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ASYMPTOTIC PROPERTIES OF KNESER SOLUTIONS TO THIRD-ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract The aim of this paper is to extend and complete the recent work by Graef et al. (J. Appl. Anal. Comput., 2021) analyzing the asymptotic properties of solutions to third-order linear delay differential equations. Most importantly, the authors tackle a particularly challenging problem of obtaining lower estimates for Kneser-type solutions. This allows improvement of existing conditions for the nonexistence of such solutions. As a result, a new criterion for oscillation of all solutions of the equation studied is established.

Keywords Third-order differential equation, delay, linear, Kneser solution, oscillation.

MSC(2010) 34C10, 34K11.

1. Introduction

With regard to many indications of the importance of third-order differential equations in the applications and the number of mathematical problems involved [6, 7], the subject of the qualitative theory for such equations has undergone rapid development. In particular, the oscillation theory of third-order functional differential equations has attracted significant attention; see the recent monographs [4, 14, 15] for a summary of the most recent results and open problems.

In this paper, we consider the third-order linear delay differential equation

$$y'''(t) + q(t)y(\tau(t)) = 0, \quad t \ge t_0 > 0, \tag{1.1}$$

where $q \in \mathcal{C}([t_0, \infty), [0, \infty))$ does not eventually vanish identically and the delay function $\tau \in \mathcal{C}([t_0, \infty), \mathbb{R})$ satisfies $\tau(t) < t$, $\tau'(t) > 0$ and $\tau(t) \to \infty$ as $t \to \infty$.

By a *solution* of (1.1) we mean a nontrivial real-valued function $y \in \mathcal{C}^3([t_0, \infty), \mathbb{R})$ satisfying (1.1) on $[T_y, \infty)$ where $T_y > t_0$ is chosen such that $\tau(t) \ge t_0$ for $t \ge T_y$. Our attention is restricted to those solutions of (1.1) that exist on some half-line $[T_y, \infty)$ and satisfy the condition $\sup\{|y(t)| : T \le t < \infty\} > 0$ for any $T > T_y$. We tacitly assume that (1.1) possesses such solutions. The oscillatory nature of the solutions is understood in the usual way, that is, a solution is termed *oscillatory* or *nonoscillatory* according to whether it does or does not have infinitely many zeros.

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It follows from a classical result of Kiguradze [13, Lemma 1.1] that the set $\mathcal{N}$ of all nonoscillatory solutions of (1.1) can be divided into the following two classes:

$$\mathcal{N}_0 = \{y(t) : (\exists t_1 \geq t_0)(\forall t \geq t_1)(y(t)y'(t) < 0,\, y(t)y''(t) > 0)\}\,,$$
$$\mathcal{N}_2 = \{y(t) : (\exists t_1 \geq t_0)(\forall t \geq t_1)(y(t)y'(t) > 0,\, y(t)y''(t) > 0)\}\,.$$

Solutions belonging to the class $\mathcal{N}_0$ are called Kneser solutions. It is known that in the absence of a delay in (1.1), i.e., if $\tau(t) = t$, the class $\mathcal{N}_0$ is always nonempty (see, e.g., [12]). Therefore, results for third-order equations have been often accomplished by introducing the concept of the so-called Property A. We say that (1.1) has *Property A* if any solution y of (1.1) is either oscillatory or is a Kneser type solution tending to zero as $t \to \infty$ (see [13]).

In the very recent paper [11], the authors showed how to extend Hanan's Kneser-type oscillation criterion from the ordinary equation

$$y'''(t) + q(t)y(t) = 0 \tag{1.2}$$

to the delay equation (1.1) so that it remains sharp for the delay Euler differential equation

$$y'''(t) + \frac{q_0}{t^3}y(\lambda t) = 0, \quad q_0 > 0, \quad \lambda \in (0,1), \quad t \geq 1. \tag{1.3}$$

Their main result is the following.

Theorem 1.1. (See [11, Theorem 2.1]) *Let* $\lambda_* := \liminf_{t\to\infty} \frac{t}{\tau(t)}$. *If*

$$\liminf_{t\to\infty} \tau^2(t)tq(t) > \begin{cases} 0, & \text{for } \lambda_* = \infty, \\ M_2, & \text{for } \lambda_* < \infty, \end{cases} \tag{1.4}$$

where $M_2 := \max\left\{-x(x-1)(x-2)\lambda_*^{x-2} : 1 < x < 2\right\}$, *then* (1.1) *has Property A.*

It is worth noting that Theorem 1.1 is unimprovable in the sense that the Euler equation (1.3) has a solution $y \in \mathcal{N}_2$ if $q_0 = M_2$ (see [11, Corollary 2.1]).

