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MULTIPLE SOLUTIONS FOR A CLASS OF MODIFIED QUASILINEAR FOURTH-ORDER ELLIPTIC EQUATIONS

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Abstract In this paper, we consider the following modified quasilinear fourth-order elliptic equations:

$$\Delta^2 u-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u-\frac{\kappa}{2}\Delta(u^2)u=f(x,u),\quad \text{in } \mathbb{R}^3,$$

where $a>0, b\geq 0, \kappa\geq 0$. Under some appropriate assumptions on $V(x)$ and $f(x,u)$, multiplicity results of two different type of solutions are established via the Mountain Pass lemma and the local minimization.

Keywords Fourth-order elliptic equations, multiple solutions, Mountain Pass lemma, local minimization.

MSC(2010) 35J35, 35J62.

1. Introduction and main results

In this paper, we consider the following modified quasilinear fourth-order elliptic equation:

$$\Delta^2 u-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u-\frac{\kappa}{2}\Delta(u^2)u=f(x,u),\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\Delta^2=\Delta(\Delta)$ is the biharmonic operator, $a>0, b\geq 0, \kappa\geq 0$. For the potential $V(x):\mathbb{R}^3\to\mathbb{R}$ we assume:

(V_1) There exist constants m, m_0 satisfying $0<m<m_0<\frac{1}{2S_2^2}$, $\inf_{x\in\mathbb{R}^3}V(x)>m-m_0$, where S_2 is defined in (2.1). Moreover, for any $M>0$, $meas\{x\in\mathbb{R}^3|V(x)\leq M\}<\infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^3$.

When $\kappa=0$, we get the following fourth-order elliptic equation:

$$\Delta^2 u-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u=f(x,u),\quad \text{in } \mathbb{R}^3. \tag{1.2}$$

Problem (1.2) is often called nonlocal because of the presence of the integral term $\int_{\mathbb{R}^3}|\nabla u|^2dx\Delta u$, which implies that the equation (1.2) is no longer a pointwise identity. Moreover, if we set $V(x)=0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega\subset\mathbb{R}^3$ in (1.2), then we get the following fourth-order elliptic equation of Kirchhoff type:

$$\begin{cases}\Delta^2 u-(a+b\int_{\mathbb{R}^3}|\nabla|^2dx)\Delta u=f(x,u) & x\in\Omega,\\ u=\nabla u=0, & x\in\partial\Omega,\end{cases} \tag{1.3}$$

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which is related to the following stationary analogue of the equation of Kirchhoff type:

$$u_{tt} + \Delta^2 u - (a + b\int_{\mathbb{R}^3} |\nabla|^2 dx)\Delta u = f(x,u) \quad \text{in } \Omega.$$

In recent years, there have been many works about the existence of nontrivial solutions to (1.2) or (1.3) by using variational methods, see [2, 5, 9, 13–16, 22–25]. In [24], Wang and An obtained the existence of nontrivial solutions to (1.3) by using the Mountain Pass lemma when $f(x,u)$ is asymptotically linear at both zero and infinity and satisfies other conditions.

In [16], Mao and Wang studied (1.2) under the conditions: the potential $V(x)$ is allowed to be sign-changing and the nonlinearity $f(x,u)$ involves the combination of convex and concave terms. They proved that (1.2) has two type of nontrivial solutions, one is obtained via the Mountain Pass lemma, the other is constructed through the local minimization.

In [22], Wang et al. studied the existence of positive solutions by using variational methods and the truncation method for the fourth order elliptic equation:

$$\Delta^2 u - (a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + cu = f(u), \qquad \text{in } \mathbb{R}^N,$$

where $N > 4$, $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies $f(t) \le C(1 + t^p)$ for some $p \in (1, \frac{N+4}{N-4})$, $\lim_{t\to 0} \frac{f(t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$.

Compared to the semilinear problem ($\kappa = 0$), the quasilinear case ($\kappa \ne 0$) becomes more complicated since the effects of the quasilinear and non-convex term $\Delta(u^2)u$ [1, 12]. One of the main difficulties of the quasilinear problem is that there is no suitable space on which the energy functional is well defined and belongs to C^1-class except for the one-dimensional case [17]. There has been several ideas and approaches used in recent years to overcome the difficulties such as by minimizations [10, 17], the Nehari or Pohozaev manifold [11, 18] and change of variables [26, 27].

Motivated by the above works, the main aim of this article is to study the existence of multiple solutions for (1.1) with sign-changing potential via the Mountain Pass lemma and the local minimization in critical point theory. To the best of our knowledge, there are few articles dealing with this type of fourth-order elliptic equation (1.1). Setting $F(x,u) = \int_0^u f(x,t)dt$ and suppose that $F(x,u) = G(x,u) + H(x,u)$, $G(x,u) \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Moreover, we suppose that G satisfies the following assumptions:

(g_1) $G(x,0) \equiv 0$ for all $x \in \mathbb{R}^3$ and there exist a real number $r_1 > 4$ and two continuous bounded functions $\eta, \zeta : \mathbb{R}^3 \to \mathbb{R}$ with $\zeta > 0$ on a bounded domain Ω such that

$$\eta(x) \le \frac{G(x,u)}{|u|^{r_1}} \le \zeta(x), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, u \ne 0,$$

and $\lim_{|u|\to\infty} \frac{G(x,u)}{|u|^{r_1}} = \zeta(x)$, uniformly in $x \in \mathbb{R}^3$.

(g_2) There exists d_0 satisfying $0 \le d_0 < \frac{1-m_0 S_2^2}{4S_2^2}$ such that

$$G(x,u) - \frac{1}{4}g(x,u)u \le d_0|u|^2, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R},$$

where $g(x,u) = G_u(x,u)$.

