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POSITIVE SOLUTIONS FOR A NONLOCAL PROBLEM WITH CRITICAL SOBOLEV EXPONENT IN HIGHER DIMENSIONS*

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Abstract This paper is devoted to a nonlocal problem involving critical Sobolev exponent and negative nonlocal term. By virtue of a cut-off technique and the concentration compactness principle, we prove the existence and asymptotic behavior of positive solutions for the considered problem. In particular, our results generalize the existence results of positive solutions to higher dimensions $N \geq 5$.

Keywords Variational methods, nonlocal problem, positive solution, critical Sobolev exponent.

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1. Introduction and main results

In this paper, we consider the following nonlocal problem

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u + |u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $a, b > 0$ are constants, $\lambda > 0$ is a parameter, $2^* = 2N/(N-2)$ is the critical Sobolev exponent and Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 5$.

Problem (1.1), in which the equation has a negative nonlocal term $-b\int_\Omega |\nabla u|^2 dx$, is a variant type of the following traditional Kirchhoff problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

which is related to the stationary analogue of the equation

$$\begin{cases} u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

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presented by Kirchhoff [6] as an extension of the classical d'Alembert's wave equation for free vibration of elastic string. We have to point that the negative sign of the nonlocal term of (1.1) causes some mathematical difficulties different from typical Kirchhoff problem, which make the study of this kind of problem particularly interesting. In recent years, many researchers pay attention to such nonlocal problems with subcritical growth

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f_\lambda(x)|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $0 < p < 2^*$ and Ω is a smooth bounded domain in $\mathbb{R}^N$. When $N \geq 1$, $2 < p < 2^*$ and $f_\lambda(x) \equiv 1$, it was proved in [19] that problem (1.3) admits two nontrivial solutions. The existence and asymptotic behavior of sign-changing solutions to (1.3) were given in [11,17]. Duan etc [1] extended the existence result of [19] to the case of $1 \leq p < 2^*$. Two positive solutions of (1.3) were obtained in [7] under $N = 3$, $0 < p < 1$ and $f_\lambda(x) = \lambda > 0$ small enough. When $N = 3$ and $f_\lambda(x) \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is a sign-changing function, the existence of positive solution of (1.3) for $1 < p < 2$ and $3 < p < 6$ were respectively established in [8, 13]. For more related results of (1.3) with general nonlinearities and its variants on whole space, we refer the readers to [3, 4, 14, 15, 20, 21] and the references therein.

However, there are few papers to deal with the critical nonlocal problems like (1.1), except [16] and [12]. For $N = 4$, by using minimization argument and mountain-pass theorem, Wang etc [16] proved the existence of two positive solutions to the problem

$$-\left(a - b\int_{\mathbb{R}^4} |\nabla u|^2 dx\right)\Delta u = \lambda g(x) + |u|^2 u, \quad x \in \mathbb{R}^4, \tag{1.4}$$

where $g(x) \in L^{4/3}(\mathbb{R}^4)$ is a nonnegative function. When $\mathbb{R}^4$ and $\lambda g(x)+|u|^2u$ are replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^4$ and $\lambda|u|^{p-2}u+Q(x)|u|^2u$ respectively, Qian [12] studied how the coefficient function $Q(x)$ of the critical term affects the number of positive solutions to problem (1.4) with Dirichlet boundary condition, via Nehari manifold method.

Considering that previous works [12,16] treat only four dimensional case, here we try to prove the existence and asymptotic behavior of positive solutions to critical problem (1.1) in higher dimensions (i.e., $N \geq 5$). Our main difficulties lie in the presence of the negative nonlocal term and the lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. We emphasize that, the arguments used in [12,16] do not apply here since their arguments heavily rely on the dimension $N = 4$. In fact, we should combine a cut-off technique (see [2, 9, 18] for some related applications) with the concentration compactness principle, in order to overcome these difficulties.

