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A NEW EFFICIENT METHOD FOR TWO CLASSES OF SINGULAR CONVOLUTION INTEGRAL EQUATIONS OF NON-NORMAL TYPE WITH CAUCHY KERNEL

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Abstract In this article, our task is to study the existence and Noethericity of solution for two classes of singular convolution integral equations with Cauchy kernels in the non-normal type case. To obtain the conditions of solvability for such equations, we establish regularity theory of solvability. By means of the theory of Fourier analysis, we will transform the equations into boundary value problems for holomorphic functions. The holomorphic solutions and conditions of solvability are obtained by using the method of complex analysis in class $\{0\}$. Moreover, we also discuss the asymptotic property of solution near nodes. Therefore, our work generalizes and improves the theories of integral equations and the classical boundary value problems for holomorphic functions.

Keywords Singular convolution integral equations, boundary value problems for holomorphic functions, Wiener-Hopf equation, dual equations, Cauchy kernel, non-normal type.

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1. Introduction

As it is well known, boundary value problems of holomorphic functions and singular integral equations are developing rapidly and have a wide range of applications. They have been widely used in elastic theory, quantum mechanics, fluid dynamics, thin shell theory, fracture mechanics, electromagnetic wave diffraction, atmospheric radiative transport, neutron migration, cybernetics, prediction of stochastic processes, and population theory, etc. The range of applications continues to expand. Singular integral equations and boundary value problems of holomorphic functions have been studied extensively in the literature (see, for instance, [2,5,7,8,15–18,34] and there references) and formed a relatively systematic theoretical system. [33]considered the Noether theory of singular integral equations in the class of Hölder functions, and obtained the general solutions and conditions of solvability. [9] first began to study integral equations with discontinuous coefficients and convolution kernel. [32] studied the solution method for some basic classes of singular integral equations in the class $\{0\}$ and put forward the concepts of function classes of (0), $< 0 >$. He set up systematically the conditions of solvability and analytical solutions of the above basic equations. More recently, [19–23] investigated some classes of

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singular integral equations with convolution kernel, and gave the theory of Noether solvability and holomorphic solutions on the whole real axis X (or, the unit circle) in the case of the normal-type.

It is often found that some functions have the exponential order decreasing (or increasing) as the independent variable tends to ∞, therefore it is especially significant to the solution of exponent decreasing (or increasing) in the equations with convolution kernel. But in general case, the functions all contain the terms of different exponential estimation, thus different boundary value problems may be produced according to different exponential addends. So it is impossible to turn it into boundary value problems by applying Fourier transform directly. For such kind of problems, [24, 25] discussed the solutions for some basic kinds of singular convolution integral equations in the function class with exponent decreasing (or increasing). By introducing auxiliary functions and some generalized convolution operators, the equations are turned into some special kinds of Riemann boundary value problems with discontinuous coefficients on two parallel lines, and such boundary value problems produced are more important in the course of application. Motivated by the above researches, this paper is devoted to the study of the following two classes of singular integral equations with Cauchy kernel and convolution kernel in the cases of the non-normal type, which is simply called C-K equation:

(1) Dual singular integral equations

$$\begin{cases} a_1 f(t) + \dfrac{b_1}{\pi i}\displaystyle\int_{\mathbb{R}} \dfrac{f(\tau)}{\tau - t} d\tau + \dfrac{c_1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} k_1(t-\tau) f(\tau) d\tau \\ \quad + \dfrac{d_1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} h_1(t-\tau)\epsilon f(\tau) d\tau = g(t), \qquad t \in \mathbb{R}^+; \\ a_2 f(t) + \dfrac{b_2}{\pi i}\displaystyle\int_{\mathbb{R}} \dfrac{f(\tau)}{\tau - t} d\tau + \dfrac{c_2}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} k_2(t-\tau) f(\tau) d\tau \\ \quad + \dfrac{d_2}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} h_2(t-\tau)\epsilon f(\tau) d\tau = g(t), \qquad t \in \mathbb{R}^-, \end{cases} \tag{1.1}$$

where a_j, b_j, c_j, d_j $(j = 1, 2)$ are constants, $k_j(t), h_j(t), g(t) \in < 0 >$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^- = \mathbb{R} - \mathbb{R}^+$. Here, we also require that $|b_1| + |b_2| \neq 0$. For equation (1.1), we want to find its solution $f(t)$ such that $f \in \{0\}$.

(2) Wiener-Hopf singular integral equation

$$\begin{aligned} &af(t) + \frac{b}{\pi i}\int_{\mathbb{R}^+} \frac{f(\tau)}{\tau - t} d\tau + \frac{c}{\sqrt{2\pi}}\int_{\mathbb{R}^+} k(t-\tau) f(\tau) d\tau + \frac{d}{\sqrt{2\pi}}\int_{\mathbb{R}^+} h(t-\tau)\epsilon f(\tau) d\tau \\ =&g(t), \quad t \in \mathbb{R}^+, \end{aligned} \tag{1.2}$$

where a, b, c, d are constants with $b \neq 0$, $k, g \in (0)$, and an unknown function $f \in \{0\}$.

The notations mentioned above can be found in Section 2.

In the course of solving equations (1.1) and (1.2), we find that the classical Bekya regularization method used in [9, 15, 19–21, 33] is no longer suitable for the case of the non-normal type. It is difficult to use only the Fourier transform technique to study equations (1.1) and (1.2) in the case of non-normal type, thus we shall introduce a new method to complete our research. On the solutions of (1.1) and (1.2), we apply the theory of Fourier analysis, the classical boundary value problems for holomorphic functions, and the principle of analytic continuation to investigate their solvability. We give a novel and effective approach, which is different from the ones in the classical regularization method. By transforming the singular convolution integral equations with Cauchy kernel and constant coefficients to Riemann boundary value problem with discontinuity, we prove the uniqueness and existence of solution and obtain the holomorphic solutions. Moreover, we also discuss the theory of Noether solvability and the asymptotic property of solutions, and generalize the index formula of the classical boundary value problems for holomorphic functions. This article is very significant for the research of developing complex analysis, integral equations, and boundary value problems for holomorphic functions.