On the other hand, it is well known that the delay argument can cause that $\mathcal{N}_0 = \mathcal{N}_2 = \emptyset$ and so (1.1) is oscillatory. In fact, the results for emptying the class $\mathcal{N}_0$ are relatively scarce, see [1,2,5,9,10], where the authors studied more or less general third-order differential and dynamic equations that include (1.1) as a particular case. In [2,10], the results were obtained by comparing the equation to first-order delay differential equations whose oscillatory character is known, resulting in conditions involving the constant $1/\mathrm{e}$. However, such an approach requires the existence of a function $\xi \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$ such that $\xi'(t) \geq 0$, $\xi(t) > t$, and $\tau(\xi(\xi(t))) < t$. Other results for the nonexistence of Kneser solutions are of the form $\limsup_{t\to\infty}(\cdot) > 1$, namely,

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \int_{u}^{t} \int_{x}^{t} q(s)\mathrm{d}s\mathrm{d}x\mathrm{d}u > 1 \tag{1.5}$$

(see [5, Theorem 4.2]), or

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s) \int_{\tau(s)}^{\tau(t)} (\tau(t) - u)\mathrm{d}u\mathrm{d}s > 1 \tag{1.6}$$

(see [1, Theorem 2.1]).

In view of the above discussion, the purpose of the present paper is to extend and complete Theorem 1.1 in two ways. Firstly, we provide additional information about the rate of convergence of nonoscillatory, say positive, Kneser solutions. The problem of obtaining lower estimates for such solutions is particularly challenging due to the alternating signs of the derivatives, although somehow possible in the ordinary case $\tau(t) = t$ (see [3, Theorem 1]). However, the approach used in [3] is not extendable to delay equations. We tackle this problem by providing a lower bound for the ratio $y(t)/y(\tau(t))$ under assumption (H) below. Secondly, based on newly obtained estimates, we improve condition (1.5) for the nonexistence of Kneser-type solutions, which, when combined with Theorem 1.1, ensures the oscillation of (1.1).

2. Preliminaries

In this section, we will present some preliminary notions that are of importance for the proofs of our main results. For convenience, we state that all functional inequalities are assumed to hold eventually, that is, they are satisfied for sufficiently large t. As usual and without loss of generality, we can assume from now on that nonoscillatory solutions of (1.1) are eventually positive.

It turns out that the following assumption plays a crucial role in our considerations.

(H) There exists a function $\alpha \in \mathcal{C}^1([t_0, \infty), \mathbb{R})$ with $\alpha' > 0$ such that

$$\alpha(\alpha(\tau(t))) = t. \tag{2.1}$$

Lemma 2.1. *Let (H) hold. Then $\alpha \circ \tau = \tau \circ \alpha$.*

Proof. From (2.1), we have $\alpha(\tau(t)) = \alpha^{-1}(t)$, i.e., $\alpha \circ \tau = \alpha^{-1}$, and using this for $t = \alpha(s)$, we get $\alpha(\tau(\alpha(s))) = s$, so $\tau(\alpha(s)) = \alpha^{-1}(s)$, i.e., $\tau \circ \alpha = \alpha^{-1}$. □

Remark 2.1. Let $\lambda \in (0, 1)$ and $k > 0$. Then for delay functions of the form $\tau(t) = \lambda t$, $\tau(t) = t^\lambda$, and $\tau(t) = t - k$, corresponding functions satisfying (2.1) are $\alpha(t) = t/\sqrt{\lambda}$, $\alpha(t) = t^{1/\sqrt{\lambda}}$, and $\alpha(t) = t + k/2$, respectively.