Let $H_0^1(\Omega)$ and $L^r(\Omega)$ denote the usual Sobolev space equipped with respect to the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$ and $|u|_r^r = \int_\Omega |u|^r dx$, respectively. Let $\to$ ($\rightharpoonup$) denote the strong (weak) convergence. We denote by $B_r(x)$ the open ball of center x and radius $r > 0$. Let S denote the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, namely,

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}}.$$

Denote by λ_1 the first positive eigenvalue of the operator $-\Delta$ on Ω. By the Hölder's inequality, we easily verify that $\lambda_1 > S/|\Omega|^{2/N}$.

Our first result is the following nonexistence of positive solution to (1.1).

Theorem 1.1. *If $\lambda \geq a\lambda_1$, then problem* (1.1) *has no positive solution.*

According to Theorem 1.1, to seek the positive solution of (1.1), it has to be in the range $\lambda \in (0, a\lambda_1)$. More precisely, our existence theorem can be stated as follows.

Theorem 1.2. *Let $\delta_* = \min\{\frac{\lambda_1(N-2)(8b)^{1/2}(aS)^{N/4}}{2^{N/4}N^{1/2}}, \frac{aS}{2|\Omega|^{2/N}}, \frac{1}{\left(16b\lambda_1^2|\Omega|(N-2)\right)^{2/(N-4)}}\}$. If $\lambda \in (a\lambda_1 - \delta_*, a\lambda_1)$, then problem* (1.1) *has at least one positive solution u_λ.*

Theorem 1.3. *Let $\{\bar{\lambda}_n\}$ be a sequence with $\bar{\lambda}_n \nearrow a\lambda_1$ as $n \to \infty$. Suppose that $u_{\bar{\lambda}_n}$ is the positive solution corresponding to $\bar{\lambda}_n$ obtained in Theorem 1.2. Then, $\lim_{n\to\infty} \|u_{\bar{\lambda}_n}\| = 0$.*

Remark 1.1. In (1.1), if $b < 0$, it reduces to the traditional Kirchhoff type problem. For this situation, the existence of positive solution when $\lambda > a\lambda_1$ has been proved in [5, Theorem 1.2]. Comparing this with our results, we see that it is quite different between $b > 0$ and $b < 0$, which indicates that the sign of nonlocal term plays an important role in the nonlocal problems.

Remark 1.2. Comparing with [12, 16], we use some new methods to extend the existence result of the case $N = 4$ to $N \geq 5$. Moreover, the asymptotic behavior of positive solutions of (1.1) is also obtained, which was not observed in [12, 16].

The paper contains two more sections. In Section 2, we present some preliminaries. In Section 3, we prove our main results.

2. Preliminaries

The energy functional corresponding to problem (1.1) is

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}.$$

As we all know, the weak solutions to (1.1) are exactly the critical points of the functional I_λ.

Define a smooth cut-off function ϕ such that

$$\begin{cases} \phi(t) = 1, & t \in [0, 1), \\ \phi(t) = 0, & t \in (2, +\infty), \\ 0 \leq \phi(t) \leq 1, & t \in [1, 2], \\ -2 \leq \phi'(t) \leq 0, & t \in [0, +\infty). \end{cases}$$

Associated with functional I_λ, we consider the following truncated functional $I_{\lambda,T}$: $H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_{\lambda,T}(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\Phi_T(u)\|u\|^4 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{2^*}|u|_{2^*}^{2^*},$$

where for each $T > 0$, $\Phi_T(u) = \phi\left(\frac{\|u\|^2}{T^2}\right)$. Clearly, $I_{\lambda,T}$ is well defined and $I_{\lambda,T} \in C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore, for any $u, v \in H_0^1(\Omega)$, one has

$$\begin{aligned}\langle I'_{\lambda,T}(u), v\rangle = &\left[a - b\Phi_T(u)\|u\|^2 - \frac{b}{2T^2}\phi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^4\right]\int_\Omega \nabla u \nabla v dx\\ &- \lambda\int_\Omega uvdx - \int_\Omega |u|^{2^*-2}uvdx.\end{aligned}$$

