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INFINITELY MANY SOLUTIONS FOR A QUASILINEAR KIRCHHOFF-TYPE EQUATION WITH HARTREE-TYPE NONLINEARITIES*

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Abstract In this paper, we consider a new kind of Kirchhoff-type equation with Hartree-type nonlinearities which is stated in the introduction. Under certain assumptions on $g(u)$, we prove that the equation has infinitely many solutions by variational methods.

Keywords Kirchhoff-type quasilinear equation, infinitely many solutions, change of variables.

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1. Introduction

In this article, we study the following quasilinear Kirchhoff-type equation

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\Delta u - a[\Delta(u^2)]u = (I_\alpha * |u|^p)|u|^{p-2}u + g(u), \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $a>0$, $b\geq 0$, $N\geq 3$, $\alpha\in(N-2,N)$, $\frac{2(N+\alpha)}{N}<p<\frac{2(N+\alpha)}{N-2}$, I_α is a Riesz potential of order $\alpha\in(N-2,N)$ defined by $I_\alpha=\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}}$, where Γ is the Gamma function. Besides we assume that the function $g\in C(\mathbb{R},\mathbb{R})$ verifies:

$(g1)$, there exists constants c_0 and $4<q<2\cdot 2^*$ such that $g(t)\leq c_0(1+|t|^{q-1})$ for all $t\in\mathbb{R}$, where $2^*=\frac{2N}{N-2}$ if $N\geq 3$, $2^*=\infty$ if $N=1$ or 2,

$(g2)$ $\lim\limits_{|t|\to+\infty}\frac{G(t)}{t^4}=+\infty$, where $G(t)=\int_0^t g(s)\mathrm{d}s$,

$(g3)$ $\widetilde{G}(t)=\frac{1}{4}g(t)t-G(t)\geq 0$, and there exists $c_1>0$ and $\sigma>\max\{1,\frac{N}{2}\}$ such that

$$|G(t)|^\sigma\leq c_1|t|^{4\sigma}\widetilde{G}(t)$$

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for t large enough,

$(g4)$ $g(-t) = -g(t)$ for all $t \in \mathbb{R}$.

Our paper was motivated by the following aspects. On one hand, the following so-called modified nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u - [\Delta(u^2)]u = h(x,u), \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

which is a quasilinear problem has attracted more and more attention of scholars. Because compared to the semilinear problem, the quasilinear problem becomes more complicated since the effects of quasilinear and non-convex term $\Delta(u^2)u$. One of the main difficulties of quasilinear problems is that there is no suitable space on which the energy functional is well defined and belongs to C^1-class. There have been serval ideas and approaches used in recent years to overcome the difficulties such as by minimizations, the Nehari or Pohozaev manifold, and change of variables [5,10]. The main idea of change of variables is that the quasilinear problem can be reduces to a semilinear one. An Orlicz space framework was used in [10] while the usual Sobolev space framework was used as the working space in [5]. By this idea and approach, many researchers prove the existence and multiplicity of solutions of quasilinear problem. Readers can see [3,7,20,22–24,28] and the references therein. Furthermore, in [11], Liu developed a perturbation method, the main idea of which is adding a regularizing term to recover the smoothness of energy functional, so the standard minimax theory can be applied. Soon after, by applying the perturbation method, in [12] the authors obtained the existence of infinitely many solutions of quasilinear problem without symmetry, and in [27], Wu proved the existence of high energy solutions for the general quasilinear problem.

Also, studying equation (1.1) is partially inspired by the Kirchhoff-type eqaution

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = g(x,u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.3}$$

where $a > 0$, $b \geq 0$, $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function and $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Problem (1.3) is a nonlocal problem due to the presence of the term $b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$, which causes some mathematical difficulties, and at the same time, makes the research of such problem particular interesting. This problem has an interesting physical context. Indeed, if we set $V(x) = 0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^3$ in (1.3), then we get the following Kirchhoff Dirichlet problem

$$\begin{cases} -(a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x)\Delta u = g(x,u), & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.4}$$

It is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|\mathrm{d}x\big)\frac{\partial^2 u}{\partial x^2} = 0$$

which was proposed by G. Kirchhoff as an extension of classical D¡¯Alembert¡¯s wave equations for free vibration of elastic strings. Kirchhoff¡¯s model takes into account the changes in length of the string produced by transverse vibrations. Later, J.L. Lions introduced a functional analysis approach. After that, (1.3) has been paid much attention to by several researchers.