Our paper is constructed as follows. In Section 2 we introduce the function classes $\{0\}$ ($< 0 >$, (0)), $\{\{0\}\}$ ($<< 0 >>$, $((0))$), and study their properties. In Sections 3 and 4, we adopt the Fourier analysis approach to convert (1.1) and (1.2) into boundary value problems for holomorphic functions with discontinuous coefficients, and we use the generalized Liouville theorem and the theory of complex analysis to solving the obtained boundary value problems of holomorphic functions. We show that the equations are solvable under certain conditions. Finally, in Section 5, we give the conclusion of the paper.

2. Preliminaries

For convenience, we now give some definitions and lemmas. We firstly present the following Fourier transform operator $\mathbb{F}$ and the inverse transform operator $\mathbb{F}^{-1}$:

$$\begin{aligned}(\mathbb{F}f)(s) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp[ist]f(t)dt;\\ (\mathbb{F}^{-1}F)(t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp[-ist]F(s)ds, \quad \forall\ f \in L^2.\end{aligned} \tag{2.1}$$

The integrals that appear in (2.1) are the Cauchy principal values integrals. For simplification, we denote (2.1) as

$$(\mathbb{F}f)(s) = F(s), \quad (\mathbb{F}^{-1}F)(t) = f(t). \tag{2.2}$$

We denote by $\tilde{H}$ the set of all functions which satisfy the Hölder condition on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. If $F(s) \in \tilde{H}$, then $F(s)$ is a continuous function on $\mathbb{R}$.

In the following, we mainly introduce the concepts of classes $\{\{0\}\}$ ($((0))$, $\ll 0 \gg$) and $\{0\}$ ((0), $< 0 >$), and point out some of their properties.

Definition 2.1. If $F(s) \in \tilde{H} \cap L^2$, we say that $F(s) \in \{\{0\}\}$.

Obviously, $\{\{0\}\} \subset L^2 \cap H$, where H is the space of Hölder continuous functions.

Definition 2.2. A function $f(t) \in \{0\}$ if $F(s) \in \{\{0\}\}$, that is,

$$\{0\} = \{f(t) | F(s) = \mathbb{F}f(t) \in \{\{0\}\}\}. \tag{2.3}$$

Definition 2.3. If (1) $F(s) = O(|s|^{-\iota})$ ($\iota > \frac{1}{2}$) for a sufficiently large number $|s|$; (2) $F(s) \in \tilde{H}$, then we call $F(s) \in ((0))^{\iota}$ or $((0))$. If $F(s) \in ((0))^{\iota}$ or $((0))$, we say that $f(t) \in (0)^{\iota}$ or (0).

Definition 2.4. If (1) $F(s) \in H^{\iota}(N_{\infty})$ ($\iota > \frac{1}{2}$), and $F(\infty) = 0$; (2) $F(s) \in \tilde{H}$, then we call $F(s) \in \ll 0 \gg^{\iota}$ or $\ll 0 \gg$, where N_{∞} denotes the neighbourhood of ∞. If $F(s) \in \ll 0 \gg^{\iota}$ or $\ll 0 \gg$, then we denote $f(t) \in < 0 >^{\iota}$ or $< 0 >$.

It is clear that $<< 0 >> \subset ((0)) \subset \{\{0\}\}$, and $< 0 > \subset (0) \subset \{0\}$. Note that, in Definitions 2.3 and 2.4, we should require that $\iota > \frac{1}{2}$, and we give the following explanation.

Because $F(s) \in \tilde{H}$ and $F(\infty) = 0$, then there is a $M \in \mathbb{R}^+$ and a sufficiently large $|s|$ such that

$$|f(s)| \leq M|s|^{-\iota},$$

so we have

$$|f(s)|^2 \leq M^2|s|^{-2\iota}.$$

Since $f(s) \in L^2(\mathbb{R})$, that is,

$$\int_{\mathbb{R}} |f(s)|^2 ds < +\infty,$$

thus

$$\int_{\mathbb{R}} M^2|s|^{-2\iota} ds < +\infty.$$

From the previous discussion, we must have $\iota > \frac{1}{2}$.

Definition 2.5. Let $f(t) \in L^2(\mathbb{R})$, then the Hilbert transform operator ϵ of $f(t)$ is defined by

$$\epsilon f(t) = \text{P.V.}\frac{1}{\pi i}\int_{\mathbb{R}} f(\tau)\frac{d\tau}{\tau - t}, \quad t \in \mathbb{R}, \tag{2.4}$$

where P.V. stands for the Cauchy principle value. For convenience, (2.4) can also be simplified as

$$\epsilon f(t) = \frac{1}{\pi i}\int_{\mathbb{R}} f(\tau)\frac{d\tau}{\tau - t}, \quad t \in \mathbb{R}. \tag{2.5}$$

Sometimes we also call ϵ as the Cauchy integral operator. By [9, 32, 33] we can verify that

$$\epsilon^2 = I.$$

where I is a unit operator.

Definition 2.6. For any function $f(t)$, the operators S and T are introduced as follows

$$Sf(t) = f(-t), \quad Tf(t) = \text{sgn}(t)f(t), \quad t \in \mathbb{R}. \tag{2.6}$$

It is clear that

$$TS + ST = 0, \quad S^2 = I, \quad T^2 = I. \tag{2.7}$$

Definition 2.7. The convolution of two functions $f(t)$ and $g(t)$ is defined by the formula

$$f * g = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t-\tau)g(\tau)d\tau, \quad t \in \mathbb{R}. \tag{2.8}$$

Obviously

$$f * g = g * f.$$

Let

$$F = \mathbb{F}f, \quad G = \mathbb{F}g,$$

by the convolution theorem [9, 33] we known that

$$\mathbb{F}(f * g) = FG.$$

Lemmas 2.1 and 2.2 are obvious facts.

Lemma 2.1. *If $f, g \in \{0\}$, then $f * g \in \{0\}$; if $f \in \{0\}$ and $g \in (0)$, then $f * g \in (0)$.*

Similarly, if $f, g \in < 0 >$, then $f * g \in < 0 >$; if $f \in \{0\}$ and $g \in < 0 >$, then $f * g \in < 0 >$.