(g_3) There exist $2 < p < \infty$ and $a_2 > 0$ such that

$$|g(x,u)| \leq a_2(1 + |u|^{p-1}), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(g_4) $\lim_{|u|\to 0} \frac{g(x,u)}{u} = 0$, uniformly in $x \in \mathbb{R}^3$.
(g_5) There exist $\mu > 4$ and $r_2 > 0$ such that

$$\mu G(x,u) \leq ug(x,u), \quad \forall x \in \mathbb{R}^3, |u| \geq r_2.$$

(g_6) $\inf_{x\in\mathbb{R}^3, |u|=r_2} G(x,u) > 0$.
Our main results read as follows.

Theorem 1.1. *Assume conditions* (V_1), (g_1) *and* (g_2) *hold,* $H(x,u) = \alpha(x)|u|^s$, *where* $1 < s < 2$, $\alpha(x) \in L^{\frac{2}{2-s}}(\mathbb{R}^3)$ *and* $\alpha(x) \geq 0$. *Then*
(i) problem (1.1) *possesses at least one nontrivial mountain-pass type of solution;*
(ii) problem (1.1) *possesses at least one nontrivial local minimum type of solution.*

Theorem 1.2. *Assume conditions* (V_1), (g_3)-(g_6) *hold,* $H(x,u) = h(x)u$, *where* $h(x) \in L^2(\mathbb{R}^3)$ *and* $h(x) \geq 0$. *Then there exists a constant* $n_0 > 0$ *such that for* $||h||_2 < n_0$,
(i) problem (1.1) *possesses at least one nontrivial mountain-pass type of solution;*
(ii) problem (1.1) *possesses at least one nontrivial local minimum type of solution.*

Compared with literature, the novelty of our results lies in two aspects. One is that problem (1.1) considered here is set in whole space and the quasilinear and non-convex term $\Delta(u^2)u$ is allowed to exist, furthermore, the nonlinearity $f(x,u)$ involves the combination of convex and concave terms which makes it very difficult to check the Mountain Pass geometry for energy functional. The other is that we obtain two different types of nontrivial solutions of the problem (1.1) via variational method. As mentioned earlier, our results extend and generalize the results obtained in [16,20,22,24].

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we are concerned with the proof of Theorem 1.1. Section 4 is devoted to dealing with the proof of Theorem 1.2.

2. Notations and Preliminaries

Throughout this paper, $L^r(\mathbb{R}^3)$ is the usual Lebesgue space whose norms we denote by $||u||_r = \left(\int_{\mathbb{R}^3} |u|^r dx\right)^{1/r}$ for $1 \leq r < \infty$, and $||u||_\infty = ess \sup_{x\in\mathbb{R}^3} |u(x)|$ for $r = \infty$. $H^2(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) | |\nabla u|, \Delta u \in L^2(\mathbb{R}^3)\}$ is the Sobolev space with standard norm. We shall denote by C, C_i,$i = 1, 2, \cdots$ for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of problem.

Let

$$E = \{u \in H^2(\mathbb{R}^3) | \int_{\mathbb{R}^3} (|\Delta u|^2 + a|\nabla u|^2 + (V(x) + m_0)u^2) dx < +\infty\}.$$

Then E is a Hilbert space with the inner product

$$(u,v) = \int_{\mathbb{R}^3} (\Delta u \Delta v + a\nabla u \nabla v + (V(x) + m_0)uv) dx,$$

and the norm

$$||u|| = (\int_{\mathbb{R}^3} (|\Delta u|^2 + a|\nabla u|^2 + (V(x) + m_0)|u|^2)dx)^{\frac{1}{2}}.$$

Since

$$\begin{aligned} ||u||_{H^2}^2 &= \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla u|^2 + |u|^2)dx \\ &\leq C \int_{\mathbb{R}^3} (|\Delta u|^2 + a|\nabla u|^2 + (V(x) + m_0)|u|^2)dx = C||u||^2, \end{aligned}$$

where $C = \max\{1, \frac{1}{a}, \frac{1}{m}\}$, and $H^2(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3), 2 \leq r \leq \infty$, then the embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for $2 \leq r \leq \infty$, and there exists $S_r > 0$ such that

$$||u||_r \leq S_r||u||, \quad \text{for all } u \in E. \tag{2.1}$$

Moreover, we have the following compactness lemma from [6].

Lemma 2.1. *Under the assumption (V_1), the embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is compact for $2 \leq r < \infty$.*

Define energy functional I on E by

$$I(u) = \frac{1}{2}||u||^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 + \frac{\kappa}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^3} F(x,u)dx - \frac{m_0}{2}\int_{\mathbb{R}^3} u^2 dx. \tag{2.2}$$

Obviously, I is a well-defined C^1 functional and satisfies

$$\begin{aligned} \langle I'(u), v\rangle = &\int_{\mathbb{R}^3} (\Delta u \Delta v + a\nabla u \nabla v + (V(x) + m_0)uv)dx + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla v dx \\ &+ \kappa \int_{\mathbb{R}^3} (uv|\nabla u|^2 + u^2 \nabla u \nabla v)dx - \int_{\mathbb{R}^3} f(x,u)v dx - m_0 \int_{\mathbb{R}^3} uv dx. \end{aligned} \tag{2.3}$$

$u \in E$ is a solution of problem (1.1) if and only if $u \in E$ is a critical point of I.