For the sake of clarity, we list the functions used in this work. That is, for $t \geq t_1 \geq t_0$ and α defined by (H), we set

$$c(t) = \frac{1}{2}\int_t^{\alpha(t)} q(s)(s-t)^2 \mathrm{d}s, \quad C(t) = c(t)c(\tau(\alpha(t))),$$

and for m, $n \in \mathbb{N}_0$,

$$\begin{cases} D_0(t) = \int_t^{\tau^{-1}(t)} q(s)\mathrm{d}s, \\ D_n(t) = D_0(t) + C(\tau^{-1}(t))D_{n-1}(\tau^{-1}(t)), \quad n \in \mathbb{N}, \end{cases} \tag{2.2}$$

and

$$\begin{cases} E_{0,n}(t) = C(\tau^{-1}(t))D_n(\tau^{-1}(t))(\tau^{-1}(t) - t) + \int_t^{\tau^{-1}(t)} \int_x^{\tau^{-1}(t)} q(s)\mathrm{d}s\mathrm{d}x, \\ E_{m,n}(t) = E_{0,n}(t) + C(\tau^{-1}(t))E_{m-1,n}(\tau^{-1}(t)), \ m \in \mathbb{N}. \end{cases} \tag{2.3}$$

Next, we have a simple result that will be needed in the proofs of our main results.

Lemma 2.2. *Let* (1.1) *possess an eventually positive solution* $y \in \mathcal{N}_0$. *Then*

$$\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0, \tag{2.4}$$

and moreover, if

$$\int_{t_0}^{\infty} t^2 q(t)\mathrm{d}t = \infty, \tag{2.5}$$

then

$$\lim_{t\to\infty} y(t) = 0. \tag{2.6}$$

Proof. For (2.4), see the first lines of the proof of [8, Proposition 4.5]. Since the proof of (2.6) proceeds in the same manner as the proof of [2, Lemma 2] except for a final interchange in the order of integration in (2.5), we omit the details. □

Remark 2.2. In [11, Lemma 3.2], we showed that (2.5) is already included in condition (1.4) for the nonexistence of $\mathcal{N}_2$-type solutions. It is also worth noting that (2.5) is necessary for Property A to hold for (1.2) (see [13, Remark 1.2]).

3. Main results

We begin this section with a lemma that provides a useful inequality in the subsequent analysis.

Lemma 3.1. *Let* (*H*) *and* (2.5) *hold and* (1.1) *possess an eventually positive solution* $y \in \mathcal{N}_0$. *Then*

$$y(t) \geq y(\tau(t))C(t). \tag{3.1}$$

Proof. Choose $t_1 \geq t_0$ so that $y(\tau(t)) > 0$ on $[t_1, \infty)$. Integrating (1.1) three times from t to ∞ and taking (2.4) and (2.6) into account, we obtain

$$y(t) = \int_t^{\infty}\int_s^{\infty}\int_u^{\infty} q(x)y(\tau(x))\mathrm{d}x\mathrm{d}u\mathrm{d}s. \tag{3.2}$$

Interchanging the order of integration in (3.2) gives

$$y(t) = \int_t^{\infty}\int_t^{x}\int_t^{u} q(x)y(\tau(x))\mathrm{d}s\mathrm{d}u\mathrm{d}x = \frac{1}{2}\int_t^{\infty} (x-t)^2 q(x)y(\tau(x))\mathrm{d}x. \tag{3.3}$$

Using the fact that $\alpha(t) > t$ (since $\tau(t) < t$, $\tau' > 0$, and $\alpha' > 0$ imply $\alpha(t) > \alpha(\tau(t)) = \tau^{-1}(t) > t$) and y is decreasing, from (3.3), we find that

$$\begin{aligned} y(t) &\geq \frac{1}{2}\int_t^{\alpha(t)} (x-t)^2 q(x)y(\tau(x))\mathrm{d}x \\ &\geq \frac{y(\tau(\alpha(t)))}{2}\int_t^{\alpha(t)} (x-t)^2 q(x)\mathrm{d}x \\ &= y(\tau(\alpha(t)))c(t). \end{aligned} \tag{3.4}$$

Shifting the independent variable in (3.4) from t to $\tau(\alpha(t))$ and using Lemma 2.1 and condition (H) in the resulting inequality yields

$$y(\tau(\alpha(t))) \geq y(\tau(\alpha(\tau(\alpha(t)))))c(\tau(\alpha(t))) = y(\tau(t))c(\tau(\alpha(t))). \tag{3.5}$$

Using (3.5) in (3.4) leads to

$$y(t) \geq y(\tau(t))c(\tau(\alpha(t)))c(t) = y(\tau(t))C(t).$$

This proves (3.1). $\square$

A simple consequence of the above result is the following.

