2(t) + x_1(t)y_2(t)z_1(t)\Big)\\
&- \alpha_2(t)\Big(x_2(t)y_2(t)z_1(t) + x_1(t)y_1(t)z_2(t)\Big) - \alpha_3(t)\Big(x_2(t)z_2^2(t) + x_1(t)z_1^2(t)\Big)\\
&+ \alpha_2(t)\Big(x_2(t)y_2(t)z_1(t) - x_2(t)y_1(t)z_2(t) - x_1(t)y_2(t)z_1(t) + x_1(t)y_1(t)z_2(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_3(t)\Big(x_1(t)y_2(t)z_2(t)-x_2(t)y_1(t)z_2(t)-x_1(t)y_2(t)z_1(t)+x_2(t)y_1(t)z_1(t)\Big)\\
&+\alpha_3(t)\Big(x_2(t)z_1(t)z_2(t)-x_1(t)z_2^2(t)-x_2(t)z_1^2(t)+x_1(t)z_1(t)z_2(t)\Big)-\frac{1}{2}\delta^2(t)\\
&-|x_2(t)-x_1(t)|^{\gamma+1}-|y_2(t)-y_1(t)|^{\gamma+1}-|z_2(t)-z_1(t)|^{\gamma+1}\Big]\\
\leq&-\mathbb{E}\Big[|x_2(t)-x_1(t)|^{\gamma+1}+|y_2(t)-y_1(t)|^{\gamma+1}+|z_2(t)-z_1(t)|^{\gamma+1}\Big].
\end{aligned}
$$

Frequently we obtain that

$$
\begin{aligned}
\mathbb{E}[\dot{V}(t)]\leq&-\mathbb{E}\Big[|x_2(t)-x_1(t)|^{\gamma+1}+|y_2(t)-y_1(t)|^{\gamma+1}+|z_2(t)-z_1(t)|^{\gamma+1}\Big]\\
\leq&-\mathbb{E}\Big[\big((x_2(t)-x_1(t))^2+(y_2(t)-y_1(t))^2+(z_2(t)-z_1(t))^2\big)^{\frac{1+\gamma}{2}}\Big]\\
=&-2^{\frac{1+\gamma}{2}}\mathbb{E}[V^{\frac{1+\gamma}{2}}(t)],
\end{aligned}
\tag{3.9}
$$

obviously $\frac{1+\gamma}{2}\in(0,1)$. Thus according to Theorem 1 in [7], we complete the proof. □

3.2. Numerical simulation

In this subsection, some numerical simulations are provided to illustrate the theoretical results.

Example 3.1. Let the drive system is deterministic, take $\sigma=2, b=1, r=0, a=-18, c=0, \beta=8(\cos t+1)$,

$$
\alpha_1(t)=\frac{2}{2t+2}-2\sin t+3+\frac{1}{\cos 3t+2},\ \alpha_2(t)=-2\sin t+3+\frac{1}{\cos 3t+2},\ \alpha_3(t)=\frac{2}{2t+2},
$$

and the controller of its response system is

$$
\begin{aligned}
\Gamma(u(t),v(t))=&\bigg(\alpha_2(t)(y_1(t)z_2(t)-y_2(t)z_1(t)),-\frac{\delta^2(t)sgn(y_2(t)-y_1(t))}{|y_2(t)-y_1(t)|},\\
&-\alpha_3(t)(x_1(t)y_2(t)-x_2(t)y_1(t))-\alpha_3(t)(x_2(t)z_1(t)-x_1(t)z_2(t))\bigg)^T.
\end{aligned}
$$

Let the Hurst parameter $H=0.7$, then Fig.10-Fig.13 of error system explain that how the two systems achieve synchronization.

Example 3.2. The only difference between this example and Example 3.1 is that the controller is replaced by $\tilde{\Gamma}(u,v,t)=(\alpha_2(t)(y_1(t)z_2(t)-y_2(t)z_1(t)),0,-\alpha_3(t)(x_1(t)y_2(t)-x_2(t)y_1(t))-\alpha_3(t)(x_2(t)z_1(t)-x_1(t)z_2(t))^T$, thus we have such following Fig.14-Fig.17.

