Journal of Applied Analysis and Computation
Volume 12, Number 5, October 2022, 1786–1800
Website:http://www.jaac-online.com
DOI:10.11948/20210301

EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATION WITH INFINITE-POINT BOUNDARY VALUE CONDITIONS*

Limin Guo[1,†], Jingbo Zhao[2,†], Shuang Dong[1] and Xinglin Hou[2]

Abstract In this paper, the existence of positive solutions for Caputo fractional differential equations boundary value problem with infinite points contained in the boundary value condition, and based on Green's function and its properties, the existence of multiple positive solutions are obtained by Avery-Peterson fixed point theorem.

Keywords Caputo fractional differential equation, multiple positive solutions, infinite-point.

MSC(2010) 34B16, 34B18.

1. Introduction

In this paper, we consider the following $p-$Laplacian fractional differential equations

$$^{c}D^{\beta}_{0^+}(\varphi_p(^{c}D^{\alpha}_{0^+}x(t))) + f(t, x(t), x'(t), \ldots, x^{(i)}(t)) = 0, \; 0 < t < 1, \tag{1.1}$$

with infinite-point boundary condition

$$x^{(j)}(0) = 0, j = 0, 1, 2, \ldots, n-1, j \neq i, x^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j x(\xi_j), ^{c}D^{\alpha}_{0^+}x(0) = 0, \tag{1.2}$$

where $0 < \beta < 1, n-1 < \alpha \leq n$, $\alpha > i+1$, $\eta_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1 (j = 1, 2 \cdots)$, $\Sigma_{j=1}^{\infty}\eta_j\xi_j < 1$, $\phi_p(s) = |s|^{p-2}s, p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in C([0, 1] \times [0, +\infty)^{i+1}, [0, +\infty))$, and $^{c}D^{\alpha}_{0^+}$ is the standard Caputo derivative.

†The corresponding author.
Email: guolimin811113@163.com(L. Guo), zhaojb@czu.cn(J. Zhao)
[1]School of science, Changzhou Institute of Technology, Liaohe, 213002 Changzhou, China
[2]School of automotive engineering, Changzhou Institute of Technology, Liaohe, 213002 Changzhou, China
*This research was supported by the National Natural Science Foundation of China (Nos. 12101086, 62101074), Changzhou Science and technology planning project(No. CJ20210133), and Project of Shandong Province Higher Educational Science and Technology Program (No. J18KA217).

Fractional-order system may have additional attractive feature over the integer-order system, for example, the analytical solutions of the systems

$$\frac{d}{dt}x(t) = at^{a-1}, {}^C D_t^\alpha x(t) = at^{a-1}, 0 < \alpha < 1$$

are $t^a + x(0)$ and $\frac{a\Gamma(a)t^{a+\alpha-1}}{\Gamma(a+\alpha)} + x(0)$, respectively. Obviously, the integer-order system is unstable for $a \in (0,1)$, the fractional dynamic system is stable as $0 < a \leq 1-\alpha$, moreover, Fractional-order systems have been shown to be more accurate and realistic than integer order models and it also provides an excellent tool to describe the hereditary properties of material and processes, particularly in viscoelasticity, electrochemistry, porous media, and so on. As a result, there has been a significant development in the study of fractional differential equations in recent years, for an extensive collection of such literature, readers can refer to [1–4, 7, 8, 12–15, 17–23]. Jong [10] studied the following m point $p-$Laplacian fractional differential equations

$$D_{0^+}^\beta \left(\varphi_p\left(D_{0^+}^\alpha u\right)\right)(t) = f(t, u(t)), 0 < t < 1,$$

with $m-$ points boundary condition

$$u(0) = 0, D_{0^+}^\gamma u(1) = \sum_{i=1}^{m-2} \xi_i D_{0^+}^\gamma u(\eta_i),$$

$$D_{0^+}^\alpha u(0) = 0, \varphi_p(D_{0^+}^\alpha u(1)) = \sum_{i=1}^{m-2} \zeta_i \varphi_p(D_{0^+}^\alpha u(\eta_i)),$$

where $1 < \alpha, \beta \leq 2, 3 < \alpha+\beta \leq 4, 0 < \gamma \leq 1, \alpha-\gamma-1 > 0$, $0 < \eta_i, \zeta_i, \xi_i < 1 (i = 1, 2, \ldots, \infty)$, $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1} < 1, \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} < 1$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s,\ p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1$, and $f \in C([0,1]\times(0,+\infty), [0,+\infty))$, $D_{0^+}^\alpha$ is the Riemann-Liouville differential fractional derivative of order α. The authors obtained the existence and uniqueness of solutions by using the fixed point theorem for mixed monotone operators. Jong [9] obtained the existence and uniqueness of positive solutions by the Banach contraction mapping principle for equation in [9]. In [17], the author considered following fractional differential equation

$$D_{0^+}^\alpha u(t) + g(t)f(t, u(t)) = 0,\ \ 0 < t < 1,$$

with infinite-point boundary condition

$$u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j),$$

where $2 < \alpha,\ n-1 < \alpha < n,\ i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{j-1} < \xi_j < \ldots < 1 (j = 1, 2, \cdots)$, f permits singularities with respect to both the time and space variables, $D_{0^+}^\alpha$ is the Riemann-Liouville differential fractional derivative of order α. According to introducing height functions, the author obtained the existence and multiplicity of positive solution theorems, and Zhang and Zhai obtained the existence and uniqueness of positive solution in [16].

Motivated by the excellent results above, in this paper, we investigate the existence of multiple positive solutions for singular fractional differential equation with infinite-point boundary value conditions (1.1,1.2). Compared with [10, 17], i order derivative is contained in the nonlinear terms, and the derivative we used is the standard Caputo fractional derivative. Compared with our paper [5], the equation in this paper is $p-$Laplacian fractional differential equation.