Theorem 3.1. *Let* (H) *and* (2.5) *hold. If*

$$\limsup_{t\to\infty} C(t) > 1, \tag{3.6}$$

then $\mathcal{N}_0 = \emptyset$ *for* (1.1).

Proof. Suppose to the contrary that (1.1) has an eventually positive solution $y \in \mathcal{N}_0$ such that $y(\tau(t)) > 0$ for $t \geq t_1 \geq t_0$. Since y is strictly decreasing and $\tau(t) < t$, we have $y(\tau(t)) > y(t)$, which in view of Lemma 3.1 contradicts (3.6). $\square$

Our next lemma is an initial step in obtaining lower estimates on solutions in $\mathcal{N}_0$.

Lemma 3.2. *Let* (H) *and* (2.5) *hold and* (1.1) *possess an eventually positive solution* $y \in \mathcal{N}_0$. *Then for any* $m, n \in \mathbb{N}_0$,

$$y''(t) \geq y(t)D_n(t) \tag{3.7}$$

and

$$-y'(t) \geq y(t)E_{m,n}(t). \tag{3.8}$$

Proof. Choose $t_1 \geq t_0$ such that $y(\tau(t)) > 0$ on $[t_1, \infty)$. In order to prove (3.7), we will proceed by induction on n. Integrating (1.1) from u to $v > u$ gives

$$y''(u) - y''(v) = \int_u^v q(s)y(\tau(s))\mathrm{d}s \geq y(\tau(v))\int_u^v q(s)\mathrm{d}s. \tag{3.9}$$

Setting $u = \tau(t)$ and $v = t$ in (3.9), we arrive at

$$y''(\tau(t)) \geq y''(t) + y(\tau(t))\int_{\tau(t)}^t q(s)\mathrm{d}s = y''(t) + y(\tau(t))D_0(\tau(t)), \tag{3.10}$$

i.e., since $y''(t) > 0$, (3.7) holds for $n = 0$. To perform the inductive step, assume that (3.7) holds for some $n \geq 0$. Substituting (3.7) into (3.10) yields

$$\begin{aligned} y''(\tau(t)) &\geq y(t)D_n(t) + y(\tau(t))D_0(\tau(t)) \\ &\overset{(3.1)}{\geq} y(\tau(t))\left[C(t)D_n(t) + D_0(\tau(t))\right] \\ &\overset{(2.2)}{=} y(\tau(t))D_{n+1}(\tau(t)), \end{aligned}$$

i.e., (3.7) holds with n replaced by $n+1$. In a similar manner, we will prove that (3.8) holds. Integrating (3.9) from x to $v > x$ with respect to u, we have

$$-y'(x) + y'(v) - y''(v)(v-x) \geq y(\tau(v))\int_x^v \int_u^v q(s)\mathrm{d}s\mathrm{d}u. \tag{3.11}$$

Letting $x = \tau(t)$ and $v = t$ in (3.11) and using (3.7), we find

$$\begin{aligned}
-y'(\tau(t)) &\geq -y'(t) + y''(t)(t-\tau(t)) + y(\tau(t))\int_{\tau(t)}^{t}\int_{u}^{t} q(s)\mathrm{d}s\mathrm{d}u \\
&\geq -y'(t) + y(t)D_n(t)(t-\tau(t)) + y(\tau(t))\int_{\tau(t)}^{t}\int_{u}^{t} q(s)\mathrm{d}s\mathrm{d}u \\
&\overset{(3.1)}{\geq} -y'(t) + y(\tau(t))E_{0,n}(\tau(t)),
\end{aligned} \tag{3.12}$$

i.e., since $y'(t) < 0$, (3.8) holds for $m = 0$. Now assume that (3.8) holds for some $m \geq 0$. Substituting (3.8) into (3.12) and taking (3.1) into account yields

$$\begin{aligned}
-y'(\tau(t)) &\geq y(t)E_{m,n}(t) + y(\tau(t))E_{0,n}(\tau(t)) \\
&\geq y(\tau(t))\left[C(t)E_{m,n}(t) + E_{0,n}(\tau(t))\right] = y(\tau(t))E_{m+1,n}(\tau(t)),
\end{aligned}$$

i.e., (3.8) holds with m replaced by $m+1$. The proof is now complete. □

With the aid of Lemmas 3.1 and 3.2, it is possible to provide upper and lower estimates for positive Kneser-type solutions.