On the other hand, if we remove the term $\Delta(u^2)u$ and set $a = 1$, $b = 0$, $g = 0$, the equation (1.1) reduces to

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \tag{1.5}$$

which is called nonlinear Choquard type equation. Its physical background can be found in [14] and references therein. Besides, readers can see [4, 6, 13, 15–19] for recent achievements.

In recent years, there exists some results about the combination of the two equations (1.2) and (1.5). For example, Yang in [29] considered the following quasilinear Choquard equation

$$-\Delta u + V(x)u - [\Delta(u^2)]u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u, \text{ in } \mathbb{R}^N, \tag{1.6}$$

where $N \geq 3$, $\mu \in (0, \frac{N+2}{2})$, $p \in (2, \frac{4N-4\mu}{N-2})$. Also by perturbation method, the authors proved the existence of positive solutions, negative solutions, and high energy solutions.

Besides, there are also some results about the combination of the two equations (1.3) and (1.5). For example, Lü in [8] studied the following Kirchhoff-type equation

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V_\lambda(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \text{ in } \mathbb{R}^3, \tag{1.7}$$

where $a > 0$, $b \geq 0$, $V_\lambda(x) = 1 + \lambda g(x)$, here $\lambda > 0$ is a parameter and $g(x)$ is a continuous potential function on $\mathbb{R}^3$,$p \in (2, 6 - \mu)$. By using the Nehari manifold and the concentration compactness principle, Lü obtained the existence of ground state solutions for (1.7) if the parameter λ is large enough.

Inspired by the works mentioned above, we consider the combination of the three type equations (1.2), (1.4) and (1.5). In our paper, by using change of variables, we prove the existence of infinitely many solutions of problem (1.1).

Our main result is as follows:

Theorem 1.1. *If (g1)-(g4) hold, then problem(1.1) possesses infinite many nontrivial solutions $\{u_n\}$.*

For the convenience of expression, hereafter, we use the following notations:

- $E := H^1(\mathbb{R}^N)$ is equipped with an equivalent norm $\|u\| = [\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x]^{\frac{1}{2}}$,
- $L^s(\mathbb{R}^N)(1 \leq s \leq \infty)$ denotes the Lebesgue space with the norm $|u|_s = (\int_{\mathbb{R}^N} |u|^s \mathrm{d}x)^{1/s}$,
- For any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, u_t is denoted as:

$$u_t = \begin{cases} 0, & t = 0, \\ \sqrt{t}u(\frac{x}{t}), & t > 0, \end{cases}$$

- For any $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$,
- $C, C_1, C_2, ...$ and $c_0, c_1, c_2, ...$ denote positive constants possibly different in different lines.

Remark 1.2. It is well known that the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is compact for $r \in [1, 2^*)$ and $E \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [1, 2^*]$.

2. Preliminaries

In this section, we will make the change of variables and some lemmas.

Problem(1.1) is the Euler-lagrange equation associated with the natural energy functional $J(u): E \to \mathbb{R}$ defined as follows

$$\begin{aligned} J(u) =& \frac{1}{2}\int_{\mathbb{R}^N} a|\nabla u|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)^2 + \int_{\mathbb{R}^N} a|u|^2|\nabla u|^2 \mathrm{d}x \\ &- \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x,\ u \in E. \end{aligned} \tag{2.1}$$

But J is not well defined in general in E. In order to overcome this difficulty, we apply the technique developed by [5,10]. We make the change of variables by $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2|f(t)|^2}}$$

on $[0, +\infty)$ and $f(-t) = -f(t)$ on $(-\infty, 0]$.

First let us recall some properties of the change of variables $f : \mathbb{R} \to \mathbb{R}$, which will be used frequently in the sequel of the paper. Proofs may be found in [5,10,20].