2. Preliminaries and lemmas

For the convenience of the reader, we first present some basic definitions and lemmas which are useful for the following research are given, and can be found in the recent literature such as [7,11]. Let $E = C^i[0,1]$ be the Banach space with the maximum norm

$$\|x\| = \max\{\|x\|_0, \|x'\|_0, \ldots, \|x^{(i)}\|_0\},$$

where $\|x\|_0 = \max_{t\in[0,1]} |x(t)|, \ldots, \|x^{(i)}\|_0 = \max_{t\in[0,1]}\{|x^{(i)}(t)|\}$.

Definition 2.1 ([7,11])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([7,11])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to R$ is given by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.3 ([7, 11])**.** The Caputo fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$${}^cD^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

where α is fractional number, $n = [\alpha]+1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([7,11])**.** *Assume that $u \in C^n[0,1]$, then*

$$I^{\alpha}_{0+}\,{}^cD^{\alpha}_{0+}u(t) = u(t) - C_1 - C_2t - \ldots - C_nt^{n-1},$$

where n is the least integer greater than or equal to α, $C_i \in R$ $(i = 1,2,\ldots,n)$.

Lemma 2.2. *Given $h \in L^1[0,1]$, then the equation*

$${}^cD^{\beta}_{0+}(\varphi_p({}^cD^{\alpha}_{0+}x(t))) + h(t) = 0,\ 0 < t < 1, \tag{2.1}$$

with boundary condition (1.2) can be expressed by

$$x(t) = \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)d\tau\right)ds,\ t\in[0,1], \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Delta\Gamma(\alpha)}\begin{cases} t^iP(s)(1-s)^{\alpha-i-1} - \Delta(t-s)^{\alpha-1},\ 0\le s\le t\le 1,\\ t^iP(s)(1-s)^{\alpha-i-1},\ 0\le t\le s\le 1,\end{cases} \tag{2.3}$$

moreover,

$$\frac{\partial G^i(t,s)}{\partial t} = \frac{1}{\Delta\Gamma(\alpha)} \begin{cases} i!P(s)(1-s)^{\alpha-i-1} - (\alpha-1)(\alpha-2)\ldots(\alpha-i)\Delta(t-s)^{\alpha-i-1}, \\ 0 \le s \le t \le 1, \\ i!P(s)(1-s)^{\alpha-i-1},\ 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

in which $P(s) = (\alpha-1)(\alpha-2)\times(\alpha-i) - \Sigma_{s\le\xi_j}\eta_j(\frac{\xi_j-s}{1-s})^{\alpha-1}(1-s)^i$ *and* $\Delta = i! - \Sigma_{j=1}^{\infty}\eta_j\xi_j^i$.

Proof. By means of the Lemma 2.1, we can reduce (2.1) to an equivalent integral equation

$$\varphi_p({}^cD_{0+}^{\alpha}x(t)) = -I_{0+}^{\beta}h(t),$$

by ${}^cD_{0+}^{\alpha}x(0) = 0$, we get $c = 0$, hence

$$\begin{aligned} {}^cD_{0+}^{\alpha}x(t) &= \varphi_q\left(-\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right) \\ &= -\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right), \end{aligned} \tag{2.5}$$

by Lemma 2.1 again, we can reduce (2.5) to an equivalent integral equation

$$\begin{aligned} x(t) = &- I_{0+}^{\alpha}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right) + d_0 + d_1 t \\ &+ \cdots + d_{i-1}t^{i-1} + d_{i+1}t^{i+1} + \ldots + d_n t^{n-1}, \end{aligned}$$

for $d_i(i = 1, 2, \ldots, n-1) \in \mathbb{R}$. From $x^{(j)}(0) = 0, j = 0, 1, 2, \ldots, n-1, j \ne i$, we have $d_j = 0, j \ne i$. Consequently, we get

$$x(t) = d_i t^i - I_{0+}^{\alpha}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right),$$

hence,

$$x^{(i)}(t) = -I_{0+}^{\alpha-i}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right) + i!d_i.$$

On the other hand, $x^{(i)}(1) = \sum_{j=1}^{\infty}\eta_j x(\xi_j)$, combining with

$$x^{(i)}(1) = i!d_i - I_{0+}^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}h(s)ds\right),$$

we get

$$\begin{aligned} d_i = &\int_0^1 \frac{(1-s)^{\alpha-i-1}}{\Gamma(\alpha-i)(i! - \sum_{j=1}^{\infty}\eta_j\xi_j^i)}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)d\tau\right)ds \\ &- \sum_{j=1}^{\infty}\eta_j\int_0^{\xi_j}\frac{(\xi_j-s)^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{j=1}^{\infty}\eta_j\xi_j^i)}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)d\tau\right)ds \\ = &\int_0^1 \frac{(1-s)^{\alpha-i-1}P(s)}{\Gamma(\alpha)\Delta}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)d\tau\right)ds, \end{aligned}$$

where $P(s) = (\alpha-1)(\alpha-2)\times(\alpha-i)-\Sigma_{s\le\xi_j}\eta_j(\frac{\xi_j-s}{1-s})^{\alpha-1}(1-s)^i$and $\Delta = i!-\Sigma_{j=1}^{\infty}\eta_j\xi_j^i$. Hence

$$\begin{aligned}x(t) =&d_i t^i - I_{0+}^{\alpha}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}h(\tau)d\tau\right)ds\\ =&-\int_0^t\frac{\Delta(t-s)^{\alpha-1}}{\Gamma(\alpha)\Delta}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}h(\tau)d\tau\right)ds\\ &+\int_0^1\frac{(1-s)^{\alpha-i-1}t^iP(s)}{\Gamma(\alpha)\Delta}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}h(\tau)d\tau\right)ds\\ =&\int_0^1G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}h(\tau)d\tau\right)ds,\end{aligned}$$