For $\lambda < a\lambda_1$, let $T = \left(\frac{a\lambda_1 - \lambda}{8b\lambda_1(N-2)}\right)^{1/2}$ and note that

$$0 \le \Phi_T(u)\|u\|^4 \le 4T^4 \quad \text{and} \quad -16T^6 \le \phi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^6 \le 0. \tag{2.1}$$

Moreover, it is easy to see that if u is a critical point of $I_{\lambda,T}$ such that $\|u\| \le T$, then u is also a critical point of I_λ. To give the proofs of our main results, we need the following three lemmas.

Lemma 2.1. *Let* $\delta_1 = \frac{\lambda_1(N-2)(8b)^{1/2}(aS)^{N/4}}{2^{N/4}N^{1/2}}$. *If* $\{u_n\} \subset H_0^1(\Omega)$ *is a sequence satisfying*

$$I_{\lambda,T}(u_n) \to c < \frac{1}{N}\left(\frac{aS}{2}\right)^{N/2} \quad \textit{and} \quad I'_{\lambda,T}(u_n) \to 0, \ \textit{as } n \to \infty, \tag{2.2}$$

then $\{u_n\}$ *has a convergent subsequence for* $\lambda \in (a\lambda_1 - \delta_1, a\lambda_1)$.

Proof. By the Sobolev inequality, (2.1) and (2.2), we have that

$$\begin{aligned}c + 1 + o(1)\|u_n\| \ge& I_{\lambda,T}(u_n) - \frac{1}{2^*}\langle I'_{\lambda,T}(u_n), u_n\rangle\\ =& \frac{a}{N}\|u_n\|^2 + \left(\frac{b}{2^*} - \frac{b}{4}\right)\Phi_T(u_n)\|u_n\|^4\\ &+ \frac{b}{2\cdot 2^* T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^6 - \frac{\lambda}{N}|u_n|_2^2\\ \ge& \left(\frac{a\lambda_1 - \lambda}{N\lambda_1}\right)\|u_n\|^2 - \frac{8b}{2^*}T^4,\end{aligned}$$

for n large enough. As $\lambda < a\lambda_1$, it follows that $\|u_n\|$ is bounded. Thus, there is a subsequence (still denoted by $\{u_n\}$) and $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H_0^1(\Omega),\\ u_n \to u, & \text{in } L^2(\Omega),\\ u_n \to u, & \text{a.e. in } \Omega.\end{cases}$$

Furthermore, according to the concentration compactness principle due to Lions [10], there exist two measures $\mu, \nu \in \mathcal{M}(\Omega)$ such that

$$|\nabla u_n|^2 \rightharpoonup d\mu \ge |\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{x_j} \quad \text{and} \quad |u_n|^{2^*} \rightharpoonup d\nu = |u|^{2^*} + \sum_{j\in J}\nu_j\delta_{x_j},$$

where J is at most a countable index set, $\{x_j\}$ is a sequence of points in Ω and δ_{x_j} is the Dirac mass at x_j. Meanwhile, for any $j \in J$, we have

$$\mu_j \ge S\nu_j^{2/2^*} \quad \text{and} \quad \mu_j, \nu_j \ge 0. \tag{2.3}$$

For any $\varepsilon > 0$ small, let $\psi_j^\varepsilon(x)$ be a smooth cut-off function centered at x_j such that

$$\begin{cases} \psi_j^\varepsilon(x) = 1, & x \in B_{\varepsilon/2}(x_j), \\ \psi_j^\varepsilon(x) = 0, & x \in B_\varepsilon^c(x_j), \\ 0 \le \psi_j^\varepsilon(x) \le 1, & x \in B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j), \\ |\nabla \psi_j^\varepsilon(x)| \le 4/\varepsilon. \end{cases}$$

It then follows from the boundedness of $\{u_n\}$ and the Hölder's inequality that,