The following lemma allows us to find *Cerami* and *Palais-Smale* type sequence. Recall that a sequence $\{u_n\} \subset E$ is said to be a *Cerami* sequence at the level $c \in \mathbb{R}$ ($(C)_c$ sequence for short) if $I(u_n) \to c$ and $(1+||u_n||)I'(u_n) \to 0$ as $n \to \infty$. I is said to satisfy the $(C)_c$ condition if any $(C)_c$ sequence has a convergent subsequence. Moreover, a sequence $\{u_n\} \subset E$ is said to be a *Palais-Smale* sequence at the level $c \in \mathbb{R}$ ($(PS)_c$ sequence for short) if $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. I is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence has a convergent subsequence.

Lemma 2.2 ([8]). *Suppose E is a real Banach space, $I \in C^1(E, \mathbb{R})$ satisfies*

$$I(0) = 0, \quad \inf_{||u||=\rho} I(u) \geq \nu > 0 \geq I(e),$$

for some $\nu, \rho > 0$ and $e \in E$ with $||e|| > \rho$. Let c be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) = e\}$, then $c \geq \nu$ is finite and I possesses a $(C)_c$ sequence at level c.

Lemma 2.3 ([4]). *Given a weakly lower semicontinuous functional $I : E \to \mathbb{R}$ on a Banach space E and a closed convex subset $X \subset E$ on which I is bounded from below, then we can find $u_0 \in X$ such that $I(u_0) = \inf_{u \in X} I(u)$.*

3. Proof of Theorem 1.1

Throughout this section, $H(x,u) = \alpha(x)|u|^s$, where $1 < s < 2$, $\alpha(x) \in L^{\frac{2}{2-s}}(\mathbb{R}^3)$ and $\alpha(x) \geq 0$. We begin with some lemmas.

Lemma 3.1. *Assume (g_1) holds. Set*

$$\aleph(u) = \int_{\mathbb{R}^3} F(x,u)dx,$$

then $\aleph$ is weakly continuous.

Proof. The proof is similar to lemma 2.2 in [4], we omit it. $\square$

Lemma 3.2 ([16]). *Let $1 < s < 2 < r, A, B > 0$, and consider the function*

$$\Phi_{A,B} = t^2 - At^s - Bt^r$$

for $t \geq 0$. Then $\max_{t\geq 0} \Phi_{A,B}(t) > 0$ if and only if $A^{r-2}B^{2-s} < d(r,s) := \frac{(r-2)^{r-2}(2-s)^{2-s}}{(r-s)^{r-s}}$. Furthermore, for $t = t_B = [\frac{2-s}{B(r-s)}]^{\frac{1}{r-2}}$, one has

$$\max_{t\geq 0} \Phi_{A,B}(t) = \Phi_{A,B}(t_B) = t_B^2[\frac{r-2}{r-s} - AB^{\frac{2-s}{r-2}}(\frac{r-s}{2-s})^{\frac{2-s}{r-2}}] > 0.$$

Lemma 3.3. *Assume (V_1) and (g_1) hold, then there exists $\rho > 0$ such that*

$$\inf_{||u||=\rho} I(u) > 0.$$

Proof. (g_1) yields

$$G(x,u) \leq \zeta^+|u|^{r_1} \text{ for all } x \in \mathbb{R}^3 \text{ and } u \in \mathbb{R},$$

which implies

$$\begin{aligned}
\int_{\mathbb{R}^3} F(x,u)dx &\leq ||\zeta^+||_\infty \int_{\mathbb{R}^3} |u|^{r_1}dx + \int_{\mathbb{R}^3} \alpha(x)|u|^s dx \\
&\leq ||\zeta^+||_\infty S_{r_1}^{r_1}||u||^{r_1} + (\int_{\mathbb{R}^3} |\alpha(x)|^{\frac{2}{2-s}}dx)^{\frac{2-s}{2}}(\int_{\mathbb{R}^3} |u|^2dx)^{\frac{s}{2}} \\
&= C_1||u||^{r_1} + ||\alpha||_{\frac{2}{2-s}}||u||_2^s \\
&\leq C_1||u||^{r_1} + ||\alpha||_{\frac{2}{2-s}}S_2^s||u||^s \\
&= C_1||u||^{r_1} + C_2||u||^s,
\end{aligned}$$

where $C_1 = ||\zeta^+||_\infty S_{r_1}^{r_1}$, $C_2 = ||\alpha||_{\frac{2}{2-s}}S_2^s$. Since

$$\begin{aligned}
I(u) &= \frac{1}{2}||u||^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2dx)^2 + \frac{\kappa}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2dx - \int_{\mathbb{R}^3} F(x,u)dx - \frac{m_0}{2}\int_{\mathbb{R}^3} u^2dx \\
&\geq \frac{1}{2}||u||^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2dx)^2 - C_1||u||^{r_1} - C_2||u||^s - \frac{m_0}{2}S_2^2||u||^2 \\
&\geq \frac{1 - m_0S_2^2}{2}||u||^2 - C_2||u||^s - C_1||u||^{r_1},
\end{aligned}$$

Lemma 3.2 together with (V_1) gives that for $\rho = t_B$ and $||u|| = \rho$,

$$I(u) \geq \frac{1 - m_0 S_2^2}{2} \Phi_{A,B}(t_B) > 0,$$

where $A = \frac{2C_2}{1-m_0 S_2^2}$, $B = \frac{2C_1}{1-m_0 S_2^2}$, it comes to the conclusion. □

Note that for any $u \in E$, since $E \hookrightarrow H^2(\mathbb{R}^3) \hookrightarrow W^{1,r}(\mathbb{R}^3)$ for $2 \leq r < \infty$,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla u|^3 dx &\leq \int_{\mathbb{R}^3} (|u|^2 + \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^2)^{\frac{3}{2}} dx \\
&\leq \int_{\mathbb{R}^3} (|u| + \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|)^3 dx \\
&\leq \int_{\mathbb{R}^3} [4 \max\{|u|, |\frac{\partial u}{\partial x_1}|, |\frac{\partial u}{\partial x_2}|, |\frac{\partial u}{\partial x_3}|\}]^3 dx \\
&\leq 4^3 \int_{\mathbb{R}^3} (|u|^3 + \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^3) dx \\
&= 4^3 ||u||^3_{W^{1,3}(\mathbb{R}^3)} \leq 4^3 \iota_3^3 ||u||^3,
\end{aligned} \tag{3.1}$$

where $\iota_r = \sup_{u \in E, ||u||=1} ||u||_{W^{1,r}(\mathbb{R}^3)}$.