Theorem 3.2. *Assume* (H) *and* (2.5) *hold and let* (1.1) *have an eventually positive solution* $y \in \mathcal{N}_0$. *Then for any* $m, n \in \mathbb{N}_0$ *there exist a sufficiently large* $t_1 \in [t_0, \infty)$ *and two constants* $k_1, k_2 > 0$ *such that for* $t \geq t_1$,

$$k_1 \exp\left(-\int_{t_1}^{t} E_{m,n}(s)\mathrm{d}s\right) \geq y(t) \geq k_2 \exp\left(-\int_{t_1}^{t} \frac{1-C(s)}{C(s)(s-\tau(s))}\mathrm{d}s\right). \tag{3.13}$$

Proof. Take $t_1 \geq t_0$ so that $y(\tau(t)) > 0$ on $[t_1, \infty)$. From (3.8), it is straightforward to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y(t)\exp\left(\int_{t_1}^{t} E_{m,n}(s)\mathrm{d}s\right)\right] \leq 0. \tag{3.14}$$

On the other hand, using the monotonicity of y', we see that

$$y(\tau(t)) = y(t) - \int_{\tau(t)}^{t} y'(s)\mathrm{d}s \geq y(t) - y'(t)(t-\tau(t)).$$

Multiplying the above inequality by $C(t)$ and taking (3.1) into account, we have

$$y(t) \geq C(t)y(\tau(t)) \geq C(t)y(t) - C(t)y'(t)(t-\tau(t)),$$

which we can rewrite as

$$-y'(t) \leq y(t)\frac{1-C(t)}{C(t)(t-\tau(t))}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y(t)\exp\left(\int_{t_1}^{t} \frac{1-C(s)}{C(t)(s-\tau(s))}\mathrm{d}s\right)\right] \geq 0. \tag{3.15}$$

In view of (3.14) and (3.15), the conclusion is immediate. □

The next result can be viewed as a direct improvement of condition (1.5).

Theorem 3.3. *Let* (H) *and* (2.5) *hold. If*

$$\limsup_{t\to\infty}\left\{C(t)\left[1+D_n(t)\frac{(t-\tau(t))^2}{2}+E_{m,n}(t)(t-\tau(t))\right]\right.$$
$$\left.+\int_{\tau(t)}^t\int_x^t\int_u^t q(s)\mathrm{d}s\mathrm{d}u\mathrm{d}x\right\}>1 \tag{3.16}$$

for some $m,n\in\mathbb{N}_0$*, then* $\mathcal{N}_0=\emptyset$*.*

Proof. Suppose to the contrary that (1.1) has an eventually positive solution $y\in\mathcal{N}_0$ such that $y(\tau(t))>0$ for $t\ge t_1\ge t_0$. Integrating (3.11) from r to $v>r$ with respect to x, we obtain

$$y(r)\ge y(v)+y''(v)\frac{(v-r)^2}{2}-y'(v)(v-r)+y(\tau(v))\int_r^v\int_x^v\int_u^v q(s)\mathrm{d}s\mathrm{d}u\mathrm{d}x.$$

Setting $r=\tau(t)$ and $v=t$ and using (3.7) and (3.8) gives

$$\begin{aligned}
y(\tau(t))&\ge y(t)+y''(t)\frac{(t-\tau(t))^2}{2}-y'(t)(t-\tau(t))\\
&\quad+y(\tau(t))\int_{\tau(t)}^t\int_x^t\int_u^t q(s)\mathrm{d}s\mathrm{d}u\mathrm{d}x\\
&\ge y(t)+y(t)D_n(t)\frac{(t-\tau(t))^2}{2}+y(t)E_{m,n}(t)(t-\tau(t))\\
&\quad+y(\tau(t))\int_{\tau(t)}^t\int_x^t\int_u^t q(s)\mathrm{d}s\mathrm{d}u\mathrm{d}x\\
&\overset{(3.1)}{\ge} y(\tau(t))\left[C(t)\left(1+D_n(t)\frac{(t-\tau(t))^2}{2}+E_{m,n}(t)(t-\tau(t))\right)\right.\\
&\quad\left.+\int_{\tau(t)}^t\int_x^t\int_u^t q(s)\mathrm{d}s\mathrm{d}u\mathrm{d}x\right].
\end{aligned}$$

Taking lim sup on both sides of this inequality, we arrive at a contradiction to (3.16). This completes the proof. □

It should be clear that analogous statements to those above for negative Kneser-type solutions are also true. The proofs of the following two theorems should be obvious.