Lemma 2.2 (see [26, 36, 38]). *The operators $\mathbb{F}, \mathbb{F}^{-1}, \epsilon, S, T$ are as the above, then we have*

$$\epsilon = \mathbb{F}T\mathbb{F}^{-1}, \quad S\mathbb{F} = \mathbb{F}S = \mathbb{F}^{-1}. \tag{2.9}$$

Lemma 2.3. *For any $s, \tau \in \mathbb{R}$, we have*

$$P.V. \int_{\mathbb{R}} \frac{\exp[ist]}{\tau - t} dt = \begin{cases} -\pi i \exp[is\tau], & s > 0, \\ 0, & s = 0, \\ \pi i \exp[is\tau], & s < 0. \end{cases} \tag{2.10}$$

Proof. We let

$$\tau - t = y,$$

then in (2.10) we have

$$\begin{aligned} &\text{P.V.} \int_{\mathbb{R}} \frac{\exp[ist]}{\tau - t} dt = \text{P.V.} \int_{\mathbb{R}} \frac{\exp[is(\tau - y)]}{y} dy \\ =& \exp[is\tau]\text{P.V.} \int_{\mathbb{R}} \frac{cossy - isinsy}{y} dy = -2i \exp[is\tau]\text{P.V.} \int_{\mathbb{R}^+} \frac{sinsy}{y} dy. \end{aligned} \tag{2.11}$$

Since

$$\int_{\mathbb{R}^+} \frac{sinsy}{y} dy = \begin{cases} \frac{\pi}{2}, & s > 0, \\ 0, & s = 0, \\ -\frac{\pi}{2}, & s < 0, \end{cases} \tag{2.12}$$

thus from (2.11), we know that (2.10) holds.

Moreover, by the generalized Liouville theorem [32, 33], we can also prove (2.10). □

For convenience, from now on we shall omit the symbol P.V..

The following Lemma 2.4 is very important, and we use it to prove some of the results in this paper.

Lemma 2.4. *Let $f \in \{0\}, F(s) = \mathbb{F}f(t)$, then we have*

$$\mathbb{F}\epsilon f = -TF. \tag{2.13}$$

Proof. Note that

$$\mathbb{F}[\epsilon f(t)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [\frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - t} d\tau] \exp[ist] dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [\frac{1}{\pi i} \int_{\mathbb{R}} \frac{\exp(ist)}{\tau - t} dt] f(\tau) d\tau. \tag{2.14}$$

According to Lemma 2.3, we get

$$\frac{1}{\pi i} \int_{\mathbb{R}} \frac{\exp[ist]}{\tau - t} dt = -\mathrm{sgn}(s) \exp[is\tau], \quad s, \tau \in \mathbb{R}. \tag{2.15}$$

From (2.14) and (2.15), we have

$$\mathbb{F}[\epsilon f(t)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-\mathrm{sgn}(s)) f(\tau) \exp[is\tau] d\tau$$
$$=(-\mathrm{sgn}(s)) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\tau) \exp[is\tau] d\tau = -\mathrm{sgn}(s) F(s).$$

This completes the proof. □

From $f \in \{0\}$, (0) or $< 0 >$, generally we could not assure that ϵf belongs to the same class. However, we have the following lemma 2.5.

Lemma 2.5. If $f \in \{0\}$, (0) or $< 0 >$, and $\mathbb{F}f(0) = 0$, then $\epsilon f \in \{0\}$, (0) or $< 0 >$.

Proof. Since $f \in \{0\}$, we know that $F = \mathbb{F}f \in \{\{0\}\}$. By using definition of $\{\{0\}\}$ and

$$\mathbb{F}f(\infty) = 0, \quad \mathbb{F}f(0) = 0, \tag{2.16}$$

we can prove

$$\int_{\mathbb{R}} |F(s)\mathrm{sgn}(s)|^2 ds < +\infty,$$

and

$$F(s)\mathrm{sgn}(s) \in \tilde{H},$$

thus $F(s)\mathrm{sgn}(s) \in \{\{0\}\}$.

It follows from Lemma 2.4 that we get $\mathbb{F}\epsilon f \in \{\{0\}\}$, therefore $\epsilon f \in \{0\}$. Similarly, we can also prove $\epsilon f \in (0), < 0 >$.

The proof is complete. □

Remark 2.1. In Lemma 2.5, it is easy to see that $\mathbb{F}f(0) = 0$ is necessary, otherwise Lemma 2.5 is invalid. It follows from [27, 40] and Lemma 2.5 that ϵ maps $\{0\}$ and $< 0 >$ into themselves respectively.

In the following Sections 3 and 4, we shall study theory of Noether solvability and methods of solution for two classes of singular integral equations of convolution type (that is, dual equations and Wiener-Hopf equation) in the case of the non-normal type.

3. Solvability of dual equations (1.1)

In this section, we focus on dual singular integral equations (1.1). To do this, we reduce equation (1.1) to the following system

$$\begin{cases} a_1 f(t) + b_1 \epsilon f(t) + c_1 k_1 * f(t) + d_1 h_1 * \epsilon f(t) = g(t), & t \in \mathbb{R}^+; \\ a_2 f(t) + b_2 \epsilon f(t) + c_2 k_2 * f(t) + d_2 h_2 * \epsilon f(t) = g(t), & t \in \mathbb{R}^-. \end{cases} \tag{3.1}$$