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega u_n \nabla u_n \nabla \psi_j^\varepsilon dx\right| \\ \le &\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{B_\varepsilon(x_j)} |\nabla u_n|^2 dx\right)^{\frac{1}{2}}\left(\int_{B_\varepsilon(x_j)} |u_n|^2 |\nabla \psi_j^\varepsilon|^2 dx\right)^{\frac{1}{2}} \\ \le &\lim_{\varepsilon\to 0} C_1 \left(\int_{B_\varepsilon(x_j)} |u|^2 |\nabla \psi_j^\varepsilon|^2 dx\right)^{\frac{1}{2}} \\ \le &\lim_{\varepsilon\to 0} C_1 \left(\int_{B_\varepsilon(x_j)} |\nabla \psi_j^\varepsilon|^N dx\right)^{\frac{1}{N}}\left(\int_{B_\varepsilon(x_j)} |u|^{2^*} dx\right)^{\frac{1}{2^*}} \\ \le &\lim_{\varepsilon\to 0} C_2 \left(\int_{B_\varepsilon(x_j)} |u|^{2^*} dx\right)^{\frac{1}{2^*}} = 0, \end{aligned} \tag{2.4}$$

where C_1 and C_2 are some positive constants, independent of ε and n. Also,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n^2 \psi_j^\varepsilon dx = \lim_{\varepsilon\to 0}\int_{B_\varepsilon(x_j)} u^2 \psi_j^\varepsilon dx = 0. \tag{2.5}$$

Moreover, since for $\lambda \in (0, a\lambda_1)$, we must have $T^2 \le \frac{a}{4b}$ and consequently,

$$a - b\Phi_T(u_n)\|u_n\|^2 - \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4 \ge a - 2bT^2 \ge \frac{a}{2}. \tag{2.6}$$

Combining (2.4)–(2.6) and the fact that $\{u_n \psi_j^\varepsilon(x)\}$ is bounded in $H_0^1(\Omega)$, we obtain

$$\begin{aligned} 0 =& \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle I'_{\lambda,T}(u_n), u_n\psi_j^\varepsilon\rangle \\ =& \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\left[a - b\Phi_T(u_n)\|u_n\|^2 - \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right]\int_\Omega |\nabla u_n|^2 \psi_j^\varepsilon dx\right. \\ &+ \left[a - b\Phi_T(u_n)\|u_n\|^2 - \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right]\int_\Omega u_n \nabla u_n \nabla \psi_j^\varepsilon dx \\ &\left. - \int_\Omega (\lambda u_n^2 + |u_n|^{2^*})\psi_j^\varepsilon dx\right\} \\ \ge& \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\frac{a}{2}\int_\Omega |\nabla u_n|^2 \psi_j^\varepsilon dx - \int_\Omega |u_n|^{2^*}\psi_j^\varepsilon dx\right\} \\ \ge& \frac{a}{2}\mu_j - \nu_j, \end{aligned}$$

which implies that $\nu_j \geq \frac{a}{2}\mu_j$. Then, it is deduced from (2.3) that

$$\nu_j \geq \left(\frac{aS}{2}\right)^{N/2} \quad \text{or} \quad \nu_j = 0.$$

Next, we show that the first alternative above does not occur. Let us argue by contradiction and assume that there is some $j_0 \in J$ such that $\nu_{j_0} \geq \left(\frac{aS}{2}\right)^{N/2}$. For $\lambda \in (a\lambda_1 - \delta_1, a\lambda_1)$, it holds

$$\begin{aligned} c + o(1) =& I_{\lambda,T}(u_n) - \frac{1}{2}\langle I'_{\lambda,T}(u_n), u_n\rangle \\ =& \frac{b}{4}\Phi_T(u_n)\|u_n\|^4 + \frac{b}{4T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^6 + \frac{1}{N}|u_n|_{2^*}^{2^*} \\ \geq& -4bT^4 + \frac{1}{N}|u_n|_{2^*}^{2^*} \\ \geq& -\frac{1}{2N}\left(\frac{aS}{2}\right)^{N/2} + \frac{1}{N}\nu_{j_0} \\ \geq& \frac{1}{N}\left(\frac{aS}{2}\right)^{N/2}, \end{aligned}$$

