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ON THE STABILITY OF BESSEL DIFFERENTIAL EQUATION

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Abstract Using power series method, Kim and Jung (2007) investigated the Hyers-Ulam stability of the Bessel differential equation, $x^2y''(x)+xy'(x)+(x^2-\alpha^2)y(x)=0$, of order non-integral number $\alpha>0$. Also Bicer and Tunc (2017) obtained new sufficient conditions guaranteeing the Hyers-Ulam stability of Bessel differential equation of order zero. In this paper, by classical integral method we will investigate the stability of Bessel differential equations of a more generalized order than previous papers. Also, we will consider a more generalized domain $(0,a)$ for any positive real number a while Kim and Jung (2007) restricted the domain near zero.

Keywords Perturbation, Hyers-Ulam stability, Bessel differential equation.

MSC(2010) 34K20, 34K30, 34K05, 34B30, 39B82.

1. Introduction

In 2007, Jung [7] analyzed the solution of the inhomogeneous Legendre differential equation using the power series method and succeeded in obtaining a partial solution by applying the result to the Hyer-Ulam stability problem for the Legendre equation. This is the first example to study the Hyers-Ulam stability of differential equations by applying the power series method. In the same year, Kim and Jung [9] applied the idea from [7] for investigating the general solution of the inhomogeneous Bessel differential equation

$$x^2y''(x)+xy'(x)+\left(x^2-\alpha^2\right)y(x)=\sum_{m=0}^{\infty}a_mx^m,$$

where α is a positive non-integral real number, and then they proved a partial solution of the Hyers-Ulam stability problem for the Bessel equation in a subclasses of analytic functions.

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Although the Bessel differential equation of order α that is an integer or half-integer is very important, Kim and Jung [9] did deal with the Bessel differential equation of order α that is not integer.

Based on the above results, the aim of our paper is to more efficiently prove the Hyers-Ulam stability of the Bessel differential equation of order α

$$x^2y''(x) + xy'(x) + \big(x^2 - \alpha^2\big)y(x) = 0, \tag{1.1}$$

where α is a nonnegative real number. And we will consider $x \in (0, a) = I$ for any positive real number a while Kim and Jung [9] restricted x near zero. Specially for the Hyers-Ulam stability of the Bessel differential equation of order zero, one can also refer the results by Biçer and Tunç [2].

In 1940, Ulam [19] asked: *When is the theorem's statement still true or nearly true despite some slight variations in the theorem's hypotheses?*

In the following year, Hyers [4] came up with the first positive answer to Ulam's question by proving the stability of the additive functional equation in Banach spaces. This became the starting point of the Hyers-Ulam stability. Since then, Hyers' result has been widely generalized in terms of the control conditions used to define the concept of an approximate solution (see [16, 17]).

Let I be a subinterval of $\mathbb{R}$, let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$, and let n be a fixed positive integer. We consider the differential equation

$$\psi\big(f(x), y(x), y'(x), y''(x), \ldots, y^{(n)}(x)\big) = 0.$$

Relatively recently, one started to study the Hyers-Ulam stability of the differential equations by asking if there exists a constant $K > 0$ such that the following statement is true for any $\varepsilon > 0$: *If an n times continuously differentiable function $z : I \to \mathbb{K}$ satisfies the inequality*

$$\Big|\psi\big(f(x), z(x), z'(x), z''(x), \ldots, z^{(n)}(x)\big)\Big| \le \varepsilon$$

for all $x \in I$, then there exists a solution $y : I \to \mathbb{K}$ of the differential equation that satisfies the inequality $|z(x) - y(x)| \le K\varepsilon$ for all $x \in I$.