Lemma 3.4. *Assume that* (V_1) *and* (g_1) *hold. Let* $\rho > 0$ *be as in Lemma* 3.3*, then there exists* $e \in E$ *with* $||e|| > \rho$ *such that* $I(e) < 0$.

Proof. Since $\zeta > 0$ on a bounded domain Ω, we can choose a function $u \in E$ such that

$$\int_{\mathbb{R}^3} \zeta(x) |u|^{r_1} dx > 0.$$

From (2.1), (3.1) and the Hölder inequality, one has

$$\begin{aligned}
I(u) =& \frac{1}{2}||u||^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 + \frac{\kappa}{2} \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx \\
&- \int_{\mathbb{R}^3} G(x,u) dx - \int_{\mathbb{R}^3} \alpha(x) |u|^s dx - \frac{m_0}{2} \int_{\mathbb{R}^3} u^2 dx \\
\leq& \frac{1}{2}||u||^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 + \frac{\kappa}{2} (\int_{\mathbb{R}^3} u^6 dx)^{\frac{1}{3}} (\int_{\mathbb{R}^3} |\nabla u|^3 dx)^{\frac{2}{3}} \\
&- \int_{\mathbb{R}^3} G(x,u) dx - \int_{\mathbb{R}^3} \alpha(x) |u|^s dx - \frac{m_0}{2} \int_{\mathbb{R}^3} u^2 dx \\
\leq& \frac{1}{2}||u||^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 + \frac{\kappa}{2} S_6^2 4^2 \iota_3^2 ||u||^4 \\
&- \int_{\mathbb{R}^3} G(x,u) dx - \int_{\mathbb{R}^3} \alpha(x) |u|^s dx - \frac{m_0}{2} \int_{\mathbb{R}^3} u^2 dx.
\end{aligned}$$

Then, it follows from (g_1) and Fatou lemma that

$$\begin{aligned}
\lim_{l \to +\infty} \frac{I(lu)}{l^{r_1}} &\leq \lim_{l \to +\infty} \sup(-\int_{\mathbb{R}^3} \frac{G(x, lu)}{l^{r_1} |u|^{r_1}} |u|^{r_1} dx) \\
&\leq -\int_{\mathbb{R}^3} \zeta(x) |u|^{r_1} dx < 0.
\end{aligned}$$

So $I(lu) \to -\infty$ as $l \to +\infty$, then there exists $e \in E$ with $||e|| > \rho$ such that $I(e) < 0$. This completes the proof. $\square$

From Lemma 2.2, 3.3 and 3.4, there exists a *Cerami* sequence $\{u_n\} \in E$ such that

$$I(u_n) \to c > 0 \text{ and } (1 + ||u_n||)I'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.2}$$

Lemma 3.5. *Assume that* (V_1), (g_1) *and* (g_2) *hold, then* $\{u_n\}$ *defined by* (3.2) *has a convergent subsequence.*

Proof. For n large enough, by (g_1) and (g_2) we have

$$\begin{aligned}
&c + 1 + ||u_n|| \geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\
=&\frac{1}{4}||u_n||^2 - \int_{\mathbb{R}^3} F(x, u_n)dx + \frac{1}{4}\int_{\mathbb{R}^3} f(x, u_n)u_n dx - \frac{m_0}{4}\int_{\mathbb{R}^3} u_n^2 dx \\
=&\frac{1}{4}||u_n||^2 + \int_{\mathbb{R}^3} (\frac{1}{4}g(x, u_n)u_n - G(x, u_n))dx + v(\frac{s}{4} - 1)\int_{\mathbb{R}^3} \alpha(x)|u_n|^s dx - \frac{m_0}{4}\int_{\mathbb{R}^3} u_n^2 dx \\
\geq&\frac{1}{4}||u_n||^2 - d_0\int_{\mathbb{R}^3} u_n^2 dx + (\frac{s}{4} - 1)||\alpha||_{\frac{2}{2-s}} S_2^s ||u_n||^s - \frac{m_0}{4}\int_{\mathbb{R}^3} u_n^2 dx \\
\geq&(\frac{1}{4} - d_0 S_2^2 - \frac{m_0 S_2^2}{4})||u_n||^2 + (\frac{s}{4} - 1)||\alpha||_{\frac{2}{2-s}} S_2^s ||u_n||^s,
\end{aligned}$$

which gives a boundedness for $\{u_n\}$.