Without loss of generality, we assume $a_1 b_2 \neq a_2 b_1$. Extending t in the first equation of (3.1) to $t \in \mathbb{R}^-$, and in the second one of (3.1) to $t \in \mathbb{R}^+$, that is, we add $\psi_-(t)$ and $\psi_+(t)$ to (3.1), then equation (3.1) can be rewritten as

$$\begin{cases} a_1 f(t) + b_1 \epsilon f(t) + c_1 k_1 * f(t) + d_1 h_1 * \epsilon f(t) = g(t) + \psi_-(t); \\ a_2 f(t) + b_2 \epsilon f(t) + c_2 k_2 * f(t) + d_2 h_2 * \epsilon f(t) = g(t) + \psi_+(t), \end{cases} \quad t \in \mathbb{R}. \tag{3.2}$$

where $\psi(t) \in \{0\}$ is an undetermined function and

$$\psi_\pm(t) = \frac{1}{2}\psi(t)(\operatorname{sgn}(t) \pm 1),$$

obviously,

$$\psi(t) = \psi_+(t) - \psi_-(t).$$

Due to Lemmas 2.4 and 2.5, we use the Fourier transforms to equation (3.2) and get

$$\begin{cases} Y_1(s)F(s) = G(s) + \Psi^-(s); \\ Y_2(s)F(s) = G(s) + \Psi^+(s), \end{cases} \quad s \in \mathbb{R}, \tag{3.3}$$

where

$$Y_j(s) = a_j - b_j \operatorname{sgn}(s) + c_j K_j(s) - d_j \operatorname{sgn}(s) H_j(s), \ j = 1, 2,$$

and

$$F(s) = \mathbb{F} f(t), \ G(s) = \mathbb{F} g(t), \ \Psi^\pm(s) = \mathbb{F}\psi_\pm(t), \ K_j(s) = \mathbb{F} k_j(t),$$
$$H_j(s) = \mathbb{F} h_j(t), \ j = 1, 2.$$

By eliminating $F(s)$ in (3.3), it gives rise to

$$\frac{1}{Y_2(s)}[G(s) + \Psi^+(s)] = \frac{1}{Y_1(s)}[G(s) + \Psi^-(s)]. \tag{3.4}$$

We denote

$$E(s) = \frac{Y_2(s)}{Y_1(s)}, \quad \Omega(s) = [\frac{Y_2(s)}{Y_1(s)} - 1]G(s),$$

so we can reduce equation (3.4) to the following boundary value problem for holomorphic function:

$$\Psi^+(s) = E(s)\Psi^-(s) + \Omega(s), \tag{3.5}$$

where $\Psi^\pm(s)$ are the boundary values of the following Cauchy singular integral

$$\Psi(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\Psi(s)}{s - z} ds, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^-. \tag{3.6}$$

Note that, using the residue theorem [33], we know that (3.6) is true. And we can verify that $\Psi^\pm(s)$ are also the one-sided Fourier transforms of $\psi(t)$:

$$\Psi^+(s) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^+} \psi(t)\exp[ist]dt;$$
$$\Psi^-(s) = -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^-} \psi(t)\exp[ist]dt.$$

It is clear that

$$\Psi^+(s) - \Psi^-(s) = \Psi(s).$$

Thus, we should only solve problem (3.5) in place of equation (1.1). Without loss of generality, we assume that

$$b_1 \neq 0, \quad a_1 \pm b_1 \neq 0. \tag{3.7}$$

Let $Y_1(s)$ have some zero-points $e_1, e_2, \cdots, e_n$ with the orders $\xi_1, \xi_2, \cdots, \xi_n$ respectively, and $Y_2(s)$ have some zero-points $d_1, d_2, \cdots, d_q$ with the orders $\eta_1, \eta_2, \cdots, \eta_q$ respectively, where ξ_j, η_j are the non-negative integers. In this case, we say that (3.5) is the boundary value problems of non-normal type.

We put

$$\Pi_1(s) = \Pi_{j=1}^n (s-e_j)^{\xi_j}, \quad \Pi_2(s) = \Pi_{j=1}^q (s-d_j)^{\eta_j}.$$

Hence, we can rewrite the problem (3.5) in the form

$$\Psi^+(s) = \frac{\Pi_2(s)}{\Pi_1(s)} D(s)\Psi^-(s) + \Omega(s), \quad s \in \mathbb{R}, \tag{3.8}$$

where

$$E(s) = \frac{\Pi_2(s)}{\Pi_1(s)} D(s), \text{ and } D(s) \neq 0, \ \forall s \in \mathbb{R}.$$

Note that, if $Y_j(s)$ $(j=1,2)$ are non-vanishing functions on $\mathbb{R}$, then (3.5) is called a boundary value problem of normal type, and this is the special case mentioned above, that is, $\Pi_1(s) = 1, \Pi_2(s) = 1$. For this case, the detailed discussion will be omitted also (see, for instance, [16, 18, 20, 21] and there references).

From (3.7), in view of the values of $a_2 \pm b_2$, we have the following two cases.

Case (1): if $|a_2+b_2| + |a_2-b_2| \neq 0$, then (3.5) is a boundary value problem for holomorphic function with nodes $s = 0, \infty$.

Case (2): if $|a_2+b_2| + |a_2-b_2| = 0$ (that is, $a_2 = b_2 = 0$), then (3.5) is a boundary value problem for holomorphic function with node $s = 0$.

Here, we only consider case (1), and case (2) can be discussed similarly. Since $f(t) \in \{0\}$, so $F(s) = \mathbb{F}f(t) \in \{\{0\}\}$. We know that $F(s)$ is continuous at $s = 0$, hence it is necessary $\Psi^\pm(s)$ are continuous at $s = 0$ and

$$G(0) + \Psi^\pm(0) = 0.$$

Returning to (3.3), we must have $F(0) = 0$ and

$$\Psi^+(+0) = \Psi^+(-0).$$

From the previous discussions, we take the limits as $s \to 0$ in (3.3), thus we get

$$(\mathbb{F}g)(0) = 0, \ \textit{i.e.}, \ G(0) = 0. \tag{3.9}$$

Moreover, we require that the solutions of (3.5) should be at least continuous along the whole real axis X and $\Psi(\infty) = 0$. Denote by

$$\frac{1}{2\pi i}\{\log|\frac{D(+0)}{D(-0)}| + i \arg \frac{D(+0)}{D(-0)}\} = \gamma_0 = \alpha_0 + i\beta_0.$$