therefore, (2.3) holds. By simple calculation, we have

$$\frac{\partial G^i(t,s)}{\partial t}=\frac{1}{\Delta\Gamma(\alpha)}\begin{cases}i!P(s)(1-s)^{\alpha-i-1}-(\alpha-1)(\alpha-2)\ldots(\alpha-i)\Delta(t-s)^{\alpha-i-1},\\ 0\le s\le t\le 1,\\ i!P(s)(1-s)^{\alpha-i-1}, 0\le t\le s\le 1,\end{cases}$$

therefore, (2.4) holds. □

Lemma 2.3. *Take $\imath,\jmath\in(0,1)$ with $\imath<\jmath$ such that*

$$P(s)\imath^i>\Delta\jmath^{\alpha-1}, i!>(\alpha-1)(\alpha-2)\times(\alpha-i)\jmath^{\alpha-i-1}$$

and take ξ_j,η_j such that

$$P(s)=(\alpha-1)(\alpha-2)\times(\alpha-i)-\Sigma_{s\le\xi_j}\eta_j(\frac{\xi_j-s}{1-s})^{\alpha-1}(1-s)^i<1,$$

then we have

$$\begin{aligned}&G(t,s)\le i!(\alpha-1)(\alpha-2)\times(\alpha-i)g(s),\ t,\ s\in[0,1],\\ &G(t,s)\ge\rho g(s)>0,\ t\in[\imath,\jmath],s\in[0,1];\\ &\frac{\partial^jG(t,s)}{\partial t^j}\le i!(\alpha-1)(\alpha-2)\times(\alpha-i),\ t,\ s\in[0,1],\\ &\frac{\partial^jG(t,s)}{\partial t^j}\ge\rho_jg(s)>0,\ t\in[\imath,\jmath],\ s\in[0,1],j=2,\ldots,i,\end{aligned}$$

where

$$\begin{aligned}&g(s)=\frac{(1-s)^{\alpha-i-1}}{\Gamma(\alpha)\Delta},\ \rho=\min\{\rho_1,\ \rho_2,\ldots,\rho_{i+1}\},\\ &\rho_1=\Delta(\imath^i-\jmath^{\alpha-1}),\ \rho_j=\Delta(i!-(\alpha-1)(\alpha-2)\times(\alpha-i)\jmath^{\alpha-i-1}),j=2,\ldots,i+1.\end{aligned}$$

Proof. By simple calcution, we get $P'(s)\ge 0, s\in[0,1]$, and so $P(s)$ is nondecreasing with respect to s. For $s\in[0,1]$, for $\alpha-i>1$, we get

$$\begin{aligned}P(s)=&(\alpha-1)(\alpha-2)\times(\alpha-i)-\Sigma_{s\le\xi_j}\eta_j(\frac{\xi_j-s}{1-s})^{\alpha-1}(1-s)^i\\ \ge&i!-\Sigma_{j=1}^{\infty}\eta_j\xi_j^i=\Delta,\end{aligned}$$

and obviously,

$$\begin{aligned} P(s) =&(\alpha-1)(\alpha-2)\times(\alpha-i)-\Sigma_{s\leq\xi_j}\eta_j\Big(\frac{\xi_j-s}{1-s}\Big)^{\alpha-1}(1-s)^i \\ \leq&(\alpha-1)(\alpha-2)\times(\alpha-i), s\in[0,1]. \end{aligned}$$

Hence, for $t,s\in[0,1]$, we have

$$\begin{aligned} G(t,s) &\leq \frac{t^iP(s)(1-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \leq \frac{P(s)(1-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \\ &\leq (\alpha-1)(\alpha-2)\times(\alpha-i)g(s) \\ &\leq i!(\alpha-1)(\alpha-2)\times(\alpha-i)g(s), \\ \frac{\partial^iG(t,s)}{\partial t^i} &\leq \frac{i!P(s)(1-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \\ &\leq i!(\alpha-1)(\alpha-2)\times(\alpha-i)g(s). \end{aligned}$$

Furthermore, for $0\leq s\leq 1$, we get

$$\begin{aligned} G(t,s) &= \frac{t^iP(s)(1-s)^{\alpha-i-1}-\Delta(t-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\ &= \frac{(t^iP(s)(1-s)^{\alpha-1}-\Delta(t-s)^{\alpha-i-1})(t-s)^i}{\Delta\Gamma(\alpha)} \\ &\geq 0, \end{aligned}$$

and, obviously, for $0\leq t\leq s\leq 1$, we get $G(t,s)\geq 0$. On the other hand, for $0\leq s\leq t\leq 1$, we have

$$\begin{aligned} \frac{\partial^iG(t,s)}{\partial t^i} &= \frac{i!P(s)(1-s)^{\alpha-i-1}-(\alpha-1)(\alpha-2)\dots(\alpha-i)\Delta(t-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \\ &\geq \frac{i!P(0)(1-s)^{\alpha-i-1}-(\alpha-1)(\alpha-2)\dots(\alpha-i)\Delta(t-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \\ &\geq \frac{\sum_{j=1}^{\infty}\eta_j\xi_j^i(\alpha-1)(\alpha-2)\dots(\alpha-i)-i!\sum_{j=1}^{\infty}\eta_j\xi_j^i(\alpha-1)}{\Delta\Gamma(\alpha)} \\ &\geq 0, \end{aligned}$$

and for $0\leq t\leq s\leq 1$, $\frac{\partial^iG(t,s)}{\partial t^i}\geq 0$ obviously holds.