which is a contradiction with $c < \frac{1}{N}\left(\frac{aS}{2}\right)^{N/2}$. This gives J is empty and thus, we conclude that $u_n \to u$ in $L^{2^*}(\Omega)$. Hence, using the Hölder's inequality, we get

$$\left|\int_\Omega |u_n|^{2^*-1}(u_n - u)dx\right| \leq |u_n|_{2^*}^{2^*-1}|u_n - u|_{2^*} \to 0, \tag{2.7}$$

as $n \to \infty$. Similarly,

$$\left|\int_\Omega u_n(u_n - u)dx\right| \leq |u_n|_2|u_n - u|_2 \to 0. \tag{2.8}$$

Since $\lim_{n\to\infty} \langle I'_{\lambda,T}(u_n), u_n - u\rangle = 0$, we can infer from (2.7) and (2.8) that

$$\lim_{n\to\infty}\left[a - b\Phi_T(u_n)\|u_n\|^2 - \frac{b}{2T^2}\phi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right]\int_\Omega \nabla u_n \nabla(u_n - u)dx = 0,$$

which yields that $\|u_n\| \to \|u\|$. This and the weak convergence of $\{u_n\}$ in $H_0^1(\Omega)$ imply that $u_n \to u$ in $H_0^1(\Omega)$. The proof of Lemma 2.1 is completed. $\square$

Lemma 2.2. *If $\lambda \in (0, a\lambda_1)$, then the functional $I_{\lambda,T}$ satisfies the mountain-pass geometry:*
(i) There exist $\rho, \alpha > 0$ such that $I_{\lambda,T}(u) \geq \alpha > 0$ for all $\|u\| = \rho$.
(ii) There exists $\bar{u} \in H_0^1(\Omega)$ with $\|\bar{u}\| > \rho$ such that $I_{\lambda,T}(\bar{u}) < 0$.

Proof. (i) By the Sobolev inequality,

$$I_{\lambda,T}(u) \geq \frac{a\lambda_1 - \lambda}{2\lambda_1}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{2^*}S^{-2^*/2}\|u\|^{2^*}.$$

From which it follows that there exist $\rho, \alpha > 0$ such that $I_{\lambda,T}(u) \geq \alpha > 0$ for all $\|u\| = \rho$.
(ii) For any $v \in H_0^1(\Omega) \setminus \{0\}$, $t > 0$, we have

$$I_{\lambda,T}(tv) \leq \frac{at^2}{2}\|v\|^2 - \frac{\lambda t^2}{2}|v|_2^2 - \frac{t^{2^*}}{2^*}|v|_{2^*}^{2^*} \to -\infty, \quad \text{as } t \to +\infty.$$

Hence, there is $t_0 > 0$ large such that $I_{\lambda,T}(\bar{u}) < 0$ and $\|\bar{u}\| > \rho$, where $\bar{u} = t_0 v$. $\square$

Lemma 2.3. *Let $e_1(x)$ be a positive eigenfunction associated with λ_1. For $\lambda \in (0, a\lambda_1)$, we have*

$$\sup_{t\geq 0} I_{\lambda,T}(te_1) \leq \frac{1}{N}|\Omega|(a\lambda_1 - \lambda)^{N/2}.$$