Lemma 2.1. *The function f(t) and its derivative satisfy the following properties:*
$(f1)$ *f is uniquely defined, C^∞, and invertible,*
$(f2)$ $|f'(t)| \le 1$ *for all* $t \in \mathbb{R}$,
$(f3)$ $|f(t)| \le |t|$, *for all* $t \in \mathbb{R}$,
$(f4)$ $f(t)/t \to 1$ *as* $t \to 0$,
$(f5)$ $f(t)/\sqrt{t} \to 2^{1/4}$, *as* $t \to +\infty$,
$(f6)$ $f(t)/2 \le tf'(t) \le f(t)$, *for all* $t > 0$,
$(f7)$ $f^2(t)/2 \le tf(t)f'(t) \le f^2(t)$, *for all* $t \in \mathbb{R}$,
$(f8)$ $|f(t)| \le 2^{1/4}|t|^{1/2}$, *for all* $t \in \mathbb{R}$,
$(f9)$ *there exits a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{1/2}, & |t| \ge 1, \end{cases}$$

$(f10)$ *there exists positive constants C_1 and C_2 such that*

$$|t| \le C_1|f(t)| + C_2|f(t)|^2,$$

for all $t \in \mathbb{R}$,
$(f11)$ *the function $f^2(t)$ is strictly convex,*
$(f12)$ $|f(t)f'(t)| \le 1/\sqrt{2}$, *for all* $t \in \mathbb{R}$.

Therefore, after the change of variables, we can write $J(u)$ as

$$\begin{aligned} I(v) =& \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |f'(v)|^2|\nabla v|^2 \mathrm{d}x\Big)^2 \\ &- \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |f(v)|^p)|f(v)|^p \mathrm{d}x - \int_{\mathbb{R}^N} G[f(v)]\mathrm{d}x. \end{aligned} \tag{2.2}$$

Lemma 2.2. (Hardy-Littlewood-Sobolev inequality [9]). *Let $0 < \alpha < N$, $p, q > 1$ and $1 \leq r < s < \infty$ be such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \ \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

1. *For any $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, one has*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \right| \leq C(N, \alpha, p) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$

2. *For any $f \in L^r(\mathbb{R}^N)$ one has*

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \leq C(N, \alpha, r) \|f\|_{L^r(\mathbb{R}^N)}.$$

From Lemma 2.1-$(f3)$, $(f12)$ and Lemma 2.2, we can get that the function $I(v)$ given by (2.2) is well defined. Our hypotheses imply that $I(v) \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle I'(v), w \rangle =& a \int_{\mathbb{R}^N} \nabla v \nabla w \mathrm{d}x + b \left(\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{1 + 2f^2(v)} \mathrm{d}x \right) \\ & \times \left(\int_{\mathbb{R}^N} \frac{\nabla v \nabla w (1 + 2f^2(v)) - 2|\nabla v|^2 f(v) f'(v) w}{[1 + 2f^2(v)]^2} \mathrm{d}x \right) \\ & - \int_{\mathbb{R}^N} (I_\alpha * |f(v)|^p) |f(v)|^{p-2} f(v) f'(v) w \mathrm{d}x - \int_{\mathbb{R}^N} g[f(v)] f'(v) w \mathrm{d}x. \end{aligned} \tag{2.3}$$

We note that if v is a critical point of the functional I, then $u = f(v)$ is a critical point of the functional J, which implies $u = f(v)$ is a solution of problem (1.1).

Lemma 2.3. (Brezis-Lieb lemma [1]). *Let $s \in (1, \infty)$ and $\{w_n\}$ be a bounded sequence in $L^s(\mathbb{R}^N)$. If $w_n \to w$ almost everywhere on $\mathbb{R}^N$, then for any $q \in [1, s]$,*

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| |w_n|^q - |w_n - w|^q - |w|^q \right|^{\frac{s}{q}} dx = 0 \tag{2.4}$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| |w_n|^{q-1} w_n - |w_n - w|^{q-1} (w_n - w) - |w|^{q-1} w \right|^{\frac{s}{q}} dx = 0. \tag{2.5}$$

Lemma 2.4. ([2,21]). *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

1. *$I(0) = 0$, $I(-u) = I(u)$ for all $u \in X$,*
2. *there exists constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \geq \alpha$,*
3. *for any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{E}) > 0$ such that $I(u) \leq 0$ on $\widetilde{X} \backslash B_R$,*

then I possess an unbounded sequence of critical values.

3. Proof of Theorem 1.1

In this section, we prove the following results and the main theorem.