If we take the integer $\mu = [\alpha_0]$, then we call μ the index of (3.8). Denote $\alpha = \alpha_0 - \mu$, then we know that $0 \leq \alpha < 1$. Again let

$$\gamma_0 - \mu = \alpha + i\beta_0 = \gamma. \tag{3.10}$$

Next we investigate the solvability of (3.8). First, we define the following (sectionally) holomorphic function:

$$X(z)=\begin{cases}(z+i)^{-\mu}\exp\{\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log D_0(s)}{s-z}ds\}, & z\in\mathbb{C}^+;\\ (z-i)^{-\mu}\exp\{\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log D_0(s)}{s-z}ds\}, & z\in\mathbb{C}^-,\end{cases} \tag{3.11}$$

where in (3.11) we have set

$$D_0(s)=(\frac{s+i}{s-i})^{\mu}D(s),$$

that is,

$$\log D_0(s)=\mu\log\frac{s+i}{s-i}+\log D(s), \tag{3.12}$$

here we have taken the definite branch of (3.12) such that

$$\log D_0(\pm 0)=\pm\mu\pi i+\log D(\pm 0).$$

In addition, the branches of (3.12) can also be obtained by the following way

$$\lim_{s\to\infty}\log\frac{s+i}{s-i}=0.$$

It is easy to verify that $X(z)$ is a canonical function of (3.8). By using Sokhotski–Plemelj formula [10, 13] to $X(z)$ in (3.11), we have

$$(s+i)^{\mu}X^+(s)=(s-i)^{\mu}D_0(s)X^-(s). \tag{3.13}$$

To solve (3.8), we need to construct the following function:

$$\Phi(z)=\begin{cases}(z+i)^{n_1}\Psi(z), & z\in\mathbb{C}^+;\\ (z-i)^{n_2}\Psi(z), & z\in\mathbb{C}^-,\end{cases} \tag{3.14}$$

where

$$n_1=\sum_{j=1}^{n}\xi_j,\quad n_2=\sum_{j=1}^{q}\eta_j.$$

Thus, (3.8) can be transformed into the following boundary value problems for holomorphic functions in the case of non-normal type:

$$\Phi^+(s)=\frac{\Pi_2(s)(s+i)^{n_1}X^+(s)}{\Pi_1(s)(s-i)^{n_2}X^-(s)}\Phi^-(s)+(s+i)^{n_1}\Omega(s). \tag{3.15}$$

For the sake of simplicity, we only solve (3.15) in the problem R_0. And by assumptions, $\Phi(z)$ should take the finite value at $z=\infty$. We again put

$$\frac{1}{2\pi i}\{\log|\frac{D(+\infty)}{D(-\infty)}|+i\arg\frac{D(+\infty)}{D(-\infty)}\}=\gamma_\infty=\alpha_\infty+i\beta_\infty.$$

Here we have taken the definite branch of $\log D(s)$ such that it is continuous at $s=\infty$, and we require that $0\le\alpha_\infty<1$. It follows from $a_1b_2\ne a_2b_1$ that $\gamma_\infty\ne 0$.

We first consider the following homogeneous problem of (3.15) given by

$$\Pi_1(s)(s-i)^{n_2}X^-(s)\Phi^+(s)=\Pi_2(s)(s+i)^{n_1}X^+(s)\Phi^-(s). \tag{3.16}$$

To do this, we consider the following function

$$\Xi(z)=\begin{cases}\dfrac{\Phi(z)}{X(z)\Pi_2(z)(z+i)^{n_1}}, & z\in\mathbb{C}^+;\\ \dfrac{\Phi(z)}{X(z)\Pi_1(z)(z-i)^{n_2}}, & z\in\mathbb{C}^-,\end{cases} \tag{3.17}$$

thus $\Xi(z)$ are holomorphic in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively. From (3.16) we know that $\Xi(z)$ is holomorphic on $\mathbb{C}$, and it is a polynomial of the degree $\mu-n_1-n_2$. By using the generalized Liouville theorem and the principle of analytic continuation [18,22], we obtain the holomorphic solutions of (3.16) as follows:

$$\Phi_*(z)=\begin{cases}X(z)\Pi_2(z)(z+i)^{n_1}P_{\mu-n_1-n_2}(z), & z\in\mathbb{C}^+;\\ X(z)\Pi_1(z)(z-i)^{n_2}P_{\mu-n_1-n_2}(z), & z\in\mathbb{C}^-,\end{cases} \tag{3.18}$$

in (3.18), when $\mu-n_1-n_2\geq 0$, $P_{\mu-n_1-n_2}(z)$ is a polynomial of the degree $\mu-n_1-n_2$ with arbitrary complex coefficients; when $\mu-n_1-n_2<0$, we have $P_{\mu-n_1-n_2}(z)\equiv 0$.

Now we solve the non-homogeneous problem (3.15). We define the following function

$$\varpi(z)=\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\Pi_1(s)\Omega(s)}{X^+(s)}\frac{ds}{s-z},\quad \forall z\in\mathbb{C}^+\cup\mathbb{C}^-.$$

According to Sokhotski–Plemelj formula and the generalized Liouville theorem, we can get the holomorphic solutions for the problem (3.15), which may have the singularity at e_j and d_k. Thus, to solve (3.15), we construct a Hermite interpolation polynomial $W_\varrho(z)$ $(\varrho=n_1+n_2-1)$ with the degree ϱ, i.e.,

$$W_\varrho(z)=B_0z^\varrho+B_1z^{\varrho-1}+\ldots+B_{\varrho-1}z+B_\varrho,$$

which has some zero-points of the orders ξ_j,η_k $(1\leq j\leq n, 1\leq k\leq q)$ at e_j,d_k respectively, where $B_t\in\mathbb{C}$ $(0\leq t\leq\varrho)$. Since $\Phi(z)$ is bounded on $\mathbb{R}$, thus $W_\varrho(z)$ is required to fulfill the following conditions:

$$\begin{aligned}\frac{d^r[\varpi(z)(z+i)^\mu]}{dz^r}|_{z=d_j}&=\frac{d^r[W_\varrho(z)]}{dz^r}|_{z=d_j};\\ \frac{d^p[\varpi(z)(z+i)^\mu]}{dz^p}|_{z=e_j}&=\frac{d^p[W_\varrho(z)]}{dz^p}|_{z=e_j},\end{aligned} \tag{3.19}$$

for any $r=1,2,\ldots,\eta_k-1,\ k=1,2,\ldots q;\ p=1,2,\ldots,\xi_j-1,\ j=1,2,\ldots,n.$