Obłoza is recognized as the first mathematician to study the Hyers-Ulam stability of linear differential equations (see [14, 15]). Then, in 1998, Alsina and Ger [1] continued the study of Obłoza's Hyers-Ulam stability of differential equations. Indeed, many mathematician investigated the Hyers-Ulam stability of the first-order linear differential equations. Some of them can be found in [5, 6, 8, 11, 18]. In 2006, Jung [6] investigated the Hyers-Ulam stability of a system of first-order linear differential equations with constant coefficients by using matrix method. Then, in 2008, Wang, Zhou and Sun [20] investigated the Hyers-Ulam stability of linear differential equations of first order by using the integral factor method.

In recent two decades, many mathematicians have tried to prove the Hyers-Ulam stability of differential equations using various techniques and they are also paying attention to the new results of the Hyers-Ulam stability of differential equations (see for example: [3, 10, 12, 13]).

2. Main Results

The solutions of Bessel differential equation (1.1) are called the Bessel functions of order α. The most important forms of Bessel differential equations occur when α

is an integer or half-integer. When α is an integer, the Bessel functions of order α are known as the cylinder functions or the cylindrical harmonics because they appear as the solution to Laplace equation in cylindrical coordinates. On the other hand, when α is a half-integer, the spherical Bessel function of order α appear as the solution to the Helmholtz equation in spherical coordinates.

Now, we are in the position to prove a type of Hyers-Ulam stability for the Bessel differential equation (1.1) of order α where α is a nonnegative real number.

Theorem 2.1. *Assume that E is a complex Banach space, α is a fixed real number, and $I = (0, a)$ is an interval with $a > 0$. Let ε be any positive real constant. Suppose $f : I \to E$ is a twice continuously differentiable function that satisfies the following conditions:*

(i) $\lim_{x\to a^-} f(x)$ *exists;*

(ii) $\lim_{x\to a^-} f'(x)$ *exists;*

(iii) $\lim_{w\to(\ln a)^-} B(t)$ *exists, where $B : (-\infty, \ln a) \to \mathbb{C}$ is a solution of the Riccati differential equation $B'(t) - B(t)^2 = e^{2t} - \alpha^2$.*

If f moreover satisfies the inequality

$$\left\|x^2 f''(x) + xf'(x) + \left(x^2 - \alpha^2\right) f(x)\right\| \le \varepsilon \tag{2.1}$$

for all $x \in I$, then there exists a solution $h \in C^2(I, E)$ of Bessel differential equation (1.1) *such that*

$$\begin{aligned}
&\|f(x) - h(x)\| \\
&\le \varepsilon \exp\left\{-\Re\left(\int_a^x \frac{1}{z} B(\ln z)dz\right)\right\} \\
&\quad\times \left[\int_a^x \frac{1}{q}\int_a^q \frac{1}{w}\exp\left\{2\Re\left(\int_a^q \frac{1}{z}B(\ln z)dz\right) - \Re\left(\int_a^w \frac{1}{z}B(\ln z)dz\right)\right\}dwdq\right]
\end{aligned}$$

for all $x \in I$.

Proof. Given any $\varepsilon > 0$, assume that a function $f \in C^2(I, E)$ satisfies inequality (2.1) for all $x \in I$. We define $J := (-\infty, \ln a)$ and the function $\phi : J \to E$ by $\phi(t) := f(e^t)$ for each $t \in J$. Then $\phi \in C^2(J, E)$ and

$$\phi'(t) = e^t f'\left(e^t\right) \quad \text{and} \quad \phi''(t) = e^t f'\left(e^t\right) + e^{2t} f''\left(e^t\right).$$

Thus, we have

$$\phi''(t) + \left(e^{2t} - \alpha^2\right)\phi(t) = e^{2t} f''\left(e^t\right) + e^t f'\left(e^t\right) + \left(e^{2t} - \alpha^2\right) f\left(e^t\right)$$

for all $t \in J$. Since $e^t \in I$ for $t \in J$, inequality (2.1) yields

$$\left\|\phi''(t) + \left(e^{2t} - \alpha^2\right)\phi(t)\right\| \le \varepsilon$$

for all $t \in J$.