For $t\in[\imath,\jmath], s\in[0,1]$, we have

$$\begin{aligned} G(t,s) &= \frac{t^iP(s)(1-s)^{\alpha-i-1}-\Delta(t-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\ &\geq \frac{\imath^iP(s)(1-s)^{\alpha-i-1}-\Delta(\jmath-\jmath s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\ &\geq \frac{[P(s)\imath^i-\Delta\jmath^{\alpha-1}](1-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \\ &\geq \frac{[\Delta(\imath^i-\jmath^{\alpha-1})(1-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} = \rho_1g(s), \end{aligned}$$

and for $t \in [\imath, \jmath], s \in [0,1]$, we have

$$\begin{aligned}
\frac{\partial^j G(t,s)}{\partial t^j} &\geq \frac{j!P(s)(1-s)^{\alpha-i-1} - \Delta(\alpha-1)(\alpha-2)\dots(\alpha-j)(t-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha-i)} \\
&\geq \frac{i!P(s)(1-s)^{\alpha-i-1} - \Delta(\alpha-1)(\alpha-2)\dots(\alpha-j)(b-bs)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \\
&= \frac{\Delta(i! - (\alpha-1)(\alpha-2)\dots(\alpha-j)b^{\alpha-i-1})(1-s)^{\alpha-i-1}}{\Delta\Gamma(\alpha)} \\
&= \rho_j g(s), \quad j = 1, \dots, i.
\end{aligned}$$

Now we define a cone P on E and an operator $A : P \to C^i[0,1]$ as follows

$$\begin{aligned}
P = &\{x \in E : \ x(t) \geq 0, \ x'(t) \geq 0, \dots, x^i(t) \geq 0, \ t \in [0,1], \\
&\min_{t \in [\imath,\jmath]} \{x^{(j)}(t)\} \geq \rho\|x\|, \ j = 0, 1, \dots, i\},
\end{aligned}$$

where $\imath, \jmath$ are the same as in Lemma 2.3, and

$$Ax(t) = \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)})d\tau\right) ds, \ x \in P.$$

By Lemma 2.3, we have that $x \in P$ is a solution of (1.1, 1.2) if it is a fixed point of A in P. □

Lemma 2.4. *The operator $A : P \to E$ is continuous.*

Proof. First, for $x \in P$, by the continuity of $G(t,s)$ and $f(s, x(s), x'(s), \dots, x^{(i)}(s))$, we have

$$Ax(t) = \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds, \ x \in P$$

is well defined on P. It thus follows from the uniform continuity of $G(t,s)$ in $[0,1] \times [0,1]$ and

$$\begin{aligned}
&|Ax(t_2) - Ax(t_1)| \\
\leq &\int_0^1 |G(t_2,s) - G(t_1,s)|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)})d\tau\right) ds
\end{aligned}$$

that $Ax \in C[0,1], \ x \in P$. Furthermore, by the uniform continuity of $G_{(t)}^{(i)}(t,s)$ for $t, s \in [0,1]$, we get

$$\begin{aligned}
Ax^{(i)}(t) = &\int_0^1 G_{(t)}^{(i)}(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds \\
&\in C[0,1].
\end{aligned}$$

On the other hand, Let $x_n \to x$ in $C^i[0,1]$, there exists $A > 0$ such that $\|x_n\| \leq A$ $(n = 1, 2, \cdots)$, and then $\|x\| \leq A$. Furthermore,

$$\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x_0, x_1, \dots, x_i)d\tau\right)$$

is continuous on $[0,1]\times(\mathbb{R}^+)^{i+1}$, so $\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f(\tau,x_0,x_1,\ldots,x_i)d\tau\right)$ is uniformly continuous on $[0,1]\times[0,A]^{i+1}$. Hence, for any $\varepsilon>0$ there exists $\delta>0$ such that, for any $s_1,\ s_2\in[0,1],\ x_0^1,\ x_0^2,\ x_1^1,\ x_1^2,\ldots,x_i^1,x_i^2\in[0,A]$, $|s_1-s_2|<\delta,|x_0^1-x_0^2|<\delta,|x_1^1-x_1^2|<\delta,\ldots,|x_i^1-x_i^2|<\delta$ we have

$$\begin{aligned}&\Big|\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f(\tau,x_0^1,x_1^1,\ldots,x_i^1)d\tau\right)\\&-\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f(\tau,x_0^2,x_1^2,\ldots,x_i^2)d\tau\right)\Big|<\varepsilon.\end{aligned}\tag{2.6}$$

By $\|x_n-x\|\to0$, then for the above $\varepsilon>0,\ \delta>0$, there exists N, when $n>N$, we have

$$|x_n(t)-x(t)|,\ |x_n'(t)-x'(t)|,\ldots,|x_n^{(i)}(t)-x^{(i)}(t)|\le\|x_n-x\|<\delta,\ \text{for any } t\in[0,1].$$

Thus, for $n>N,\ s\in[0,1]$, by (2.6), we have

$$\begin{aligned}&|(Ax_n)(t)-(Ax)(t)|\\=&\Big|\int_0^1G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau_1)^{\beta-1}f(\tau_1,x_n(\tau),x_n'(\tau),\ldots,x_n^{(i)}(\tau))d\tau\right)ds\\&-\int_0^1G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau_1)^{\beta-1}f(\tau_1,x(\tau),x'(\tau),\ldots,x^{(i)}(\tau))d\tau\right)\Big|\\\le&\varepsilon\int_0^1g(s)ds,\end{aligned}$$

and

$$\begin{aligned}&|(Ax_n)^{(i)}(t)-(Ax)^{(i)}(t)|\\=&\Big|\int_0^1\frac{\partial^iG(t,s)}{\partial t^i}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau_1)^{\beta-1}f(\tau,x_n(\tau),x_n'(\tau),\ldots,x_n^{(i)}(\tau))d\tau\right)\\&-\int_0^1\frac{\partial G^i(t,s)}{\partial t^i}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau_1)^{\beta-1}f(\tau,x(\tau),x_n'(\tau),\ldots,x_n^{(i)}(\tau))d\tau\right)\Big|\\\le&\varepsilon\int_0^1g(s)ds,\end{aligned}$$

hence, we get $\|Ax_n-Ax\|_0\to0,\|(Ax_n)'-(Ax)'\|_0\to0,\ldots,\ \|(Ax_n)^{(i)}-(Ax)^{(i)}\|_0\to0\ (n\to\infty)$. That is $\|Ax_n-Ax\|\to0(n\to\infty)$, i.e. A is continuous in the space$(E,\|.\|)$. □