Similar to the discussion in [16–18], we need to define the following function:

$$Y(z)=\begin{cases}\dfrac{X(z)(z+i)^{n_1}[\varpi(z)(z+i)^\mu-W_\varrho(z)]}{\Pi_1(z)}, & z\in\mathbb{C}^+;\\ \dfrac{X(z)(z-i)^{n_2}[\varpi(z)(z+i)^\mu-W_\varrho(z)]}{\Pi_2(z)}, & z\in\mathbb{C}^-.\end{cases} \tag{3.20}$$

By means of the boundary value problems for holomorphic functions and of a system of linear algebraic equations, we can prove that $Y(z)$ is the particular solution of (3.15). In view of linearity, we obtain the holomorphic solution of (3.15) as follows

$$\Phi(z) = Y(z) + \Phi_*(z), \tag{3.21}$$

and we can also write $\Phi(z)$ as the explicit solution:

$$\Phi(z)=\begin{cases} \dfrac{X(z)(z+i)^{n_1}}{\Pi_1(z)}[\varpi(z)(z+i)^{\mu} - W_\varrho(z) + \Pi_1(z)\Pi_2(z)P_{\mu-n_1-n_2}(z)], \ z\in\mathbb{C}^+; \\ \dfrac{X(z)(z-i)^{n_2}}{\Pi_2(z)}[\varpi(z)(z+i)^{\mu} - W_\varrho(z) + \Pi_1(z)\Pi_2(z)P_{\mu-n_1-n_2}(z)], \ z\in\mathbb{C}^-. \end{cases} \tag{3.22}$$

Now we investigate the asymptotic property of solutions for the problem (3.15). First, we discuss the behaviors of solution near $s = 0$. Since $\Phi(s)$ is continuous at $s = 0$, by [13, 28, 29] we can obtain

$$X^{\pm}(s) = \exp\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log D_0(t)}{t-z}dt\}(E_0(s))^{\pm\frac{1}{2}}. \tag{3.23}$$

If $s = 0$ is an ordinary node, that is, $0 < \alpha < 1$, then we know that γ given by (3.10) is a non-integer, so

$$\gamma \neq 0, \quad \exp[\gamma\pi i] \neq 1. \tag{3.24}$$

It is easy to verify that, in the neighbourhood of $s = 0$,

$$\begin{aligned} \Phi^+(+0) &= \frac{i^{n_1-1}\exp[-\gamma\pi i]}{2\sin\gamma\pi}[\Omega(+0) - \exp(-3\gamma\pi i)\Omega(-0)]; \\ \Phi^+(-0) &= \frac{i^{n_1-1}\exp[-2\gamma\pi i]}{2\sin\gamma\pi}[\Omega(+0) - \exp(-3\gamma\pi i)\Omega(-0)]. \end{aligned} \tag{3.25}$$

From

$$\Phi^+(+0) = \Phi^+(-0)$$

it follows that

$$\Omega(+0) = \exp[-3\gamma\pi i]\Omega(-0). \tag{3.26}$$

If $s = 0$ is a special node, that is, $\alpha = 0$, then we require that (3.9) as well as the following condition of solvability are satisfied:

$$\frac{i^{\mu-1}}{2\pi\Pi_1(0)\Pi_2(0)}\int_{\mathbb{R}} \frac{\Pi_1(s)\Omega(s)}{X^+(s)}\frac{ds}{s} = \frac{c_0}{\Pi_1(0)\Pi_2(0)} - b_0, \tag{3.27}$$

where b_0, c_0 are the constant terms of $P_{\mu-n_1-n_2}(z)$, $W_\varrho(z)$, respectively.

Next, we investigate the asymptotic property and conditions of solvability at $s = \infty$. It is easy to prove that, near $s = \infty$,

$$X(s) = s^{-\alpha_\infty}\chi(s) \ (s\to\infty), \tag{3.28}$$

where $\chi(s) \in H(N_\infty)$, i.e., $\chi(s)$ satisfies the Hölder condition in the neighbourhood N_∞ of ∞.

If $s = \infty$ is an ordinary node, then $0 < \alpha_\infty < 1$, $\gamma_\infty \neq 0$. From (3.9) and $\Omega(s) \in \tilde{H}$, we have $\varpi(s) \in \tilde{H}$, so, when $\frac{1}{2} < \alpha_\infty < \iota < 1$, we obtain the asymptotic property of solution at $s = \infty$,

$$X^+(s)\varpi(s) = o(s^{-\alpha_\infty}) \ (s\to\infty); \tag{3.29}$$

when $\frac{1}{2} < \iota \leq \alpha_\infty < 1$, by [11, 18, 25] we get

$$X^+(s)\varpi(s) = O(s^{-\alpha_\infty+\varepsilon})\ (s \to \infty), \tag{3.30}$$

where $\varepsilon > 0$ is arbitrarily small such that $\alpha_\infty - \varepsilon > \frac{1}{2}$. Again denote

$$A(s) = [\Pi_2(s)P_{\mu-n_1-n_2}(s) - \frac{W_\varrho(s)}{\Pi_1(s)}]X^+(s)(s+i)^{-\mu+n_1}. \tag{3.31}$$

Therefore, when $\mu - n_1 - n_2 \geq 0$, we have

$$A(s) = O(s^{-\alpha_\infty})\ (s \to \infty); \tag{3.32}$$

and when $\mu - n_1 - n_2 < 0$, in order guarantee that $\Phi(z)$ is bounded at $z = \infty$, (3.19) should be changed to

$$B_0 = B_1 = \ldots = B_{n_1+n_2-\mu-1} = 0. \tag{3.33}$$

Thus, when $\alpha_\infty > \frac{1}{2}$, we have

$$\Phi^+(s) = o(s^{-v})\ (s \to \infty), \tag{3.34}$$

where $v > \min\{\iota, \alpha_\infty - \varepsilon\}$; when $\alpha_\infty \leq \frac{1}{2}$, the discussions may be made fully analogous to those in [12, 26, 37].