3.1. Lemmas

Lemma 3.1. *Suppose* $(g1)-(g3)$ *are satisfied, then* $I(v)$ *satisfies* $(C)_c$ *condition.*

Proof. Let $\{v_n\} \subset E$ such that $I(v_n) \to c$ and $(1+\|v_n\|)I'(v_n) \to 0$. Then there exists a constant $C_3 > 0$ such that

$$I(v_n) - \frac{1}{4}\langle I'(v_n), v_n\rangle \leq C_3. \tag{3.1}$$

Set $u_n = f(v_n)$. Then we first prove $\{u_n\}$ is bounded in E. Indeed, set $\varphi_n = \frac{f(v_n)}{f'(v_n)}$, then there is a constant $C_4 > 0$ such that $\|\varphi_n\| \leq C_4\|v_n\|$. Since $\{v_n\}$ is a Cerami sequence of I, from (3.1) we can obtain

$$\begin{aligned} C_5 \geq & I(v_n) - \frac{1}{4}\langle I'(v_n), \varphi_n\rangle \\ = & \frac{a}{4}\int_{\mathbb{R}^N} (f(v_n))^2|\nabla v_n|^2 \mathrm{d}x + \frac{p-2}{4p}\int_{\mathbb{R}^N} (I_\alpha * |f(v_n)|^p)|f(v_n)|^p \mathrm{d}x \\ & + \int_{\mathbb{R}^N} [\frac{1}{4}g(f(v_n))f(v_n) - G(f(v_n))]\mathrm{d}x \\ \geq & \frac{a}{4}\int_{\mathbb{R}^N} (f(v_n))^2|\nabla v_n|^2 \mathrm{d}x, \end{aligned} \tag{3.2}$$

which implies that $\{\|u_n\|\}$ is bounded in E and

$$C_6 \geq \int_{\mathbb{R}^N} \widetilde{G}(f(v_n))\mathrm{d}x = \int_{\mathbb{R}^N} \widetilde{G}(u_n)\mathrm{d}x. \tag{3.3}$$

Therefore, passing to a subsequence, we can assume that there exists a $u \in X$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \\ u_n \to u \text{ in } L^s(\mathbb{R}^N), \ \forall\, s \in [1, 2^*) \\ u_n \to u \text{ a.e. on } \mathbb{R}^N. \end{cases}$$

Next, we claim that $\int_{\mathbb{R}^N} |\nabla u_n^2|^2 \mathrm{d}x$ is bounded. If $\int_{\mathbb{R}^N} |\nabla u_n^2|^2 \mathrm{d}x$ is unbounded, we can assume that, up to a subsequence, $\int_{\mathbb{R}^N} |\nabla u_n^2|^2 \mathrm{d}x \to \infty$. Then we set

$$A_n^4 = \int_{\mathbb{R}^N} |\nabla u_n^2|^2 \mathrm{d}x \text{ and } w_n = \frac{u_n}{A_n}.$$

Then $w_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in [1, 2^*)$, $w_n \to 0$ a.e. on $\mathbb{R}^N$ and $\{w_n^2\}$ is bounded in E. Hence $\{w_n\}$ is bounded in $L^{2\cdot 2^*}$. By interpolation, we have $w_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in [1, 2\cdot 2^*)$. Since $I(v_n) = J(f(v_n)) = J(u_n)$, together with (2.1) and boundedness of $\{u_n\}$, we can get that

$$\begin{aligned} \int_{\mathbb{R}^N} G(u_n)\mathrm{d}x = & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x\right)^2 \\ & + \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_n^2|^2 \mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x - J(u_n) \\ \geq & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x + \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_n^2|^2 \mathrm{d}x - C_7 - J(u_n), \end{aligned}$$

which implies

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\frac{G(u_n)}{A_n^4}\geq\frac{a}{4}. \tag{3.4}$$

Similar to the idea in [24], for $0\leq a<b$, let $\Omega_n(a,b)=\{x\in\mathbb{R}: a\leq|u_n(x)|<b\}$. For any $0<\varepsilon<\frac{1}{16}$ and any $r_1>0$, there exists $N_0>0$ such that

$$\begin{aligned}\int_{\Omega_{(0,r_1)}}\frac{|G(u_n)|}{|u_n|^4}|w_n|^4\mathrm{d}x&\leq\int_{\Omega_{(0,r_1)}}\frac{c_0(|u_n|+|u_n|^p)}{|u_n|^4}|w_n|^4\mathrm{d}x\\&\leq(c_0+c_0r_1^{p-1})\int_{\Omega_{(0,r_1)}}\frac{|w_n|}{A_n^3}\mathrm{d}x<\varepsilon\end{aligned}\tag{3.5}$$