Proof. By the Hölder's inequality and the fact $\|e_1\|^2 = \lambda_1 |e_1|_2^2$,

$$\begin{aligned}\sup_{t\geq 0} I_{\lambda,T}(te_1) \leq & \sup_{t\geq 0}\left\{\frac{a}{2}t^2\|e_1\|^2 - \frac{\lambda}{2}t^2|e_1|_2^2 - \frac{1}{2^*}t^{2^*}|e_1|_{2^*}^{2^*}\right\} \\ = & \sup_{t\geq 0}\left\{\frac{a\lambda_1 - \lambda}{2}t^2|e_1|_2^2 - \frac{1}{2^*}t^{2^*}|e_1|_{2^*}^{2^*}\right\} \\ \leq & \sup_{t\geq 0}\left\{\frac{a\lambda_1 - \lambda}{2}t^2|\Omega|^{\frac{2^*-2}{2^*}}|e_1|_{2^*}^2 - \frac{1}{2^*}t^{2^*}|e_1|_{2^*}^{2^*}\right\}.\end{aligned}$$

For $t \geq 0$, set

$$h(t) = \frac{a\lambda_1 - \lambda}{2}|\Omega|^{\frac{2^*-2}{2^*}}|e_1|_{2^*}^2 t^2 - \frac{1}{2^*}|e_1|_{2^*}^{2^*}t^{2^*}.$$

By easy calculation, we obtain that $h(t)$ attains its maximum at

$$t_{max} = \left(\frac{(a\lambda_1 - \lambda)|\Omega|^{\frac{2^*-2}{2^*}}|e_1|_{2^*}^2}{|e_1|_{2^*}^{2^*}}\right)^{1/(2^*-2)}$$

and

$$\sup_{t\geq 0} I_{\lambda,T}(te_1) \leq h(t_{max}) = \frac{1}{N}|\Omega|(a\lambda_1 - \lambda)^{N/2}.$$

This ends the proof of Lemma 2.3. $\square$

3. Proofs of main results

Proof of Theorem 1.1. If $u \in H_0^1(\Omega)$ is a positive solution of (1.1), it follows that

$$\begin{aligned}-\lambda\int_\Omega ue_1 dx = & \left(a - b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \Delta u e_1 dx + \int_\Omega u^{2^*-1}e_1 dx \\ > & \left(a - b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \Delta u e_1 dx \\ = & -\lambda_1\left(a - b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega ue_1 dx \\ > & -a\lambda_1\int_\Omega ue_1 dx,\end{aligned}$$

where e_1 is defined as in Lemma 2.3. Therefore, we obtain that, for all $\lambda \geq a\lambda_1$, problem (1.1) has no positive solution. $\square$

Proof of Theorem 1.2. By Lemma 2.2, using the mountain-pass theorem, there is a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $I_{\lambda,T}(u_n) \to c_\lambda$ and $I'_{\lambda,T}(u_n) \to 0$ for

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\lambda,T}(\gamma(t)) \geq \alpha > 0,$$

where

$$\Gamma := \{\gamma \in C\left([0,1], H_0^1(\Omega)\right) : \ \gamma(0) = 0 \text{ and } I_{\lambda,T}(\gamma(1)) < 0\}.$$

By Lemma 2.3, it is easy to see that $c_\lambda < \frac{1}{N}|\Omega|(a\lambda_1 - \lambda)^{N/2}$. Let $\delta_2 = \frac{aS}{2|\Omega|^{2/N}}$ and note that when $\lambda \in (a\lambda_1 - \delta_2, a\lambda_1)$, it holds

$$\frac{1}{N}|\Omega|(a\lambda_1 - \lambda)^{N/2} < \frac{1}{N}\left(\frac{aS}{2}\right)^{N/2}.$$