If $s = \infty$ is a special node, then $\alpha_\infty = 0$ but $\gamma_\infty \neq 0$, one can translate it into the case that $\alpha_\infty < \frac{1}{2}$. Similar arguments can be used (see, e.g., [10, 17, 26, 29]). Thus, when $\mu > 0$, we obtain

$$A(s) = o(s^{-\alpha_\infty})\ (s \to \infty), \tag{3.35}$$

in this case, (3.15) has a solution; when $\mu < 0$, since $z = -i$ is a singular point of $\Phi(z)$, in order to eliminate the singularity, we must have

$$\int_{\mathbb{R}} \frac{\Pi_1(s)\Omega(s)}{X^+(s)(s+i)^t} ds = 0, \quad \forall\ t = 1, 2, \ldots, |\mu|; \tag{3.36}$$

and when $\mu = 0$, the following conditions (3.37) are fulfilled:

$$\begin{aligned} &(d_k + i)\varpi(d_k) = b_0; \\ &(e_j + i)\varpi(e_j) = c_0, \end{aligned} \tag{3.37}$$

for any $k = 1, 2, \ldots, q;\ j = 1, 2, \ldots, n$.

Moreover, since $\Phi(z) \in \{\{0\}\}$, thus $\Phi(z)$ is continuous at d_k, e_j $(k = 1, 2, \ldots q; j = 1, 2, \ldots, n)$, then we also have the following conditions of solvability

$$\begin{aligned} &\int_{\mathbb{R}} \frac{\Pi_1(s)\Omega(s)[X^+(s)]^{-1}}{(s-d_k)^r} ds = 0; \\ &\int_{\mathbb{R}} \frac{\Pi_1(s)\Omega(s)[X^+(s)]^{-1}}{(s-e_j)^p} ds = 0, \end{aligned} \tag{3.38}$$

for any $r = 1, 2, \ldots, \eta_k,\ k = 1, 2, \ldots q;\ p = 1, 2, \ldots, \xi_j,\ j = 1, 2, \ldots, n$.

In conclusion, we have the following results.

Theorem 3.1. *Under suppositions $b_1 \neq 0$ and $a_1 \pm b_1 \neq 0$, in the case of non-normal type, the necessary condition of solvability to equation (3.1) is (3.9) in class $\{0\}$.*

(1) When $\mu - n_1 - n_2 > 0$, (3.1) has $\mu - n_1 - n_2$ linearly independent solutions; when $\mu - n_1 - n_2 \leq 0$, (3.1) has the unique solution.

(2) Let $s = 0$ be an ordinary node, one requires that (3.26) holds; let $s = 0$ be a special node, when (3.9) and (3.27) hold, (3.1) has a solution.

(3) Let $s = \infty$ be an ordinary node, if $\alpha_\infty > \frac{1}{2}$, then (3.29), (3.30), and (3.34) hold. When $\mu - n_1 - n_2 \geq 0$, (3.32) holds, then (3.1) has $\mu - n_1 - n_2$ linearly independent solutions; when $\mu - n_1 - n_2 < 0$, (3.33) holds. If $\alpha_\infty \leq \frac{1}{2}$, the only difference lies in that, we write $P_{\mu-n_1-n_2-1}(s)$ instead of $P_{\mu-n_1-n_2}(s)$ in (3.22), then (3.1) has $\mu - n_1 - n_2 - 1$ linearly independent solutions.

Let $s = \infty$ be a special node, when $\mu > 0$, (3.35) holds; when $\mu < 0$, (3.36) holds; when $\mu = 0$, (3.37) holds. Moreover, the solvable condition (3.38) should also be supplemented, then (3.1) has a unique solution.

Assume that (2) and (3) are fulfilled, then (3.1) has a general solution

$$f(t) = \mathbb{F}^{-1}F(s), \tag{3.39}$$

where $F(s)$ is given by (3.3). It is also obvious that the solution $f \in \{0\}$, actually $f \in < 0 >$.

Remark 3.1. In (3.1), if $a_1 b_2 = a_2 b_1$, we know that γ_∞ and γ may be zero, then in which cases the analysis is even simpler. Further discussions will be omitted also.

4. Solvability of Wiener-Hopf equation (1.2)

The method used in Section 3 is applicable to solving Wiener-Hopf type singular integral equation. After simplification, equation (1.2) may be written as

$$af_+(t) + b\epsilon f_+(t) + ck * f_+(t) + dh * \epsilon f_+(t) = g(t), \quad t \in \mathbb{R}^+. \tag{4.1}$$

In order to give a solution of equation (4.1), we need to extend (4.1) to $t \in \mathbb{R}^-$, that is, the right-hand side of (4.1) is augmented with $f_-(t)$, so (4.1) can be written as the following form

$$af_+(t) + b\epsilon f_+(t) + ck * f_+(t) + dh * \epsilon f_+(t) = g(t) + f_-(t), \quad t \in \mathbb{R}. \tag{4.2}$$

Applying the Fourier transforms to (4.2), and by Lemmas 2.4 and 2.5, we may get

$$F^+(s) = \frac{1}{a - b\mathrm{sgn}(s) + cK(s) - d\mathrm{sgn}(s)H(s)}[F^-(s) + G(s)], \quad s \in \mathbb{R}, \tag{4.3}$$

where

$$F^\pm(s) = \mathbb{F}f_\pm(t),\ G(s) = \mathbb{F}g(t),\ K(s) = \mathbb{F}k(t),\ H(s) = \mathbb{F}h(t).$$

So we easily find that (4.3) is equivalent to (4.2). Denote

$$E(s) = \frac{1}{a - b\mathrm{sgn}(s) + cK(s) - d\mathrm{sgn}(s)H(s)}.$$

Now we again give the definitions of $\gamma_\infty, \gamma_0, \mu$ as follows

$$\gamma_\infty = \alpha_\infty + i\beta_\infty = \frac{1}{2\pi i}\log\frac{E(-\infty)}{E(+\infty)} = \frac{1}{2\pi i}\log\frac{a-b}{a+b}. \tag{4.4}$$

Note that $\log E(s)$ is taken to be continuous branch such that it is continuous at $s = \infty$ and $0 \leq \alpha_\infty < 1$. It is not difficult to prove that $\gamma_\infty \neq 0$ since $b \neq 0$. We also have

$$\gamma_0 = \alpha_0 + i\beta_0 = \frac{1}{2\pi i}\log\frac{E(-0)}{E(+0)} = \frac{1}{2\pi i}\log\frac{a-b+cK(0)-dH(0)}{a+b+cK(0)+dH(0)}. \tag{4.5}$$