for all $n>N_0$. On the other hand, if $r_0>0$ is sufficiently large, and if we set $\sigma'=\frac{\sigma}{\sigma-1}$, from $(g3)$ and (3.3) we have

$$\begin{aligned}\int_{\Omega_{(r_1,+\infty)}}\frac{|G(u_n)|}{|u_n|^4}|w_n|^4\mathrm{d}x&\leq\left(\int_{\Omega_{(r_1,+\infty)}}(\frac{|G(u_n)|}{|u_n|^4})^\sigma\mathrm{d}x\right)^{\frac{1}{\sigma}}\left(\int_{\Omega_{(r_1,+\infty)}}|w_n|^{4\sigma'}\mathrm{d}x\right)^{\frac{1}{\sigma'}}\\&\leq c_1^{\frac{1}{\sigma}}\left(\int_{\Omega_{(r_1,+\infty)}}\widetilde{G}(u_n)\mathrm{d}x\right)^{\frac{1}{\sigma}}\left(\int_{\Omega_{(r_1,+\infty)}}|w_n|^{4\sigma'}\mathrm{d}x\right)^{\frac{1}{\sigma'}}\\&\leq C_8\left(\int_{\Omega_{(r_1,+\infty)}}|w_n|^{4\sigma'}\mathrm{d}x\right)^{\frac{1}{\sigma'}}<\varepsilon\end{aligned}\tag{3.6}$$

for all n. Combining (3.5) and (3.6) we have

$$\int_{\mathbb{R}^N}\frac{|G(u_n)|}{A_n^4}=\int_{\Omega_{(0,r_1)}}\frac{|G(u_n)|}{|u_n|^4}|w_n|^4\mathrm{d}x+\int_{\Omega_{(r_1,+\infty)}}\frac{|G(u_n)|}{|u_n|^4}|w_n|^4\mathrm{d}x<2\varepsilon<\frac{1}{8}$$

for all $n>N_0$, which contradicts (3.4). This shows $\{\int_{\mathbb{R}^N}|\nabla u_n^2|^2\mathrm{d}x\}$ is bounded. Hence $\{|u_n|_{2\cdot2^*}\}$ is bounded. Besides, $(g1)$ implies that $\{\int_{\mathbb{R}^N}G(f(v_n))\mathrm{d}x\}$ is bounded. Since

$$\begin{aligned}\frac{a}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x=&I(v_n)-\frac{b}{4}\left(\int_{\mathbb{R}^N}|f'(v_n)|^2|\nabla v_n|^2\mathrm{d}x\right)^2\\&+\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|f(v_n)|^p)|f(v_n)|^p\mathrm{d}x+\int_{\mathbb{R}^N}G[f(v_n)]\mathrm{d}x\\\leq&I(v_n)+\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|f(v_n)|^p)|f(v_n)|^p\mathrm{d}x+\int_{\mathbb{R}^N}G[f(v_n)]\mathrm{d}x,\end{aligned}$$

it is obviously that $\{v_n\}$ is bounded E. Therefore, by$(f2)$, $(f8)$, $(f12)$ and $(g1)$, a standard argument shows $\{v_n\}$ has a convergent subsequence in E. □

Let $\{e_j\}$ be a total orthonormal basis of E and define $X_j=\mathbb{R}e_j$, $Y_k=\oplus_{j=1}^kX_j$, $Z_k=\overline{\oplus_{j=k+1}^\infty X_j}$, $k\in\mathbb{Z}$, and Y_k is finite dimensional.

Lemma 3.2. *Suppose $(g1)$ holds, then there exists constants m, ρ, $\alpha>0$ such that $I|_{S_\rho\cap Z_m}\geq\alpha$.*

Proof. From Lemma 3.8 in [26], we know that for any $s\in[1,2^*)$, $\beta_k(s)=\sup_{v\in Z_k,\ \|v\|=1}|v|_s\to0$. Thus, we can choose an integer $m>1$ such that $0<\beta_m(1)\ll$

1, $0 < \beta_m(p/2) \ll 1$ and

$$|v|_1 \leq \beta_m(1)\|v\|, |v|_{p/2} \leq \beta_m(p/2)\|v\| \leq c_3\|v\|, \forall v \in Z_m. \tag{3.7}$$