We now set

$$\delta(t) := \phi''(t) + \left(e^{2t} - \alpha^2\right)\phi(t) \quad \text{and} \quad \phi_1(t) := \phi'(t) + B(t)\phi(t)$$

for each $t \in J$, where $B : J \to \mathbb{C}$ is a solution of the Riccati differential equation $B'(t) - B(t)^2 = e^{2t} - \alpha^2$. Then

$$\begin{aligned}
\phi_1'(t) &= \phi''(t) + B'(t)\phi(t) + B(t)\phi'(t) \\
&= \delta(t) - \left(e^{2t} - \alpha^2\right)\phi(t) + B'(t)\phi(t) + B(t)\phi'(t) \\
&= \delta(t) + \left(B'(t) - \left(e^{2t} - \alpha^2\right)\right)\phi(t) + B(t)\phi'(t) \\
&= \delta(t) + B^2(t)\phi(t) + B(t)\phi'(t) \\
&= \delta(t) + B(t)\left(B(t)\phi(t) + \phi'(t)\right) \\
&= \delta(t) + B(t)\phi_1(t)
\end{aligned}$$

for any $t \in J$. Hence, it follows that

$$-B(\tau)\phi_1(\tau) + \phi_1'(\tau) = \delta(\tau)$$

for all $\tau \in J$.

Multiplying the above identity by $\exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}$, we have

$$\begin{aligned}
&-B(\tau)\exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\phi_1(\tau) + \exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\phi_1'(\tau) \\
&= \exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau),
\end{aligned}$$

which implies that

$$\frac{d}{d\tau}\left(\exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\phi_1(\tau)\right) = \exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)$$

for all $\tau \in J$.

Integrating both sides of the last equality with respect to τ, we get

$$\begin{aligned}
&\int_{\ln a}^{p}\frac{d}{d\tau}\left(\exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\phi_1(\tau)\right)d\tau = \int_{\ln a}^{p}\exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau \\
\Longleftrightarrow &\exp\left\{-\int_{\ln a}^{p} B(s)ds\right\}\phi_1(p) - \phi_1(\ln a) = \int_{\ln a}^{p}\exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau,
\end{aligned}$$

for any $p \in J$, where we set $\phi_1(\ln a) := \lim\limits_{w \to (\ln a)^-} \phi_1(w)$. Then we have

$$\phi_1(\ln a) = \lim_{w \to (\ln a)^-}\left(\phi'(w) + B(w)\phi(w)\right) = \lim_{w \to (\ln a)^-}\left(e^w f'\left(e^w\right) + B(w) f\left(e^w\right)\right)$$

and $w \to (\ln a)^-$ implies that $e^w \to a^-$. Also, the existence of $\phi_1(\ln a)$ is guaranteed by conditions (i), (ii) and (iii), and consequently

$$\phi_1(p) = \exp\left\{\int_{\ln a}^{p} B(s)ds\right\}\left(\phi_1(\ln a) + \int_{\ln a}^{p}\exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau\right) \tag{2.2}$$

for all $p \in J$.

We recall that $\phi_1(p) = \phi'(p) + B(p)\phi(p)$ and we multiply this equality by $\exp\left\{\int_{\ln a}^{p} B(s)ds\right\}$ to get

$$\exp\left\{\int_{\ln a}^{p} B(s)ds\right\}\phi_1(p) = \exp\left\{\int_{\ln a}^{p} B(s)ds\right\}\phi'(p) + \exp\left\{\int_{\ln a}^{p} B(s)ds\right\}B(p)\phi(p),$$

which implies that

$$\exp\left\{\int_{\ln a}^{p} B(s)ds\right\}\phi_1(p) = \frac{d}{dp}\left(\exp\left\{\int_{\ln a}^{p} B(s)ds\right\}\phi(p)\right)$$

for all $p \in J$.