Then choose the integer $\mu = [\alpha_0]$, we say that μ is the index of (4.3). We denote γ, α as in (3.10). It follows from $F^+(\infty) = 0$ that $F^-(\infty) = F(\infty) = 0$. Assume that $(E(s))^{-1}$ has some zero-points $w_1, w_2, \cdots, w_t$ with the orders $\phi_1, \phi_2, \cdots, \phi_t$ respectively, then (4.3) can also be written as

$$F^+(s) = \frac{(s+i)^n}{V(s)}E_1(s)F^-(s) + E(s)G(s), \tag{4.6}$$

where

$$\sum_{j=1}^{t}\phi_j = n, \quad V(s) = \Pi_{j=1}^{t}(s-w_j)^{\phi_j},$$

and

$$E(s) = \frac{(s+i)^n}{V(s)}E_1(s), \quad E_1(s) \neq 0, \quad \forall s \in \mathbb{R}.$$

Without loss of generality, we only discuss the case

$$a \pm b \neq 0.$$

Other cases can be discussed similarly. In this case, (4.6) is also a boundary value problem for holomorphic function with nodes $s = 0, \infty$. It is easy to see that (4.1) and (4.6) have the same solutions. By means of the classical Riemann-Hilbert boundary value approach and the generalized Liouville theorem, we can obtain the general solution of (4.6).

Firstly, we consider the homogeneous problem of (4.6) (that is, $G(s) \equiv 0$):

$$F^+(s) = \frac{(s+i)^n}{V(s)}E_1(s)F^-(s). \tag{4.7}$$

So we again use the results obtained in [15, 32, 33] and Section 3, we can get a general solution of (4.7) as follows:

$$F_1(z) = \begin{cases} X(z)(z+i)^n P_{\mu-n-1}(z), & z \in \mathbb{C}^+, \\ X(z)V(z)P_{\mu-n-1}(z), & z \in \mathbb{C}^-. \end{cases} \tag{4.8}$$

Similarly, according to the extended residue theorem [33], we can verify that $F_2(z)$ given by (4.9) is the particular solution of (4.6):

$$F_2(z) = \begin{cases} \dfrac{X(z)(z+i)^{n-\mu}}{V(z)}[\Upsilon(z)(z+i)^\mu - W_\varrho(z)], & z \in \mathbb{C}^+, \\ \dfrac{X(z)}{(z+i)^\mu}[\Upsilon(z)(z+i)^\mu - W_\varrho(z)], & z \in \mathbb{C}^-, \end{cases} \tag{4.9}$$

in which we have put

$$\Upsilon(z) = \frac{z+i}{2\pi i} \int_{\mathbb{R}} \frac{V(s)G(s)E(s)ds}{X^+(s)(s+i)^{n+1}(s-z)}, \quad \forall z \in \mathbb{C}^+ \cup \mathbb{C}^-, \tag{4.10}$$

here $P_{\mu-n-1}(z)$ is a polynomial of the degree $\mu-n-1(\geq 0)$, which contains arbitrary complex coefficients, and $W_\varrho(z)$ is a Hermite interpolation polynomial with the degree $n-1$. Note that $X(z)$ given by (3.11) is the canonical function of (4.6) and satisfies

$$X^+(s) = E_1(s)X^-(s).$$

Thus, by means of the solvability theory of linear equations, we obtain the general solution of (4.6) as follows

$$F(z) = F_1(z) + F_2(z). \tag{4.11}$$

Therefore, using the results in [5,15,33], we get the following conclusions (1)-(3).

(1) If s=0 is an ordinary node, then (3.9) must be fulfilled.

(2) If $s = 0$ is a special node, under (3.9) and the following condition of solvability

$$b_0 = \frac{d_0 - \Upsilon(0)i^\mu}{V(0)}, \tag{4.12}$$

(4.6) has a solution.

(3) At the node $s = \infty$, if the following conditions (4.13) and (4.14) are also fulfilled:

$$\int_{\mathbb{R}} \frac{V(s)G(s)}{X^+(s)(s+i)^n(s-w_k)^r} ds = 0, \quad r = 1, 2, \ldots, \alpha_k; \ k = 1, 2, \ldots, t, \tag{4.13}$$

and

$$\int_{\mathbb{R}} \frac{V(s)G(s)E(s)}{X^+(s)(s+i)^{n+j}} ds = 0, \quad j = 1, 2, \ldots, |\mu|, \tag{4.14}$$

then all the results as stated in Section 3 remain true.

From the above analysis, we now formulate the main results about the solvability of equation (4.1) in the case of the non-normal type.

Theorem 4.1. *Under suppositions $a \pm b \neq 0$, (3.9) is the necessary condition of solvability to equation (4.1) in class $\{0\}$. Assume that (4.12)-(4.14) are satisfied, then (4.1) has the general solution given by*

$$f(t) = \mathbb{F}^{-1}F(s),$$

where $F(s)$ is given by (4.11), and $f(t) \in \{0\}$ (or (0)).

Remark 4.1. In equations (1.1) and (1.2), if $k_1, k_2, g \in <0>$ (or $k \in (0)$), then $f \in <0>$ (or $f \in (0)$). Similarly, if $k_1, k_2, g \in <0>^\iota$ (or $k \in (0)^\iota$), then $f \in <0>^\iota$ (or $f \in (0)^\iota$), where $0 < \iota < 1$.

5. Conclusions

In this article, we dealt with two classes of singular convolution integral equations with Cauchy kernel in the case of non-normal type. By using the boundary-value problems of holomorphic functions and method of complex analysis, we obtain the expression of solution and conditions of solvability for two kinds of equations in some classes of functions. On the other hand, we can study the stability of solutions for equations (1.1) and (1.2), and we can also consider the solvability of equations (1.1) and (1.2) in Clifford analysis (see [1,3,4,6,11,12,14,30,31,35,37,39]).

Moreover, in recent years, there appear the singular integral equations with the order of a singular point higher than the special dimension in the fields of aeromechanics, electron optics, fracture mechanics and others. Some mathematicians have made a series of useful research and have obtained many important achievements. Therefore, we need further discuss and study these problems.

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