Lemma 2.5. *$A:P\to P$ is completely continuous.*

Proof. From Lemma 2.3, we have $(Ax)^{(j)}(t)\ge0$, j=0, 1,2,...,i, $t\in[0,1]$ and

$$\begin{aligned}&\max_{t\in[0,1]}(Ax)^{(j)}(t)\\=&\max_{t\in[0,1]}\int_0^1\frac{\partial^jG(t,s)}{\partial t^j}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f(\tau,x(\tau),x'(\tau),\ldots,x^{(i)}(\tau))d\tau\right)ds\\\le&\int_0^1\max_{t\in[0,1]}\frac{\partial^jG(t,s)}{\partial t^j}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f(\tau,x(\tau),x'(\tau),\ldots,x^{(i)}(\tau))d\tau\right)ds\end{aligned}$$

$$\leq \int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds,$$

so

$$\begin{aligned}&\|Ax\|_0,\ \|(Ax)'\|_0, \dots, \|(Ax)^{(i)}\|_0\\ &\leq \int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds.\end{aligned}$$

Consequently

$$\begin{aligned}\|Ax\| &= \max\{\|Ax\|_0, \|(Ax)'\|_0, \dots, \|(Ax)^{(i)}\|_0\}\\ &\leq \int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds.\end{aligned}$$

Hence, for all $x \in P$, we have

$$\begin{aligned}&\min_{t\in[\imath,\jmath]}(Ax)^{(j)}(t)\\ &= \min_{t\in[\imath,\jmath]}\int_0^1 \frac{\partial^j G(t,s)}{\partial t^j}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds\\ &\geq \int_0^1 \min_{t\in[\imath,\jmath]}\frac{\partial^j G(t,s)}{\partial t^j}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds\\ &\geq \int_0^1 \rho_j g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds\\ &\geq \rho_{j+1}\int_0^1 \frac{\partial^{(j)} G(t,s)}{\partial t^j}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds\\ &\geq \rho_{j+1}\int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds\\ &\geq \rho_{j+1}\|Ax\|\\ &\geq \rho\|Ax\|, j = 0, 1, 2, \dots, i.\end{aligned}$$

Thus, $A(P) \subset P$.

Next we will proof that AV is relatively compact in E for bounded $V \subset P$. For $x \in V,\ t \in [0,1]$, for $s \in [0,1]$, we have

$$\begin{aligned}|Ax(t)| &= \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \dots, x^{(i)}(\tau))d\tau\right) ds\\ &\leq \int_0^1 \frac{i!(\alpha-1)(\alpha-2)\dots(\alpha-i)}{\Delta\Gamma(\alpha)}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\varphi(\tau)d\tau\right) ds\\ &\leq \frac{i!(\alpha-1)(\alpha-2)\dots(\alpha-i)}{\Delta\Gamma(\alpha)}\varphi_q\left(\frac{M}{\Gamma(\beta+1)}\right).\end{aligned}$$

Similarly, we can derive

$$|(Ax)^{(i)}(t)| \leq \frac{i!(\alpha-1)(\alpha-2)\dots(\alpha-i)}{\Delta\Gamma(\alpha)}\varphi_q\left(\frac{M}{\Gamma(\beta+1)}\right),\ t \in [0,1],$$

these show that AV is bounded in E. Next we will verify that $(AV)^{(i)}$ is equicontinuous. Let $t_1, t_2 \in [0,1], t_1 < t_2, x \in V$, we get

$$\begin{aligned}
&|(Ax)^{(i)}(t_2) - (Ax)^{(i)}(t_1)| \\
=&\left|\int_0^1 \frac{i!P(s)(1-s)^{\alpha-i-1}}{\Gamma(\alpha)\Delta} \times \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \ldots, x^{(i)}(\tau))d\tau\right) ds\right. \\
&- \int_0^{t_2} \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-i)-(t_2-s)^{\alpha-2}}{\Gamma(\alpha)} \\
&\times \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \ldots, x^{(i)}(\tau))d\tau\right) ds \\
&- \int_0^1 \frac{i!P(s)(1-s)^{\alpha-i-1}}{\Gamma(\alpha)\Delta} \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \ldots, x^{(i)}(\tau))d\tau\right) ds \\
&- \int_0^{t_1} \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-i)-(t_1-s)^{\alpha-2}}{\Gamma(\alpha)} \\
&\times \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \ldots, x^{(i)}(\tau))d\tau\right) ds| \\
=&\left|\frac{1}{\Gamma(\alpha-i)}\int_0^{t_2} (t_2-s)^{\alpha-2} \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \ldots, x^{(i)}(\tau))d\tau\right) ds\right. \\
&\left.-\frac{1}{\Gamma(\alpha-i)}\int_0^{t_1} (t_1-s)^{\alpha-2} \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau), x'(\tau), \ldots, x^{(i)}(\tau))d\tau\right) ds\right| \\
\leq& \varphi_q(\Lambda\|\varphi(\tau)\|_{q_1})\frac{1}{\Gamma(\alpha-i)}(\int_0^{t_2} (t_2-s)^{\alpha-i-1} ds - \int_0^{t_1} t_1 - s)^{\alpha-i-1}) ds \\
=&\frac{\varphi_q(\Lambda\|\varphi(\tau)}{\Gamma(\alpha-i+1)}(t_2^{\alpha-i} - t_1^{\alpha-i}).
\end{aligned}$$

From above and the uniform continuity of $t^{\alpha-i}$ on $[0,1]$, and together with the theorem 1.2.7 of [6], we can derive than AV is relatively compact in $C^i[0,1]$, so we can get $A : P \to P$ is completely continuous. □

Definition 2.4. The map α is said to be non-negative continuous concave functional on P, provided $\alpha : P \to R^+$is continuous and

$$\alpha(tx + (1-t)y) \geq \alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$, $t \in [0,1]$.