For any $v \in Z_m$ with $\|v\| = \rho < 1$, by $(f3)$, $(f8)$, $(g1)$ and (3.7), we have

$$\begin{aligned}
I(v) = & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |f'(v)|^2|\nabla v|^2 \mathrm{d}x\right)^2 \\
& - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |f(v)|^p)|f(v)|^p \mathrm{d}x - \int_{\mathbb{R}^N} G[f(v)]\mathrm{d}x \\
\geq & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |f(v)|^p)|f(v)|^p \mathrm{d}x - \int_{\mathbb{R}^N} G[f(v)]\mathrm{d}x \\
\geq & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - \frac{C_9}{2p}\int_{\mathbb{R}^N} (I_\alpha * |v|^{\frac{p}{2}})|v|^{\frac{p}{2}} \mathrm{d}x - \int_{\mathbb{R}^N} (c_0|f(v)| + c_0|f(v)|^q)\mathrm{d}x \\
\geq & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - C_9\|v\|^p - \int_{\mathbb{R}^N} (c_0|v| + 2^{\frac{1}{4}}c_0|v|^{q/2})\mathrm{d}x \\
\geq & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - C_9\|v\|^p - c_0\beta_m(1)\|v\| - 2^{\frac{1}{4}}c_0\beta_m(q/2)\|v\|^{q/2} \\
\geq & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x - C_9\|v\|^p - c_0\beta_m(1)\|v\| - c_4\|v\|^{q/2} \\
= & \rho(\frac{a\rho}{2} - C_9\rho^{p-1} - c_0\beta_m(1) - c_4\rho^{\frac{q-2}{2}}) > 0.
\end{aligned} \tag{3.8}$$

Here we use the fact that $0 < \beta_m(1) \ll 1$, which implies that $0 < \beta_m(1) < \frac{a\rho}{2c_0} - \frac{C_9}{c_0}\rho^{p-1} - \frac{c_4}{c_0}\rho^{\frac{q-2}{2}}$ if m suitable large. Thus, we complete the proof. □

Lemma 3.3. *Under assumptions $(g1)$ and $(g2)$, for any infinite dimensional subspace $\widetilde{E} \subset E$, there holds $I(v_t) \to -\infty$ as $\|v_t\| \to \infty$, $v_t \in \widetilde{E}$.*

Proof. Because

$$\begin{aligned}
I(v_t) = & \frac{at^{N-1}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{bt^{2N-2}}{4}\left(\int_{\mathbb{R}^N} |f'(v)|^2|\nabla v|^2 \mathrm{d}x\right)^2 \\
& - \frac{t^{N+\alpha}}{2p}\int_{\mathbb{R}^N} (I_\alpha * |f(v)|^p)|f(v)|^p \mathrm{d}x - t^N\int_{\mathbb{R}^N} G[f(\sqrt{t}v)]\mathrm{d}x,
\end{aligned}$$

and $\alpha > N - 2$, we can easily get $I(v_t) \to -\infty$ as $t \to +\infty$. □

Corollary 3.4. *If $(g1)$ and $(g2)$ hold, then for any finite dimensional subspace $\widetilde{E} \subset E$, there exists $R = R(\widetilde{E}) > 0$ such that $I(u) \leq 0$, $\forall u \in \widetilde{E}, \|u\| \geq R$.*

3.2. Proof of Theorem 1.1

Let $X = E$, $Y = Y_m$, $Z = Z_m$. Then it is obviously that $I(0) = 0$ and $(g4)$ implies that I is even. By Lemma 3.1-3.3 and Corollary 3.4, all conditions of Lemma 2.4 are satisfied. Thus, I possesses a sequence of critical points $\{v_n\}$ such that $I(v_n) \to \infty$ as $n \to \infty$. Namely, problem (1.1) possesses a sequence of nontrivial solutions $\{u_n\}$ such that $J(u_n) \to \infty$ as $n \to \infty$.

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References

[1] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations I*, Arch. Ration. Mech. Anal., 1983, 82, 313—346.

[2] T. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear Anal., 1983, 7, 241–273.

[3] J. M. Bezerra do Ö, O. H. Miyagaki and S. H. M. Soares, *Soliton solutions for quasilinear Schrödinger equations with critical growth*, J. Differential Equations, 2010, 248, 722–744.