Set $\bar{\delta} = \min\{\delta_1, \delta_2\}$, then by Lemma 2.1, if $\lambda \in (a\lambda_1 - \bar{\delta}, a\lambda_1)$, we know that $\{u_n\}$ possesses a convergent subsequence. Thus, we may assume that $u_n \to u_\lambda$ in $H_0^1(\Omega)$. As a consequence, for $\lambda \in (a\lambda_1 - \bar{\delta}, a\lambda_1)$, one has

$$\frac{1}{N}|\Omega|(a\lambda_1 - \lambda)^{N/2} \geq I_{\lambda,T}(u_\lambda) = \lim_{n\to\infty} I_{\lambda,T}(u_n) = c_\lambda \geq \alpha > 0, \tag{3.1}$$

and $I'_{\lambda,T}(u_\lambda) = 0$, namely, u_λ is a nontrivial critical point of $I_{\lambda,T}$. We next show that $\|u_\lambda\| \leq T$. To proceed, let $\delta_* = \min\{\bar{\delta}, \frac{1}{\left(16b\lambda_1^2|\Omega|(N-2)\right)^{2/(N-4)}}\}$. Since for $\lambda \in (a\lambda_1 - \delta_*, a\lambda_1)$ we have

$$\frac{(a\lambda_1 - \lambda)T^2}{N\lambda_1} - \frac{8bT^4}{2^*} = \frac{(a\lambda_1 - \lambda)^2}{16b\lambda_1^2 N(N-2)} \geq \frac{1}{N}(a\lambda_1 - \lambda)^{N/2}|\Omega|, \quad (\text{necessarily } N \geq 5)$$

then it follows from (3.1) that

$$\begin{aligned}
\frac{(a\lambda_1 - \lambda)T^2}{N\lambda_1} - \frac{8bT^4}{2^*} \geq & I_{\lambda,T}(u_\lambda) - \frac{1}{2^*}\langle I'_{\lambda,T}(u_\lambda), u_\lambda\rangle \\
= & \frac{a}{N}\|u_\lambda\|^2 + \left(\frac{1}{2^*} - \frac{1}{4}\right) b\Phi_T(u_\lambda)\|u_\lambda\|^4 \\
& + \frac{b}{2\cdot 2^* T^2}\phi'\left(\frac{\|u_\lambda\|^2}{T^2}\right)\|u_\lambda\|^6 - \frac{\lambda}{N}|u_\lambda|_2^2 \\
\geq & \frac{a}{N}\|u_\lambda\|^2 - \frac{8}{2^*}bT^4 - \frac{\lambda}{N}|u_\lambda|_2^2 \\
\geq & \frac{a\lambda_1 - \lambda}{N\lambda_1}\|u_\lambda\|^2 - \frac{8}{2^*}bT^4,
\end{aligned}$$

which provides that $\|u_\lambda\| \leq T$ and hence, u_λ is a nontrivial solution to (1.1). Similar to the proof of [16, Theorem 1.2], we can further derive that u_λ is a positive solution. □

Proof of Theorem 1.3. For any sequence $\{\bar{\lambda}_n\}$ with $\bar{\lambda}_n \nearrow a\lambda_1$ as $n \to \infty$, let $u_{\bar{\lambda}_n}$ be the positive solution of problem (1.1) with $\lambda = \bar{\lambda}_n$ obtained in Theorem 1.2. Then, there holds

$$\begin{aligned}
c_{\bar{\lambda}_n} &= I_{\bar{\lambda}_n}(u_{\bar{\lambda}_n}) - \frac{1}{2^*}\langle I'_{\bar{\lambda}_n}(u_{\bar{\lambda}_n}), u_{\bar{\lambda}_n}\rangle \\
&\geq \frac{a\lambda_1 - \bar{\lambda}_n}{N\lambda_1}\|u_{\bar{\lambda}_n}\|^2 + \left(\frac{1}{2^*} - \frac{1}{4}\right) b\|u_{\bar{\lambda}_n}\|^4 \\
&\geq \left(\frac{1}{2^*} - \frac{1}{4}\right) b\|u_{\bar{\lambda}_n}\|^4.
\end{aligned}$$

Combing this with the fact that $0 \leq c_{\bar{\lambda}_n} \leq \frac{1}{N}|\Omega|(a\lambda_1 - \bar{\lambda}_n)^{N/2}$, we conclude that $\lim_{n\to\infty}\|u_{\bar{\lambda}_n}\| = 0$. □

References

[1] Y. Duan, X. Sun and H. Li, *Existence and multiplicity of positive solutions for a nonlocal problem*, J. Nonlinear Sci. Appl., 2017, 10, 6056–6061.