Definition 2.5. The map β is said to be non-negative continuous convex functional on P, provided $\beta : P \to R^+$ is continuous and

$$\beta(tx + (1-t)y) \leq \beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$, $t \in [0,1]$.

3. Main result

Let φ, θ be non-negative continuous convex functional on P, ϕ be non-negative continuous concave functional on P, and ψ be non-negative continuous functional on P. Then for non-negative number $e,\ f,\ g,\ h$, we define the following convex sets:

$$
\begin{aligned}
&P(\varphi, h) = \{x \in P | \varphi(x) < h\},\\
&P(\varphi, \phi, f, h) = \{x \in P | \phi(x) \geq f,\ \varphi(x) \leq h\},\\
&P(\varphi, \theta, \phi, f, g, h) = \{x \in P | f \leq \phi(x), \theta(x) \leq g,\ \varphi(x) \leq h,\\
&R(\varphi, \psi, e, h) = \{x \in P | e \leq \psi(x),\ \varphi(x) \leq h\}.
\end{aligned}
$$

We will apply the following fixed point theorem of Avery and Peterson to solve the problem $(1.1, 1.2)$.

Lemma 3.1 ([25]). *Let P is a cone of E. φ, θ be non-negative continuous convex functional on P, ϕ be non-negative continuous concave functional on P, and ψ be non-negative continuous functional on P, $\psi(\mu x) \leq \mu\psi(x),\ 0 \leq \mu \leq 1$, such that for some positive numbers L and h, satisfy*

$$\phi(x) \leq \psi(x) \text{ and } \|x\| \leq L\varphi(x)$$

for all $x \in \overline{P(\varphi, h)}$. If

$$A : \overline{P(\varphi, h)} \to \overline{P(\varphi, h)}$$

is completely continuous, and there exist positive number $e,\ f,\ g$ with $e < f$, such that the following conditions are satisfied:

(S_1) $\{x \in P(\varphi, \theta, \phi, f, g, h) : \phi(x) > f\} \neq \phi$, and $\phi(Ax) > f$, for $x \in P(\varphi, \theta, \phi, f, g, h)$;

(S_2) $\phi(Ax) > f$, for $x \in P(\varphi, \phi, f, h)$ and $\theta(Ax) > g$;

(S_3) $0 \notin R(\varphi, \psi, e, h)$ and $\psi(Ax) < e$, for $x \in R(\varphi, \psi, e, h)$ with $\psi(x) = e$. Then A at least exist three fixed points $x_1,\ x_2,\ x_3$, such that

$$\varphi(x_i) \leq h,\ \text{for } i = 1,\ 2,\ 3,$$

and

$$f < \phi(x_1),\ e < \psi(x_2),\ \phi(x_2) < f,\ \psi(x_3) < e.$$

Let convex functions $\psi(x) = \theta(x) = \varphi(x) = \|x\|$ on P, define a concave function $\phi(x) = \min\{\min_{t\in[\imath,\jmath]} |x(t)|, \min_{t\in[\imath,\jmath]} |x'(t)|, \ldots, \min_{t\in[\imath,\jmath]} |x^{(i)}(t)|\}$, where $\imath,\ \jmath$ are the same as in lemma 2.3.

Theorem 3.1. *Assume that there exist positive numbers $e,\ f,\ g,\ h$ with $f > e$, $g > \max\{\frac{1}{\rho}, e^{1-\frac{\imath}{2}}\}f,\ h > \frac{r}{\rho Q}f$ and $h \geq g$, such that*

(H_3) $\varphi_q(f(t, x, x', \ldots, u^{(i)})) < \frac{h}{r}$, for $(t, x, x', \ldots, x^{(i)}) \in [0, 1] \times [0, h]^{i+1}$;

(H_4) $\varphi_q(f(t, x, x', \ldots, x^{(i)})) \geq \frac{f}{\rho Q}$, for $(t, x, x', \ldots, x^{(i)}) \in [\imath, \jmath] \times [f, g]^{i+1}$;

(H_5) $\varphi_q(f(t, x, x', \ldots, x^{(i)})) < \frac{e}{r}$, for $(t, x, x', \ldots, x^{(i)}) \in [0, 1] \times [0, e]^{i+1}$,

where $r = \varphi_q(\frac{1}{\Gamma(\beta+1)}) \int_0^1 g(s)ds,\ Q = \int_\imath^\jmath g(s)ds$. The problem $(1.1, 1.2)$ has at least three fixed points $x_1,\ x_2,\ x_3$ satisfying $\|x_i\| \leq h,\ i = 1,\ 2,\ 3$, and $f < \phi(x_1),\ e < \psi(x_2),\ \phi(x_2) < f, \psi(x_3) < e$.

Proof. Let $x \in \overline{P(\varphi, h)}$. By condition (H_3), we get

$$
\begin{aligned}
\|Ax\|_0 = \max_{t\in[0,1]} |Ax(t)| \leq& \int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x, x', \ldots, x^{(i)})d\tau\right)ds\\
\leq& \varphi_q(\frac{1}{\Gamma(\beta+1)})\frac{h}{r}\int_0^1 g(s)ds \leq h,
\end{aligned}
$$

$$
\begin{aligned}
\|(Ax)^{(i)}\|_0 &= \max_{t\in[0,1]}\{|\frac{\partial^i(Ax)(t)}{\partial t^i}|\} \\
&\leq \int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau,x,x',\ldots,x^{(i)})d\tau\right)ds \\
&\leq \varphi_q(\frac{1}{\Gamma(\beta+1)})\frac{h}{r}\int_0^1 g(s)ds \leq h,
\end{aligned}
$$

consequently, we obtain $\varphi(Ax) = \|Ax\| \leq h$. This, together with Lemma 2.4 and 2.5, means that $A:\overline{P(\varphi,h)} \to \overline{P(\varphi,h)}$ is completely continuous.