Comparing with the above two examples, we can find that both the two response systems can achieve synchronization with the drive system, however there are also some differences between them. First, as in Fig.10, there exists obvious oscillations in the sample path of the response system with controller in Example 3.1, but the other controller shows a smooth sample path in Fig.14. In addition, the convergence rate of the later controller is much faster. In conclusion, whether it is for simplicity or efficiency, the second controller is a better choice.

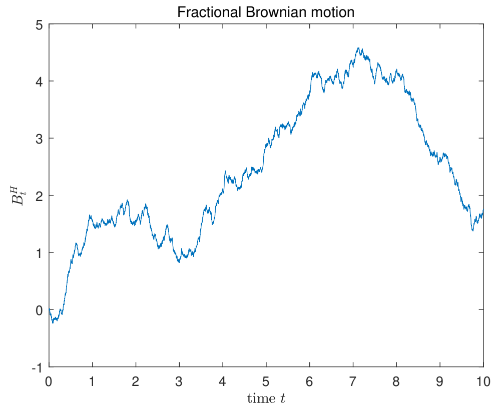

Figure 10. Trajectory of fBm

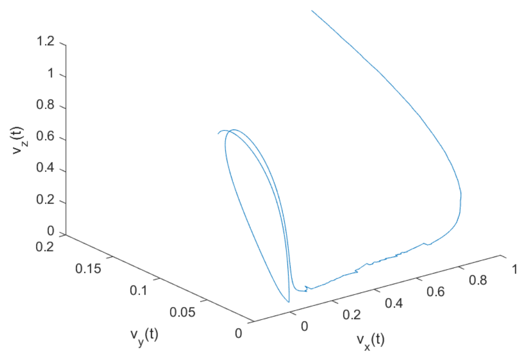

Figure 11. Sample path of response system

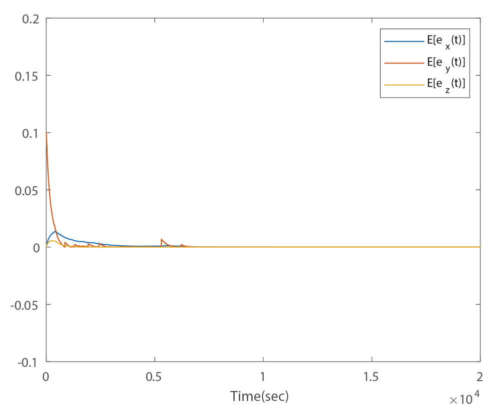

Figure 12. Time responses of the synchronization error

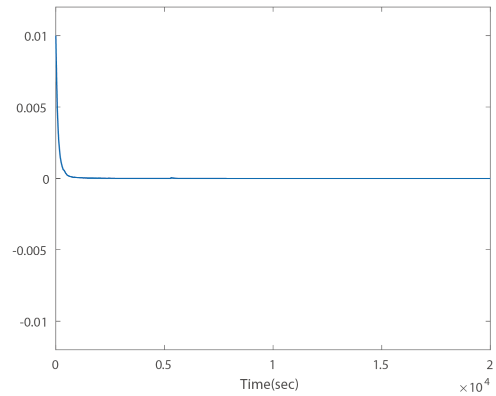