Therefore, by integrating both sides of the last equality with respect to p, we have

$$\int_{\ln a}^{t} \exp\left\{\int_{\ln a}^{p} B(s)ds\right\}\phi_1(p)dp = \exp\left\{\int_{\ln a}^{t} B(s)ds\right\}\phi(t) - \phi(\ln a), \tag{2.3}$$

for any $t < p < \ln a$, where we set $\phi(\ln a) := \lim\limits_{w\to(\ln a)^-} \phi(w) = \lim\limits_{x\to a^-} f(x)$, whose existence is guaranteed by condition (i). By (2.2), we have

$$\begin{aligned}
&\int_{\ln a}^{t} \exp\left\{\int_{\ln a}^{p} B(s)ds\right\}\phi_1(p)dp \\
&= \int_{\ln a}^{t} \exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\left(\phi_1(\ln a) + \int_{\ln a}^{p} \exp\left\{-\int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau\right)dp \\
&= \int_{\ln a}^{t} \exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\phi_1(\ln a)dp \\
&\quad + \int_{\ln a}^{t}\int_{\ln a}^{p} \exp\left\{2\int_{\ln a}^{p} B(s)ds - \int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau dp
\end{aligned}$$

and by combining the last equality and (2.3), we obtain

$$\begin{aligned}
&\exp\left\{\int_{\ln a}^{t} B(s)ds\right\}\phi(t) - \phi(\ln a) \\
&= \int_{\ln a}^{t} \exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\phi_1(\ln a)dp \\
&\quad + \int_{\ln a}^{t}\int_{\ln a}^{p} \exp\left\{2\int_{\ln a}^{p} B(s)ds - \int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau dp,
\end{aligned}$$

which implies that

$$\begin{aligned}
\phi(t) &= \exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\phi(\ln a) \\
&\quad + \exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\int_{\ln a}^{t} \exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\phi_1(\ln a)dp \\
&\quad + \exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\int_{\ln a}^{t}\int_{\ln a}^{p} \exp\left\{2\int_{\ln a}^{p} B(s)ds - \int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau dp.
\end{aligned} \tag{2.4}$$

Now we define the function $g : J \to E$ by

$$\begin{aligned}
g(t) &:= \exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\phi(\ln a) \\
&\quad + \exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\int_{\ln a}^{t} \exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\phi_1(\ln a)dp.
\end{aligned}$$

Then we have $g \in C^2(J,E)$ and

$$g'(t) = -B(t)\exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\phi(\ln a) + \exp\left\{\int_{\ln a}^{t} B(s)ds\right\}\phi_1(\ln a)$$
$$- B(t)\exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\int_{\ln a}^{t}\exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\phi_1(\ln a)dp$$

and

$$\begin{aligned} g''(t) =& -\left(B'(t) - B(t)^2\right)\exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\phi(\ln a) \\ & - \left(B'(t) - B(t)^2\right)\exp\left\{-\int_{\ln a}^{t} B(s)ds\right\}\int_{\ln a}^{t}\exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\phi_1(\ln a)dp \\ =& -\left(e^{2t} - \alpha^2\right)\exp\left\{-\int_{\ln a}^{t} B(s)ds\right\} \\ & \times \left(\phi(\ln a) + \int_{\ln a}^{t}\exp\left\{2\int_{\ln a}^{p} B(s)ds\right\}\phi_1(\ln a)dp\right). \end{aligned}$$

Note that

$$g''(t) = \left(\alpha^2 - e^{2t}\right)g(t) \tag{2.5}$$

for any $t \in J$.