Next, we prove that the sequence $\{u_n\}$ has a convergent subsequence. Going if necessary to a subsequence, there exists $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } E; \\ u_n \to u & \text{in } L^r(\mathbb{R}^3)\ (2 \leq r < \infty); \\ u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{3.3}$$

Since $(1 + ||u_n||)I'(u_n) \to 0$, we have

$$\begin{aligned}
\langle I'(u_n), u_n\rangle =&||u_n||^2 + b(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2 + 2\kappa\int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx \\
&- \int_{\mathbb{R}^3} f(x, u_n)u_n dx - m_0\int_{\mathbb{R}^3} u_n^2 dx = o(1), \\
\langle I'(u_n), v\rangle =&(u_n, u) + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \nabla u dx \\
&+ \kappa\int_{\mathbb{R}^3} (u_n u|\nabla u_n|^2 + u_n^2 \nabla u_n \nabla u)dx - \int_{\mathbb{R}^3} f(x, u_n)u dx - m_0\int_{\mathbb{R}^3} u_n u dx \\
=&o(1),
\end{aligned}$$

so in order to prove that $||u_n|| \to ||u||$, we just need to check

$$\int_{\mathbb{R}^3} f(x, u_n)u_n dx - \int_{\mathbb{R}^3} f(x, u_n)u dx = o(1), \tag{3.4}$$

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} \nabla u_n \nabla u dx = o(1), \tag{3.5}$$

$$\int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} u_n u|\nabla u_n|^2 dx = o(1), \tag{3.6}$$

$$\int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2 dx - \int_{\mathbb{R}^3} u_n^2 \nabla u_n \nabla u dx = o(1), \tag{3.7}$$

and

$$\int_{\mathbb{R}^3} u_n^2 dx - \int_{\mathbb{R}^3} u_n u dx = o(1). \tag{3.8}$$

In fact, by (3.3) and the Hölder inequality, it is easy to check (3.4)-(3.8) hold. This completes the proof. □

Proof of Theorem 1.1. (i) As a consequence of Lemma 3.3-3.5, using Lemma 2.2, we get the desired result.

(ii) Since $\alpha(x) \geq 0$, it is easy to take a $\varphi \in E$ such that $\int_{\mathbb{R}^3} \alpha(x)|\varphi|^s dx > 0$, it follows from (g_1) that for $t > 0$ sufficiently small,

$$\begin{aligned}
I(t\varphi) =& \frac{t^2}{2}||\varphi||^2 + \frac{bt^4}{4}(\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx)^2 + \frac{\kappa t^4}{2}\int_{\mathbb{R}^3} \varphi^2|\nabla\varphi|^2 dx \\
&- \int_{\mathbb{R}^3} G(x, t\varphi)dx - t^s \int_{\mathbb{R}^3} \alpha(x)|\varphi|^s dx - \frac{m_0 t^2}{2}\int_{\mathbb{R}^3} \varphi^2 dx \\
\leq& \frac{t^2}{2}||\varphi||^2 + \frac{bt^4}{4}(\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx)^2 + \frac{\kappa t^4}{2}\int_{\mathbb{R}^3} \varphi^2|\nabla\varphi|^2 dx \\
&- t^{r_1}\int_{\mathbb{R}^3} \eta(x)|\varphi|^{r_1} dx - t^s \int_{\mathbb{R}^3} \alpha(x)|\varphi|^s dx - \frac{m_0 t^2}{2}\int_{\mathbb{R}^3} \varphi^2 dx \\
<& 0.
\end{aligned}$$

It follows from Lemma 2.3 that the minimum of the functional I on any closed ball in E with center 0 and radius $\hat{r} < \rho$ satisfying

$$I(u) \geq 0 \text{ for all } u \in E \text{ with } ||u|| = \hat{r}$$

is achieved in the corresponding open ball and thus yields a nontrivial solution u_0 of (1.1) satisfying $I(u_0) < 0$ and $||u_0|| < \hat{r} < \rho$. This completes the proof. □

4. Proof of Theorem 1.2

Throughout this section, $H(x, u) = h(x)u$, where $h(x) \in L^2(\mathbb{R}^3)$ and $h(x) \geq 0$. In order to deduce our result, we need the following lemmas.

Lemma 4.1. *Assume (V_1), (g_3) and (g_4) hold, then there exist some constants $\rho, \nu, n_0 > 0$ such that $I(u) \geq \nu > 0$ with $||u|| = \rho$ for all $u \in E$ and h satisfying $||h||_2 < n_0$.*

Proof. By (g_3) and (g_4), there exists $C_\epsilon > 0$ such that $|g(x, u)| \leq \epsilon|u| + C_\epsilon|u|^{p-1}$, and for all $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$, one has

$$|G(x, u)| \leq \frac{\epsilon}{2}|u|^2 + \frac{C_\epsilon}{p}|u|^p. \tag{4.1}$$

It follows from (2.1), (2.2), (4.1) and the Hölder inequality that

$$I(u) \geq \frac{1}{2}||u||^2 - \int_{\mathbb{R}^3} \frac{\epsilon}{2}|u|^2 + \frac{C_\epsilon}{p}|u|^p dx - ||h||_2||u||_2 - \frac{m_0}{2}||u||_2^2$$

$$\geq \frac{1}{2}||u||^2 - \frac{\epsilon}{2}S_2^2||u||^2 - \frac{C_\epsilon}{p}S_p^p||u||^p - S_2||h||_2||u|| - \frac{m_0}{2}S_2^2||u||^2 \tag{4.2}$$
$$= ||u||[(\frac{1}{2} - \frac{\epsilon}{2}S_2^2 - \frac{m_0}{2}S_2^2)||u|| - \frac{C_\epsilon}{p}S_p^p||u||^{p-1} - S_2||h||_2].$$

Taking $\epsilon = \frac{1}{2S_2^2} - m_0$ and setting $y(t) = \frac{1}{4}t - \frac{C_\epsilon}{p}S_p^p t^{p-1}$ for $t \geq 0$. By direct calculations, we see that $\max_{t\geq 0} y(t) = y(\rho) > 0$, where $\rho = (\frac{p}{4(p-1)C_\epsilon S_p^p})^{\frac{1}{p-2}} > 0$. Then it follows from (4.2) that, if $||h||_2 < n_0 = \frac{y(\rho)}{2S_2}$, there exists $\nu > 0$ such that $\inf_{||u||=\rho} I(u) \geq \nu > 0$. This completes the proof. □