[4] M. Chimenti and J. Van Schaftingen, *Nodal solutions for the Choquard equation*, J. Funct. Anal., 2016, 271, 107–135.

[5] M. Colin and L. Jeanjean, *Solutions for quasilinear Schrödinger equation: a dual approach*, Nonlinear Anal., 2004, 56, 213–226.

[6] P. Chen and X. Liu, *Ground states for Kirchhoff equation with Hartree-type nonlinearities*, J. Math. Anal. Appl., 2019, 473, 587–608.

[7] X. Fang and A. Szulkin, *Multiple solutions for a quasilinear Schrödinger equation*, J. Differential Equations, 2013, 254, 2015–2032.

[8] D. Lü, *A note on Kirchhoff-type equations with Hartree-type nonlinearities*, Nonlinear Annal., 2014, 99, 35–48.

[9] E. H. Lieb and M. Loss, *Analysis*, second ed, Grad. Stud. Math, vol 14, American Mathematical Scoiety, Province, RL, 2001.

[10] J. Liu, Y. Wang and Z. Wang, *Solutions for quasilinear Schrödinger equations II*, J. Differential Equations, 2003, 187, 473–493.

[11] X. Liu, J. Liu and Z. Wang, *Quasilinear elliptic equations via perturbation method*, Proc. Amer. Math. Soc., 2013, 141, 253–263.

[12] X. Liu and F. Zhao, *Existence of infinitely many solutions for quasilinear equations perturbed from symmetry*, Adv. Nonlinear Stud., 2013, 13, 965–978.

[13] L. Ma and Z. Lin, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch Ration. Mech. Aral., 2010, 195, 455–467.

[14] I. M. Moroz, R. Penrose and P. Tod, *Spherically-symmetric solutions of Schrödinger-Newton equations*, Classical Quantum Gravity, 1998, 15, 2733–2742.

[15] V. Morozand and J. Van Schaftingen, *Ground states of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal., 2013, 265, 153–184.

[16] V. Moroz and J. Van Schaftingen, *Existence of ground states for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc., 2015, 367, 6557–6579.

[17] V. Moroz and J. Van Schaftingen, *Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains*, J. Differential Equations, 2013, 254, 3089–3145.

[18] V. Moroz and J. Van Schaftingen, *Ground states of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent*, Commun. Contemp. Math., 2015, 17, 1550005.

[19] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl., 2017, 19, 773–813.

[20] J. Marcos do Ö and U. Devero, *Solitary waves for a class of quasilinear Schrödinger equations in demension two*, Calc. Var. Partial Differential Equations, 2010, 38, 275–315.

[21] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, in: CBMS Reg. Conf. Ser. in Math., vol. 65, Amer. Math. Soc., Province, RI, 1986.

[22] E. B. Silva and G. F. Vieira, *Quasilinear asymptotically periodic Schrödinger equations with critical growth*, Calc. Var. Partial Differential Equations, 2010, 39, 1–33.

[23] E. B. Silva and G. F. Vieira, *Quasilinear asymptotically periodic Schrödinger equations with subcritical growth*, Nonlinear Anal., 2010, 72, 2935–2949.

[24] X. Tang, *Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity*, J. Math. Anal. Appl., 2013, 401, 407–415.

[25] X. Tang, *New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation*, Adv. Nonlinear Stud., 2014, 14, 349–361.

[26] M. Willem, *Minimax Theorems*, Proress in Nonlinear Differential Equations and Their Applications 24, Birkhäuser, Boston, MA, 1996.

[27] X. Wu and K. Wu, *Existence of positive solutions, negative solutions and high energy solutions for quasilinear elliptic equations on* $\mathbb{R}^N$, Nonlinear Anal., Real Word Appl., 2014, 16, 48–64.

[28] X. Wu, *Multiple solutions for quasilinear Schrödinger equations with a parameter*, J. Differential Equations, 2014, 256, 2619–2632.

[29] X. Yang, W. Zhang and F. Zhao, *Existence and multiplicity of solutions for a quasilinear Choquard equation via perturbation method*, J. Math. Phys., 2018, 59, 081503.

[30] J. Zhang, X. Tang and W. Zhang, *Existence of infinitely many solutions for a quasilinear elliptic equation*, Appl. Math. Lett., 2014, 37, 131–135.