According to (2.4) and the definition of g, we have

$$\begin{aligned} \phi(t) - g(t) =& \exp\left\{-\int_{\ln a}^{t} B(s)ds\right\} \\ & \times \int_{\ln a}^{t}\int_{\ln a}^{p}\exp\left\{2\int_{\ln a}^{p} B(s)ds - \int_{\ln a}^{\tau} B(s)ds\right\}\delta(\tau)d\tau dp \end{aligned}$$

for all $p \in J$. Thus, using the fact that $\|\delta(\tau)\| \le \varepsilon$, we obtain

$$\begin{aligned} \|\phi(t) - g(t)\| \le& \varepsilon\exp\left\{-\Re\left(\int_{\ln a}^{t} B(s)ds\right)\right\} \\ & \times \left[\int_{\ln a}^{t}\int_{\ln a}^{p}\exp\left\{2\Re\left(\int_{\ln a}^{p} B(s)ds\right) - \Re\left(\int_{\ln a}^{\tau} B(s)ds\right)\right\}d\tau dp\right] \end{aligned}$$

for any $t \in J$.

Since $x \in I$ if and only if $\ln x \in J$, it follows from the last inequality that

$$\begin{aligned} & \|\phi(\ln x) - g(\ln x)\| \\ \le& \varepsilon\exp\left\{-\Re\left(\int_{\ln a}^{\ln x} B(s)ds\right)\right\} \\ & \times \left[\int_{\ln a}^{\ln x}\int_{\ln a}^{p}\exp\left\{2\Re\left(\int_{\ln a}^{p} B(s)ds\right) - \Re\left(\int_{\ln a}^{\tau} B(s)ds\right)\right\}d\tau dp\right] \\ =& \varepsilon\exp\left\{-\Re\left(\int_{a}^{x}\frac{1}{z}B(\ln z)dz\right)\right\} \\ & \times \left[\int_{a}^{x}\frac{1}{q}\int_{a}^{q}\frac{1}{w}\exp\left\{2\Re\left(\int_{a}^{q}\frac{1}{z}B(\ln z)dz\right) - \Re\left(\int_{a}^{w}\frac{1}{z}B(\ln z)dz\right)\right\}dwdq\right], \end{aligned} \tag{2.6}$$

for any $0 < q < a$, where $q = e^p$. Now, by defining the function $h : I \to E$ as $h(x) := g(\ln x)$, we have $h \in C^2(I, E)$ and

$$h'(x) = \frac{1}{x} g'(\ln x).$$

Further, it follows from (2.5) that

$$h''(x) = \frac{1}{x^2} g''(\ln x) - \frac{1}{x^2} g'(\ln x) = \frac{\alpha^2 - x^2}{x^2} g(\ln x) - \frac{1}{x^2} g'(\ln x)$$

for each $x \in I$. So, we obtain

$$x^2 h''(x) + x h'(x) + \big(x^2 - \alpha^2\big) h(x) = 0$$

for all $x \in I$, which implies that h is a solution of Bessel differential equation (1.1).

Taking into account the fact that $\phi(x) = f(e^x)$ for all $x \in J$, we see that $f(x) = \phi(\ln x)$ for each $x \in I$. By (2.6) we obtain

$$\begin{aligned} &\|f(x) - h(x)\| \\ &\leq \varepsilon \exp\left\{ -\Re\left(\int_a^x \frac{1}{z} B(\ln z) dz \right) \right\} \\ &\quad \times \left[\int_a^x \frac{1}{q} \int_a^q \frac{1}{w} \exp\left\{ 2\Re\left(\int_a^q \frac{1}{z} B(\ln z) dz \right) - \Re\left(\int_a^w \frac{1}{z} B(\ln z) dz \right) \right\} dw dq \right] \end{aligned}$$

for all $x \in I$. □

3. Example

In this section we will consider the Bessel differential equation of order $\frac{1}{2}$. In this case, the Riccati differential equation $B'(t) - B(t)^2 = e^{2t} - \alpha^2 = e^{2t} - \frac{1}{4}$ has a particular solution, $B(t) = ie^t + \frac{1}{2}$.