Lemma 4.2. *Assume that* (V_1) *and* (g_3)*-*(g_6) *hold. Let* $\rho > 0$ *be as in Lemma* 4.1*, then there exists* $e \in E$ *with* $||e|| > \rho$ *such that* $I(e) < 0$*.*

Proof. For any $x \in \mathbb{R}^3, |z| \geq r_2$, set $\xi(t) = G(x, \frac{z}{t})t^\mu, \forall t \in [1, \frac{|z|}{r_2}]$. By (g_5), one has

$$\xi'(t) = t^{\mu-1}[\mu G(x, t^{-1}z) - t^{-1}zg(x, t^{-1}z)] \leq 0.$$

Hence, $\xi(1) \geq \xi(\frac{|z|}{r_2})$, that is

$$G(x, u) \geq G(x, \frac{r_2}{|z|}z)\frac{|z|^\mu}{r_2^\mu} \geq \inf_{x\in\mathbb{R}^3, ||u||=r_2} G(x, u)\frac{|z|^\mu}{r_2^\mu} \geq C_3|z|^\mu \tag{4.3}$$

for any $x \in \mathbb{R}^3, |z| \geq r_2$. By (g_4), there exists $\delta \leq r_2$ such that

$$|\frac{g(x, z)z}{z^2}| = |\frac{g(x, z)}{z}| \leq 1,$$

for all $x \in \mathbb{R}^3, 0 < |z| < \delta$. It follows from (g_3) that there exists a positive constant M_1 such that

$$|\frac{g(x, z)z}{z^2}| \leq \frac{a_2(1 + |z|^{p-1})|z|}{z^2} \leq M_1,$$

and so

$$g(x, z)z \geq -(M_1 + 1)|z|^2$$

for all $x \in \mathbb{R}^3, 0 < |z| < \delta$. Using the definition of $G(x, u)$, we have

$$G(x, z) \geq -\frac{1}{2}(M_1 + 1)|z|^2 \tag{4.4}$$

for all $x \in \mathbb{R}^3, 0 < |z| < \delta$. Setting $C_4 = \frac{1}{2}(M_1 + 1) + C_3$, we obtain from (4.3) and (4.4) that

$$G(x, z) \geq C_3|z|^\mu - C_4|z|^2 \tag{4.5}$$

for a.e. $x \in \mathbb{R}^3, z \in \mathbb{R}$. Since $E \hookrightarrow L^2(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$ is a separable Hilbert space, E has a countable orthogonal basis $\{e_j\}$. Set $E_k = \text{span}\{e_1, e_2, \cdots, e_k\}$. Then $E = E_k \oplus E_k^\perp$ and E_k is finite-dimensional space. Moreover, for any finite dimensional subspace $\tilde{E} \subset E$, there is a positive integral number n such that $\tilde{E} \subset E_n$. Hence, by (3.1), (4.5) and the assumptions on $h(x)$, one has

$$I(u) = \frac{1}{2}||u||^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 + \frac{\kappa}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx$$
$$- \int_{\mathbb{R}^3} G(x, u)dx - \int_{\mathbb{R}^3} h(x)udx - \frac{m_0}{2}\int_{\mathbb{R}^3} u^2 dx$$

$$
\begin{aligned}
&\leq \frac{1}{2}||u||^2 + \frac{bC_5}{4}||u||^4 + \frac{\kappa}{2}S_6^2 4^2 \iota_3^2 ||u||^4 \\
&\quad - C_3||u||_\mu^\mu + C_4||u||_2^2 + \int_{\mathbb{R}^3} h(x)|u|dx + \frac{m_0}{2}||u||_2^2 \\
&\leq C_6||u||^2 + C_7||u||^4 - C_8||u||^\mu + \int_{\mathbb{R}^3} h(x)|u|dx
\end{aligned}
$$

for all $u \in E_n$, where in the last inequality we use the equivalence of all norms on the finite dimensional subspace E_n. Consequently, by $\mu > 4$, there exists a point $e \in E$ with $||e|| > \rho$ such that $I(e) < 0$. This completes the proof. $\square$

Lemma 4.3. *Assume* (V_1) *and* (g_3)*-*(g_6) *hold. Then any* (PS) *sequence of* I *is bounded if* $||h|| < n_0$.

Proof. Consider a sequence $\{u_n\}$ which satisfies $I(u_n) \to c$ and $\langle I'(u_n), u_n \rangle \to 0$ as $n \to \infty$. If $\{u_n\}$ is unbounded in E, we can assume $||u_n|| \to \infty$ as $n \to \infty$. Set $\omega_n = \frac{u_n}{||u_n||}$, then $||\omega_n|| = 1$ and $||\omega_n||_s \leq S_s$ for $s \in [2, \infty)$. Going if necessary to a subsequence, there exists $\omega \in E$ such that

$$
\begin{cases}
\omega_n \rightharpoonup \omega & \text{in } E; \\
\omega_n \to \omega & \text{in } L^r(\mathbb{R}^3) \ (2 \leq r < \infty); \\
\omega_n(x) \to \omega(x) & \text{a.e. in } \mathbb{R}^3.
\end{cases}
\tag{4.6}
$$

Set $\mathit{\Pi} = \{x \in \mathbb{R}^3 | \omega(x) \neq 0\}$. If $meas(\mathit{\Pi}) > 0$, then $|u_n| \to +\infty$ a.e. $x \in \mathit{\Pi}$ as $n \to \infty$. It follows from (4.5) that

$$
g(x, u_n)u_n \geq C_9|u_n|^\mu - C_{10}|u_n|^2
$$

for a.e. $x \in \mathbb{R}^3$ and all $u_n \in \mathbb{R}$. Hence

$$
\int_{\mathbb{R}^3} \frac{g(x, u_n)u_n}{||u_n||^\mu} dx \geq C_9||\omega_n||_\mu^\mu - C_{10}\frac{||\omega_n||_2^2}{||u_n||^{\mu-2}}. \tag{4.7}
$$