Theorem 3.4. *Let* (H) *and all the assumptions of Theorem 1.1 hold. Then* (1.1) *has Property A, and moreover, any positive Kneser solution satisfies* (3.13)*.*

Theorem 3.5. *Let* (H) *and all the assumptions of Theorem 1.1 hold. If* (3.6) *or* (3.16) *is satisfied, then* (1.1) *is oscillatory.*

We conclude this paper with an example of our results.

Example 3.1. Consider the Euler delay differential equation (1.3). Clearly,

$$q(t)=\frac{q_0}{t^3},\quad \tau(t)=\lambda t,\quad\text{and}\quad \alpha(t)=\frac{1}{\sqrt{\lambda}}t.$$

By straightforward computations, we have

$$
\begin{aligned}
&c(t) = c := \frac{q_0}{2}\left[\ln\frac{1}{\sqrt{\lambda}} - 2\left(1-\sqrt{\lambda}\right) + \frac{1-\lambda}{2}\right],\\
&C(t) = C := c^2,\\
&D_n(t) = \frac{D_n}{t^2}, \quad n \in \mathbb{N}_0, \quad D_n := \begin{cases} \frac{q_0(1-\lambda^2)}{2}, & n = 0,\\ D_0 \sum_{i=0}^{n} (C\lambda^2)^i, & n \in \mathbb{N}. \end{cases}\\
&E_{m,n}(t) = \frac{E_{m,n}}{t} \quad m,n \in \mathbb{N}_0, \quad E_{m,n} := \begin{cases} CD_n(1-\lambda)\lambda + \frac{q_0(1-\lambda)^2}{2}, & m = 0,\\ E_{0,n} \sum_{i=0}^{m} (C\lambda)^i, & m \in \mathbb{N}. \end{cases}
\end{aligned}
$$

By Theorem 3.1, we conclude that $\mathcal{N}_0 = \emptyset$ for (1.3) provided

$$C > 1. \tag{C$_1$}$$

If (C_1) fails, it is possible to find limits of the sequences $\{D_n\}$ and $\{E_{m,n}\}$:

$$D = \lim_{n\to\infty} D_n = \frac{q_0(1-\lambda^2)}{2(1-C\lambda^2)} \quad \text{and} \quad E = \lim_{m,n\to\infty} E_{m,n} = CD(1-\lambda)\lambda + \frac{q_0}{2}(1-\lambda)^2.$$

By Theorem 3.2, a positive solution $y \in \mathcal{N}_0$ of (1.3) satisfies

$$k_1 t^{-E+\varepsilon} \geq y(t) \geq k_2 t^{-\frac{1-C}{C(1-\lambda)}}$$

for any $\varepsilon > 0$. By Theorem 3.3, if

$$\frac{q_0 m}{2} + C\left(1 + \frac{D(1-\lambda)^2}{2} + E(1-\lambda)\right) > 1, \quad m := \ln\frac{1}{\lambda} - 2(1-\lambda) + \frac{1-\lambda^2}{2}, \tag{C$_2$}$$

then $\mathcal{N}_0 = \emptyset$, while existing conditions (1.5) and (1.6) require

$$\frac{q_0 m}{2} > 1 \tag{ex$_1$}$$

and

$$\frac{q_0\lambda^2}{2}\left[\ln\frac{1}{\lambda} + 2\left(1-\frac{1}{\lambda}\right) - \frac{1}{2}\left(1-\frac{1}{\lambda^2}\right)\right] > 1, \tag{ex$_2$}$$

respectively. If we set e.g., $\lambda = 0.5$, condition (C_2) requires only $q_0 > 26.8938$ for $\mathcal{N}_0 = \emptyset$, while ($\mathrm{ex}_1$) and ($\mathrm{ex}_2$) require $q_0 > 29.3482$ and $q_0 > 41.4192$, respectively.

Remark 3.1. In this paper, the asymptotic properties of Kneser solutions are established via a novel approach. Most importantly, we provide a lower non-zero bound for a positive Kneser solution of (1.1), which was not known so far, as was pointed out in [11] and left as an open problem. The strength of our results essentially depends on the quantity $C(t)$ in (3.1). Hence, an interesting problem for further research is to sharpen the estimate (3.1). Another problem that currently remains open is to find a similar relation for Kneser solutions of (1.1) without requiring (H).

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