So, we have

$$\Re\left(\int_a^x \frac{1}{z} B(\ln z) dz \right) = \Re\left(i(x-a) + \frac{1}{2} \ln \frac{x}{a} \right) = \frac{1}{2} \ln \frac{x}{a}.$$

Therefore, if the function $f(x)$ satisfies the assumptions of Theorem 2.1, then there exists a solution $h(x)$ of the Bessel differential equation of order $\frac{1}{2}$ such that

$$\begin{aligned} \|f(x) - h(x)\| &\leq \varepsilon \sqrt{\frac{a}{x}} \left[\int_a^x \frac{1}{a} \int_a^q \frac{1}{w} \sqrt{\frac{a}{w}} \, dw dq \right] \\ &= \frac{2\varepsilon}{\sqrt{x}} \left[\int_a^x \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{q}} \right) dq \right] \\ &= 2\varepsilon \left[\sqrt{\frac{x}{a}} - 2 + \sqrt{\frac{a}{x}} \right] \\ &= 2\varepsilon \left(\sqrt[4]{\frac{a}{x}} - \sqrt[4]{\frac{x}{a}} \right)^2 \end{aligned}$$

for all $x \in I$.

Hence, by Theorem 2.1, we obtain the following corollary.

Corollary 3.1. *Assume that E is a complex Banach space and $I = (0, a)$ is an interval with $a > 0$. Let ε be any positive real constant. Suppose $f : I \to E$ is a twice continuously differentiable function that satisfies the following conditions:*

(i) $\lim_{x \to a^-} f(x)$ *exists;*

(ii) $\lim_{x \to a^-} f'(x)$ *exists;*

(iii) $\lim_{w \to (\ln a)^-} B(t)$ *exists, where $B : (-\infty, \ln a) \to \mathbb{C}$ is a solution of the Riccati differential equation $B'(t) - B(t)^2 = e^{2t} - \alpha^2$.*

If f moreover satisfies the inequality

$$\left\| x^2 f''(x) + x f'(x) + \left(x^2 - \frac{1}{4} \right) f(x) \right\| \leq \varepsilon$$

for all $x \in I$, then there exists a solution $h \in C^2(I, E)$ of Bessel differential equation (1.1) *with $\alpha = \frac{1}{2}$ such that*

$$\|f(x) - h(x)\| \leq 2\varepsilon \left(\sqrt[4]{\frac{a}{x}} - \sqrt[4]{\frac{x}{a}} \right)^2$$

for all $x \in I$.

4. Discussion

Bessel differential equation arises when we find separable solutions of the Laplace equation and the Helmholtz equation, in cylindrical or spherical coordinates. So, Bessel differential equation is important for many problems of wave propagation and static potentials. In solving problems in cylindrical coordinate systems, one obtains Bessel functions, the solution of the Bessel differential equation, of integer order $\alpha = n$, and in spherical problems, one obtains the functionss of half-integer order $\alpha = n + \frac{1}{2}$. For example, electromagnetic waves in a cylindrical waveguide, heat conduction in a cylindrical object, dynamics of floating bodies, diffusion problems on a lattice, etc.

Therefore, we investigated the stability of Bessel differential equations of a more generalized order than previous paper [9]. And in Theorem 2.1 we assumed that $f \in C^2(I, E)$ and f satisfies the inequality

$$\left\| x^2 f''(x) + x f'(x) + \left(x^2 - \alpha^2 \right) f(x) \right\| \leq \varepsilon$$

for all $x \in I$, while Kim and Jung [9] assumed that f is an analytic function and f satisfies the inequality (2.1) and

$$x^2 f''(x) + x f'(x) + \left(x^2 - \alpha^2 \right) f(x) = \sum_{m=0}^{\infty} a_m x^m$$

with $|\sum_{m=0}^{\infty} a_m x^m| \leq K \sum_{m=0}^{\infty} |a_m x^m|$ for some constant K.

In fact, to use the method with power series, Kim and Jung [9] did need an infinitely differentiable function $f(x)$, while we need a twice continuously differentiable function $f(x)$.

At last, we considered the Bessel differential equations on more general domain, $(0, a)$ where a is a positive real number, while Kim and Jung [9] studied the differential equations near zero to use the method with power series.

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