Since $\mu > 4$ and

$$
\begin{aligned}
\frac{\langle I'(u_n), u_n \rangle}{||u_n||^\mu} =& \frac{1}{||u_n||^{\mu-2}} + \frac{b(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2}{||u_n||^\mu} + \frac{2\kappa \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx}{||u_n||^\mu} \\
&- \int_{\mathbb{R}^3} \frac{g(x, u_n)u_n}{||u_n||^\mu} dx - \int_{\mathbb{R}^3} h(x)\frac{u_n}{||u_n||^\mu} dx - \frac{m_0 \int_{\mathbb{R}^3} u_n^2 dx}{||u_n||^\mu},
\end{aligned}
$$

one has $\int_{\mathbb{R}^3} \frac{g(x,u_n)u_n}{||u_n||^\mu} dx \to 0$ as $n \to \infty$. Passing the limit $n \to \infty$ in (4.7), we have

$$
0 = \lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{g(x, u_n)u_n}{||u_n||^\mu} dx \geq C_9||\omega||_\mu^\mu > 0,
$$

which is a contradiction. Hence, $meas(\mathit{\Pi}) = 0$, that is, $\omega(x) = 0$ a.e. $x \in \mathbb{R}^3$. It follows from (g_3)-(g_5) that

$$
|ug(x, u) - \mu G(x, u)| \leq C_{11}u^2, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}.
$$

Thus, for $||h||_2 < n_0$,

$$\begin{aligned}
&\frac{1}{||u_n||^2}[I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle] \\
\geq&(\frac{1}{2} - \frac{1}{\mu}) + \frac{1}{||u_n||^2}\int_{\mathbb{R}^3}[\frac{1}{\mu}g(x, u_n)u_n - G(x, u_n)]dx \\
&+ (\frac{1}{\mu} - 1)\frac{1}{||u_n||^2}\int_{\mathbb{R}^3} h(x)u_n dx + (\frac{1}{\mu} - \frac{1}{2})m_0\frac{1}{||u_n||^2}\int_{\mathbb{R}^3} u_n^2 dx \\
\geq&(\frac{1}{2} - \frac{1}{\mu}) - \frac{C_{11}}{\mu}\int_{\mathbb{R}^3}\omega_n^2 dx + (\frac{1}{\mu} - 1)\frac{n_0 S_2}{||u_n||} + (\frac{1}{\mu} - \frac{1}{2})m_0 S_2^2.
\end{aligned} \tag{4.8}$$

Passing the limit $n \to \infty$ in (4.8), there holds $0 \geq (\frac{1}{2} - \frac{1}{\mu}) + (\frac{1}{\mu} - \frac{1}{2})m_0 S_2^2 > 0$, which yields to a contradiction. Hence, $\{u_n\}$ is bounded in E. □

Lemma 4.4. *Let (V_1) and (g_3)-(g_4) hold and $\{u_n\}$ is a bounded (PS) sequence of I, then $\{u_n\}$ has a strongly convergent subsequence in E.*

Proof. Since $\{u_n\}$ is a bounded sequence in E, going if necessary to a subsequence, there exists $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{in } E; \\ u_n \to u \quad \text{in } L^r(\mathbb{R}^3)\ (2 \leq r < \infty); \\ u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{4.9}$$

By an elementary computation,

$$\begin{aligned}
&\langle I'(u_n) - I'(u), u_n - u\rangle \\
=&||u_n - u||^2 - \int_{\mathbb{R}^3}(g(x, u_n) - g(x, u))(u_n - u)dx - m_0\int_{\mathbb{R}^3}(u_n - u)^2 dx \\
&+ b\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\int_{\mathbb{R}^3}|\nabla(u_n - u)|^2 dx \\
&+ b(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx - \int_{\mathbb{R}^3}|\nabla u|^2 dx)\int_{\mathbb{R}^3}\nabla u\nabla(u_n - u)dx \\
&+ \kappa\int_{\mathbb{R}^3}(|u_n|^2\nabla u_n - |u|^2\nabla u)\nabla(u_n - u)dx + \kappa\int_{\mathbb{R}^3}(|\nabla u_n|^2 u_n - |\nabla u|^2 u)(u_n - u)dx \\
\geq&||u_n - u||^2 - \int_{\mathbb{R}^3}(g(x, u_n) - g(x, u))(u_n - u)dx - m_0\int_{\mathbb{R}^3}(u_n - u)^2 dx \\
&+ b(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx - \int_{\mathbb{R}^3}|\nabla u|^2 dx)\int_{\mathbb{R}^3}\nabla u\nabla(u_n - u)dx \\
&+ \kappa\int_{\mathbb{R}^3}(|u_n|^2 - |u|^2)\nabla u_n\nabla(u_n - u)dx + \kappa\int_{\mathbb{R}^3}|u|^2|\nabla(u_n - u)|^2 dx \\
&+ \kappa\int_{\mathbb{R}^3}(|\nabla u_n|^2 - |\nabla u|^2)u(u_n - u)dx + \kappa\int_{\mathbb{R}^3}|\nabla u_n|^2(u_n - u)^2 dx \\
\geq&||u_n - u||^2 - \int_{\mathbb{R}^3}(g(x, u_n) - g(x, u))(u_n - u)dx - m_0\int_{\mathbb{R}^3}(u_n - u)^2 dx \\
&+ b(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx - \int_{\mathbb{R}^3}|\nabla u|^2 dx)\int_{\mathbb{R}^3}\nabla u\nabla(u_n - u)dx
\end{aligned}$$

$$+\kappa\int_{\mathbb{R}^3}(|u_n|^2-|u|^2)\nabla u_n\nabla(u_n-u)dx+\kappa\int_{\mathbb{R}^3}(|\nabla u_n|^2-|\nabla u|^2)u(u_n-u)dx. \tag{4.10}$$