[2] W. Gan and S. Liu, *Multiple positive solutions of a class of Schrödinger-Poisson equation involving indefinite nonlinearity in* $\mathbb{R}^3$, Appl. Math. Lett., 2019, 93(1), 111–116.

[3] M. Hamdani, N. Chung and M. Aminlouee, *Infinitely many solutions for a new class of Schrödinger-Kirchhoff type equations in* $\mathbb{R}^N$ *involving the fractional* p*-Laplacian*, J. Elliptic Parabol. Equ., 2021, 7, 243–267.

[4] M. Hamdani, A. Harrabi, F. Mtiri and D. Repovš, *Existence and multiplicity results for a new* $p(x)$*-Kirchhoff problem*, Nonlinear Anal., 2020, 190, 111598.

[5] Y. Huang, Z. Liu and Y. Wu, *On Kirchhoff type equations with critical Sobolev exponent and Naimen's open problems*, Mathematics, 2015, 7, 97–114.

[6] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[7] C. Lei, C. Chu and H. Suo, *Positive solutions for a nonlocal problem with singularity*, Electron. J. Differential Equations, 2017, 85, 1–9.

[8] C. Lei, J. Liao and H. Suo, *Multiple positive solutions for a class of nonlocal problems involving a sign-changing potential*, Electron. J. Differential Equations, 2017, 9, 1–8.

[9] Y. Li, F. Li and J. Shi, *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differential Equations, 2012, 253(7), 2285–2294.

[10] P. L. Lions, *The concentration-compactness principle in the calculus of variations: the limit case II*, Annales De L Institut Henri Poincare Non Linear Analysis, 1985, 1, 145–201.

[11] X. Qian, *Ground state sign-changing solutions for a class of nonlocal problem*, J. Math. Anal. Appl., 2021, 495, 124753.

[12] X. Qian, *Multiplicity of positive solutions for a class of nonlocal problem involving critical exponent*, Electron. J. Qual. Theory Differ. Equ., 2021, 57, 1–14.

[13] X. Qian and W. Chao, *Existence of positive solutions for nonlocal problems with indefinite nonlinearity*, Bound. Value Probl., 2020, 40, 1–13.

[14] X. Qian and W. Chao, *Existence and concentration of ground state solutions for a class of nonlocal problem in* $\mathbb{R}^N$, Nonlinear Anal., 2021, 203, 112170.

[15] Z. Tang and Z. Ou, *Infinitely many solutions for a nonlocal problem*, J. Appl. Anal. Comput., 2020, 10(5), 1912–1917.

[16] Y. Wang, H. Suo and C. Lei, *Multiple positive solutions for a nonlocal problem involving critical exponent*, Electron. J. Differential Equations, 2017, 275, 1–11.

[17] Y. Wang and X. Yang, *Infinitely many solutions for a new Kirchhoff-type equation with subcritical exponent*, Appl. Anal., 2022, 101(3), 1038–1051.

[18] L. Yang, Z. Liu and Z. Ouyang, *Multiplicity results for the Kirchhoff type equations with critical growth*, Appl. Math. Lett., 2017, 63(4), 118–123.

[19] G. Yin and J. Liu, *Existence and multiplicity of nontrivial solutions for a nonlocal problem*, Bound. Value Probl., 2015, 26, 1–7.

[20] B. Zhang, B. Ge and X. Cao, *Multiple solutions for a class of new* $p(x)$*-Kirchhoff problem without the Ambrosetti-Rabinowitz conditions*, Mathematics, 2020, 8, 1–13.

[21] J. Zhang and Z. Zhang, *Existence of nontrivial solution for a nonlocal problem with subcritical nonlinearity*, Adv. Difference Equations, 2018, 359(1), 1–8.