Take $x(t) = fe^{t-0.5\imath},\ t\in[0,1]$. According to simple calculation, we can have that $x\in P$, $\|x\|<g$, and $\phi(x)>f$, so

$$\{x\in P(\varphi,\theta,\phi,f,g,h):\ f<\phi(x)\}\neq\emptyset.$$

For $x\in P(\varphi,\theta,\phi,f,g,h)$, by (H_4), we get

$$
\begin{aligned}
\phi(Ax) &= \min\left\{\min_{t\in[\imath,\jmath]}|Ax(t)|, \min_{t\in[\imath,\jmath]}|(Ax)'(t)|,\ldots,\min_{t\in[\imath,\jmath]}|(Ax)^{(i)}(t)|\right\} \\
&\geq \rho\int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau,x,x',\ldots,x^{(i)})d\tau\right)ds \\
&> \int_\imath^\jmath \rho g(s)\frac{f}{\rho Q}ds = \jmath,
\end{aligned}
$$

which shows that condition S_1 is satisfied.

Take $x\in P(\varphi,\phi,f,h)$ and $\|Ax\|>g$. Since $Ax\in P$, we obtain

$$
\begin{aligned}
\phi(Ax) &= \min\left\{\min_{t\in[\imath,\jmath]}|Ax(t)|, \min_{t\in[\imath,\jmath]}|(Ax)'(t)|,\ldots,\min_{t\in[\imath,\jmath]}|(Ax)^{(i)}(t)|\right\} \\
&\geq \rho\|Ax\| \geq \rho g > f,
\end{aligned}
$$

which means that condition (S_2) holds.

Next we will verify condition (S_3) holds. For $\psi(0)=0$, we have $0\in R(\varphi,\psi,e,h)$. Let $x\in R(\varphi,\psi,e,h)$ and $\psi(x)=\|x\|=e$, by (H_5), we get

$$
\begin{aligned}
\|Ax\|_0 &= \max_{t\in[0,1]}|Ax(t)| \leq \int_0^1 g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau,x,x',\ldots,x^{(i)})d\tau\right)ds \\
&< \frac{e}{r}\varphi_q(\frac{1}{\Gamma(\beta+1)})\int_0^1 g(s)ds \leq e, \\
\|(Ax)'\|_0 &= \max_{t\in[0,1]}\left|\frac{\partial(Ax)(t)}{\partial t}\right| \\
&\leq \int_0^1 g(s)g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau,x,x',\ldots,x^{(i)})d\tau\right)ds \\
&< \frac{e}{r}\varphi_q(\frac{1}{\Gamma(\beta+1)})\int_0^1 g(s)ds \leq e,
\end{aligned}
$$

$\cdots$

$$\begin{aligned}\|(Ax)^{(i)}\|_0 &= \max_{t\in[0,1]}\left|\frac{\partial^i (Ax)(t)}{\partial t^i}\right| \\ &\le \int_0^1 g(s)g(s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau,x,x',\ldots,x^{(i)})d\tau\right)ds \\ &< \frac{e}{r}\varphi_q(\frac{1}{\Gamma(\beta+1)})\int_0^1 g(s)ds \le e.\end{aligned}$$

Consequently, we have $\psi(Ax)=\|Ax\|<e$. Then, condition (S_3) holds. Consequently, we have $\psi(Ax)=\|Ax\|<e$. Then, condition (S_3) holds.

By Lemma 3.1, we can get that $(1.1, 1.2)$ exists at least three positive solutions $x_1^*,\ x_2^*,\ x_3^*$, satisfying

$$\|x_i^*\|\le h,\ i=1,\ 2,\ 3,\ \text{and}\ f<\phi(x_1^*),\ e<\psi(x_2^*),\ \phi(x_2^*)<f,\ \psi(x_3^*)<e.$$

□

4. An example

Consider the following infinite-point boundary value problem:

$$\begin{cases} {}^cD_{0+}^{\frac{1}{2}}(\varphi_p({}^cD^{\frac{7}{2}}x(t)))+f(t,x(t),x''(t))=0,\ 0<t<1,\\ x(0)=x'(0)=x'''(0)=0,\ x''(1)=\displaystyle\sum_{j=1}^{\infty}\frac{1}{j^2}x(\frac{1}{j}),{}^c D^{\frac{7}{2}}x(0)=0,\end{cases} \tag{4.1}$$

where $\varphi_q(f(t,x_1,x_2))$ is continuous in $[0,1]\times R^+\times R^+$, $\varphi_q(f(t,x_1,x_2))\le\frac{499\times 3}{\sqrt{\pi}}$, for $(t,x_1,x_2)\in[0,1]\times R^+\times R^+$, such that

$$\varphi_q(f(t,x_1,x_2))=\begin{cases}\frac{1}{2\sqrt{\pi}}(x_1^2+x_2^2),\ (t,x_1,x_2)\in(0,1]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}],\\ \frac{499}{2\sqrt{\pi}}(\sqrt[6]{x_1}+\sqrt[6]{x_2}),\ (t,x_1,x_2)\in(0,1]\times[1,20]\times[1,20],\\ \frac{499}{2\sqrt{\pi}},\ (t,x_1,x_2)\in(0,1]\times[100,\infty)\times[100,\infty).\end{cases}$$