Clearly, from (4.9), there hold

$$\langle I'(u_n)-I'(u),u_n-u\rangle\to 0, \tag{4.11}$$

$$\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx-\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)dx\to 0 \tag{4.12}$$

and

$$\int_{\mathbb{R}^3}(u_n-u)^2dx\to 0 \tag{4.13}$$

as $n\to\infty$. Applying (3.1), (4.9) and the Hölder inequality, we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3}(|u_n|^2-|u|^2)\nabla u_n\nabla(u_n-u)dx\Big|\\
\le&\int_{\mathbb{R}^3}|u_n-u||u_n+u||\nabla u_n||\nabla(u_n-u)|dx\\
\le&\Big(\int_{\mathbb{R}^3}|u_n-u|^6dx\Big)^{\frac16}\Big(\int_{\mathbb{R}^3}|u_n+u|^6dx\Big)^{\frac16}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^3dx\Big)^{\frac13}\Big(\int_{\mathbb{R}^3}|\nabla(u_n-u)|^3dx\Big)^{\frac13}\\
\le&C_{12}||u_n-u||_6\to 0
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3}(|\nabla u_n|^2-|\nabla u|^2)u(u_n-u)dx\Big|\\
\le&\int_{\mathbb{R}^3}|\nabla u_n|^2|u||u_n-u|dx+\int_{\mathbb{R}^3}|\nabla u|^2|u||u_n-u|dx\\
\le&\Big(\int_{\mathbb{R}^3}|\nabla u_n|^3dx\Big)^{\frac23}\Big(\int_{\mathbb{R}^3}|u|^6dx\Big)^{\frac16}\Big(\int_{\mathbb{R}^3}|u_n-u|^6dx\Big)^{\frac16}\\
&+\Big(\int_{\mathbb{R}^3}|\nabla u|^3dx\Big)^{\frac23}\Big(\int_{\mathbb{R}^3}|u|^6dx\Big)^{\frac16}\Big(\int_{\mathbb{R}^3}|u_n-u|^6dx\Big)^{\frac16}\\
\le&C_{13}||u_n-u||_6\to 0,
\end{aligned} \tag{4.15}$$

as $n\to\infty$. In addition, it follows from (4.9) and the Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^3}(g(x,u_n)-g(x,u))(u_n-u)dx\\
\le&\int_{\mathbb{R}^3}(|g(x,u_n)|+|g(x,u)|)|u_n-u|dx\\
\le&\epsilon\int_{\mathbb{R}^3}(|u_n|+|u|)|u_n-u|dx+C_\epsilon\int_{\mathbb{R}^3}(|u_n|^{p-1}+|u|^{p-1})|u_n-u|dx\\
\le&\epsilon(||u_n||_2+||u||_2)||u_n-u||_2+C_\epsilon(||u_n||_p^{p-1}+||u||_p^{p-1})||u_n-u||_p\\
\le&C_{14}||u_n-u||_2+C_{15}||u_n-u||_p\to 0,
\end{aligned} \tag{4.16}$$

as $n\to\infty$. Thus, it follows from (4.10)-(4.16) that $||u_n-u||\to 0$. This completes the proof. □

Proof of Theorem 1.2. (i) As a consequence of Lemma 4.1-4.4, using Mountain Pass lemma, we get the desired result.

(ii) Since $h(x) \geq 0$, it is easy to take a $\varphi \in E$ such that $\int_{\mathbb{R}^3} h(x)\varphi dx > 0$, it follows from (4.5) that for $t > 0$ sufficiently small,

$$\begin{aligned}
I(t\varphi) =& \frac{t^2}{2}||\varphi||^2 + \frac{bt^4}{4}(\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx)^2 + \frac{\kappa t^4}{2}\int_{\mathbb{R}^3} \varphi^2|\nabla\varphi|^2 dx \\
& - \int_{\mathbb{R}^3} G(x, t\varphi)dx - t\int_{\mathbb{R}^3} h(x)\varphi dx - \frac{m_0 t^2}{2}\int_{\mathbb{R}^3} \varphi^2 dx \\
\leq & \frac{t^2}{2}||\varphi||^2 + \frac{bt^4}{4}(\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx)^2 + \frac{\kappa t^4}{2}\int_{\mathbb{R}^3} \varphi^2|\nabla\varphi|^2 dx \\
& - C_3 t^\mu \int_{\mathbb{R}^3} |\varphi|^\mu dx + C_4 t^2 \int_{\mathbb{R}^3} |\varphi|^2 dx - t\int_{\mathbb{R}^3} h(x)\varphi dx - \frac{m_0 t^2}{2}\int_{\mathbb{R}^3} \varphi^2 dx \\
<& 0.
\end{aligned}$$

Then, we get $c_0 = \inf_{u\in \bar{B}_\rho} I(u) < 0$, where ρ is given by Lemma 4.1, $B_\rho = \{u \in E|||u|| < \rho\}$. It follows from Ekeland variational principle that there exists a sequence $\{u_n\} \subset \bar{B}_\rho$ such that $c_0 \leq I(u_n) \leq c_0 + \frac{1}{n}$ and $I(u) \geq I(u_n) - \frac{1}{n}||u - u_n||$ for all $u \in \bar{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded (PS) sequence of I. In view of Lemma 4.4, we obtain that there exists a function $u_0 \in E$ such that $I'(u_0) = 0$, $I(u_0) = c_0 < 0$. This completes the proof. □

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