Figure 13. Time responses of mean values

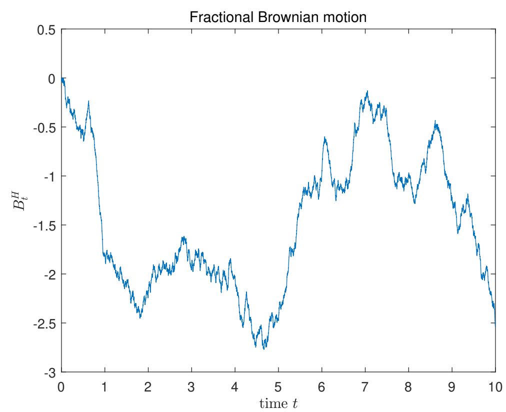

Figure 14. Trajectory of fBm

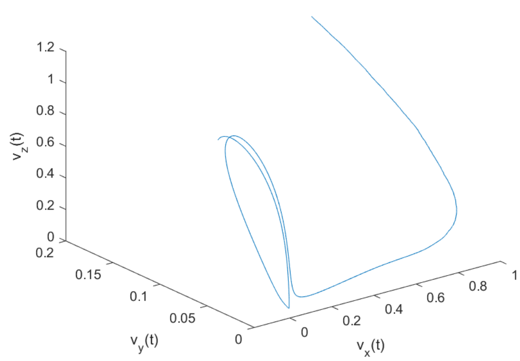

Figure 15. Sample path of response system

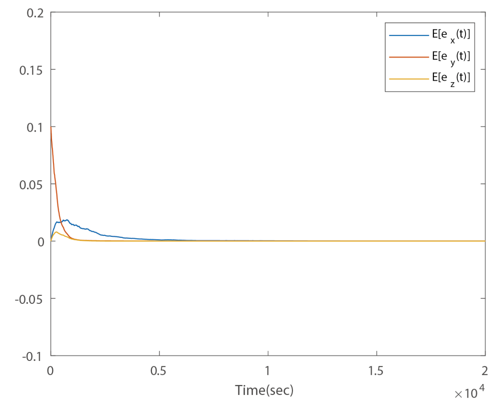

Figure 16. Time responses of the synchronization error

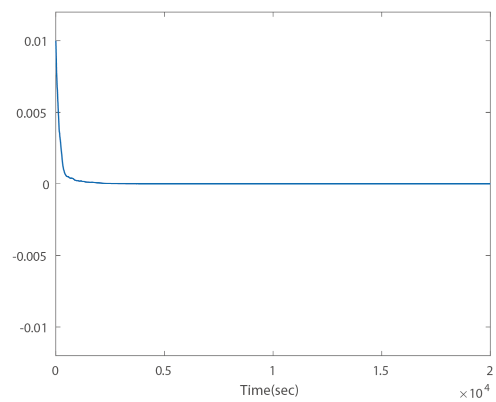

Figure 17. Time responses of mean values

Remark 3.1. In fact, if $H = \frac{1}{2}$, which implies that the error system (1.5) is driven by standard Brownian motion, synchronization will also be realised like following two pictures.

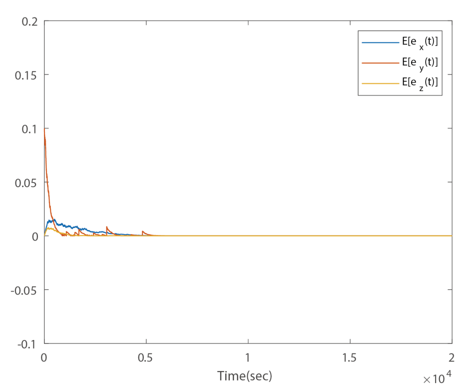

Figure 18. Times responses with controller Γ

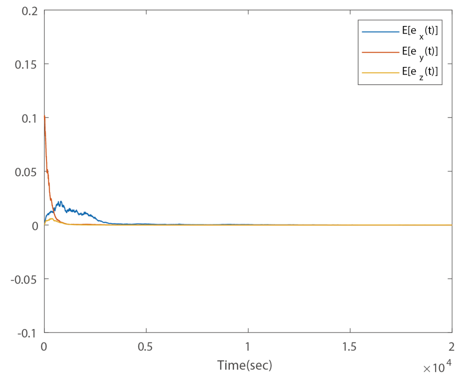

Figure 19. Times responses with controller $\tilde{\Gamma}$

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