By means of Theorem 3.1, we take $\alpha=\frac{7}{2},\beta=\frac{1}{2},a=\frac{2}{5},b=\frac{4}{9},\rho_1=\Delta(a^2-b^{\frac{5}{2}})\ge\Delta(a-b^{\alpha-1})\approx 0.0283\Delta,\rho_2=P(s)-\Delta(\alpha-1)b^{\alpha-2}\ge\Delta(1-2.5(\frac{4}{9})^{\frac{3}{2}})\approx 0.2592\Delta\ge 0,\Delta=i!-\Sigma_{j=1}^{\infty}\eta_j\xi_j^i=2!-\Sigma_{j=1}^{\infty}\eta_j\xi_j^2\approx 0.9177,\ P(s)=(\alpha-1)(\alpha-2)-\Sigma_{s\le\xi_j}\eta_j(\frac{\xi_j-s}{1-s})^{\alpha-1}(1-s)^2\le 1$, apparently, $\rho_2<\rho_1\ \rho=\rho_2,\ \eta_j=\frac{1}{j^2},\xi_j=\frac{1}{j}$, $g(s)=\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha)\Delta}=\frac{1}{\Gamma(\frac{7}{2})\Delta}(1-s)^{\frac{1}{2}},\ r=\varphi_q(\frac{1}{\Gamma(\beta+1)})\int_0^1 g(s)ds=\frac{3}{2\Gamma(\varphi_q(\frac{2}{\sqrt{\pi}}))\frac{7}{2}\Delta},\ L=\int_a^b g(s)ds=\frac{2}{3\Gamma(\frac{7}{2})\Delta}[(\frac{3}{5})^{\frac{3}{2}}-(\frac{5}{9})^{\frac{3}{2}}]$. Let $e=\frac{1}{2},\ f=1,\ g=20,\ d=500$. By direct calculation, we can obtain that the conditions of Theorem 3.1 are satisfied. So, the BVP (4.1) exists at least three positive solutions x_1^*,x_2^*,x_3^* satisfying

$$x_i^*\le 500,\ i=1,\ 2,\ 3,$$

and

$$1<\phi(x_1^*),\ \frac{1}{2}<\|x_2^*\|,\ \phi(x_2^*)<1,\ \|x_3^*\|<\frac{1}{2}.$$

Acknowledgements

We would like to thank you to typeset your article according the above style very closely in advance.

References

[1] A. Cabada and Z. Hamdi, *Nonlinear fractional differential equations with integral boundary value conditions*, Appl. Math. Comput., 2014, 228(2012), 251–257.

[2] L. Guo and L. Liu, *Unique iterative positive solutions for a singular p-laplacian fractional differential equation system with infinite-point boundary conditions*, Boundary Value Problems, 2019, 2019(1).

[3] L. Guo, L. Liu and Y. Feng, *Uniqueness of iterative positive solutions for the singular infinite-point p-laplacian fractional differential system via sequential technique*, Nonlinear Analysis: Modelling and Control, 2020, 25(5).

[4] L. Guo, L. Liu and Y. Wang, *Maximal and minimal iterative positive solutions for p-laplacian hadamard fractional differential equations with the derivative term contained in the nonlinear term*, AIMS. Math., 2021, 6(11), 12583–12598.

[5] L. Guo, J. Zhao,L. Liao and L. Liu, *Existence of multiple positive solutions for a class of infinite-point singular p-Laplacian fractional differential equation with singular source terms*, Nonlinear Anal.: Model. Control, 2022, 27(4), 609–629.

[6] L. Guo and L. Liu, *Maximal and minimal iterative positive solutions for singular infinite-point p-laplacian fractional differential equations*, Nonlinear Analysis: Modelling and Control, 2018, 23(6), 851–865.

[7] P. Hentenryck, R. Bent and E. Upfal, *An introduction to the Fractional Calculus and Fractional Differential Equations* , Wiley, New York, 1993.

[8] M. Jleli and B. Samet, *Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method*, Nonlinear Anal.: Model. Control, 2015, 20(3), 367–376.

[9] K. Jong, *Existence and uniqueness of positive solutions of a kind of multi-point boundary value problems for nonlinear fractional differential equations with p-laplacian operator*, Mediterr. J. Math., 2018, 2018, 129.

[10] K. Jong, H. Choi and Y. Ri, *Existence of positive solutions of a class of multi-point boundary value problems for p-laplacian fractional differential equations with singular source terms*, Commun. Nonlinear Sci. Numer. Simulat., 2019, 72, 272–281.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science BV, Amsterdam, 2006.

[12] S. Liang and J. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equation*, Nonlinear Anal., 2009, 71, 5545–5550.

[13] D. Ma, *Positive solutions of multi-point boundary value problem of fractional differential equation*, Arab. J. Math. Sci., 2015, 21(2), 225–236.

[14] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.

[15] Y. Wang, L. Liu, X. Zhang and Y. Wu, *Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection*, Appl. Math. Comput., 2015, 258, 312–324.

[16] C. Zhai and L. Wang, *Some existence,uniqueness results on positive solutions for a fractional differential equation with infinite-point boundary conditions*, Nonlinear Anal. Model. Control, 2017, 22, 566–577.

[17] X. Zhang, *Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions*, Appl. Math. Lett., 2015, 39, 22–27.

[18] X. Zhang, L. Liu, B. Wiwatanapataphee and Y. Wu, *The eigenvalue for a class of singular p-laplacian fractional differential equations involving the riemann–stieltjes integral boundary condition*, Appl. Math. Comput., 2014, 235(4), 412–422.

[19] X. Zhang, L. Liu, Y. Wu and B. Wiwatanapataphee, *The spectral analysis for a singular fractional differential equation with a signed measure*, Appl. Math. Comput., 2015, 257, 252–263.

[20] X. Zhang, Z. Shao and Q. Zhong, *Positive solutions for semipositone* $(k, n-k)$ *conjugate boundary value roblems with singularities on space variables*, Appl. Math. Lett., 2017, 72, 50–57.

[21] X. Zhang, L. Wang and Q. Sun, *Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter*, Appl. Math. Comput., 2014, 226, 708–718.

[22] X. Zhang and Q. Zhong, *Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions*, Fract. Calc. Appl. Anal., 2017, 20(6), 1471–1484.

[23] X. Zhang and Q. Zhong, *Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables*, Appl. Math. Lett., 2018, 80, 12–19.

[24] R. W. Leggett and L. R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ.math.j., 1979, 28, 673-688.

[25] R.I. Avery and A.C. Peterson, *Three positive fixed points of nonlinear operators on order Banach spaces*, Comput. Math. Appl., 2001, 42, 313-322.