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NONEXISTENCE OF STABLE SOLUTIONS OF THE WEIGHTED LANE-EMDEN SYSTEM

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Abstract The aim of this paper is to study the stability of the positive solutions of the weighted Lane-Emden system. By applying the structure of the m-biharmonic weighted equation, we prove the nonexistence of positive stable solutions for the case $0 < p < 1 < p^{-1} < \theta$.

Keywords Stable solutions, Lane-Emden system, $m-$biharmonic equation, Nonexistence.

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1. Introduction

The famous Lane-Emden equation

$$-\Delta u = u^p, \quad x \in \mathbb{R}^N \tag{1.1}$$

has played the important role in the development of nonlinear analysis in last decades.

It is well known that, the stable solutions of the corresponding Lane-Emden equation and system, or the biharmonic equation (see [13]), which have been widely studied by many experts. For the corresponding second order equation

$$-\Delta u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

Farina [8] obtained the optimal Liouville type result for solutions stable at infinity. Indeed, he proved that a smooth nontrival solution to (1.2) exists, if $1 < p < p_{JL}$ and $N \geqslant 2$. Here p_{JL} stands for Joseph-Lundgren exponent (see [12]). However, Farina's technique may fail to do completely classify the stable solutions and finite Morse index solutions of the biharmonic equation($p > 1$)

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

To solve this problem, Dávila, Dupaigne, Wang and Wei [5] give a complete classification of stable and finite Morse index solutions (whether positive or sign changing), in the full exponent range. They derive a monotone formula for solutions of equation (1.3) by using Pohozaev identity and simplify the problem to the non-existence of stable homogeneous solutions.

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Equation(1.1) and (1.3) are special cases of the Lane-Emden system

$$\begin{cases} -\Delta u = v^p, \ \text{ in } \mathbb{R}^N, \\ -\Delta v = u^\theta, \ \text{ in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

With $p, \theta > 0$ among which, the problem of existence and nonexistence (known as the Liouville theorem) of stable solutions has attracted wide attention [10,11], but has not yet fully answered.

The famous Lane-Emden conjecture says that the system(1.4) admit no positive classical solutions in subcritical case

$$\frac{1}{p+1} + \frac{1}{\theta+1} > \frac{N-2}{N}. \tag{1.5}$$

Moreover, we can check readily that if $p\theta > 1$,

$$(1.5) \Leftrightarrow N < 2 + \alpha + \beta = \frac{2(p+1)(\theta+1)}{p\theta - 1} \tag{1.6}$$

where $\alpha = \frac{2(p+1)}{p\theta-1}, \quad \beta = \frac{2(\theta+1)}{p\theta-1}$.

In [19] this system admits no radial solutions in dimensions $N \leq 2$. Souplet [21] proved the Lane-Emden conjecture in dimension $N = 3$. Cowan [2] proved the non-existence, up to translations, of the stability (radial or not) if $p, \theta \geqslant 2$ and $N \leqslant 10$. Moreover Chen, Dupaigne and Ghergu [4] showed the stability of radial solutions when $p, \theta \geqslant 1$. They proved that if $p, \theta \geqslant 1$, then a radial solution is unstable if and only if $N \leqslant 10$, or $N \geqslant 11$ and

$$\left[\frac{(N-2)^2 - (\alpha-\beta)^2}{4}\right]^2 < p\theta\alpha\beta(N-2-\alpha)(N-2-\beta).$$

Recently, Hu [15] drived Liouville type theorem for the weighted Lane-Emden system

$$\begin{cases} -\Delta u = |x|^\beta v, \ \text{ in } \mathbb{R}^N, \\ -\Delta v = |x|^\alpha u^p, \ \text{ in } \mathbb{R}^N. \end{cases}$$

by using Pohozaev identity to construct a monotonicity formula and revealing their certain equivalence relation.

In general weighted Lane-Emden equations

$$\begin{cases} -\Delta u = \rho v^p, u > 0, \ \text{ in } \mathbb{R}^N, \\ -\Delta v = \rho u^\theta, v > 0, \ \text{ in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

for $\rho = (1+|x|^2)^{\frac{\alpha}{2}}$ with $p, \theta > 1$ in (1.7) was considered, Liouville type theorems for the classical positive stable solutions in higher space dimension was established by the authors [3,7]. In 2017, Hajlaoui, Harrabi and Mtiri [14] improved the previous works [9,16] and mainly obtained a new comparison property which is key to deal with the case $1 < \theta \leqslant \frac{4}{3}$.

As a generalization, the weighted elliptic system with the advection term is generally of the form

$$\begin{cases} -\omega\Delta u - \nabla\omega\cdot\nabla u = \omega_1 v^p, \text{ in } \mathbb{R}^N, \\ -\omega\Delta v - \nabla\omega\cdot\nabla v = \omega_2 u^\theta, \text{ in } \mathbb{R}^N, \end{cases} \tag{1.8}$$

has also been studied recently, see [17] and the references there in for example. For $\omega = (1+|x|^2)^{\frac{\delta}{2}}, \omega_1 = (1+|x|^2)^{\frac{\alpha}{2}}$ and $\omega_2 = (1+|x|^2)^{\frac{\beta}{2}}$ with $\alpha, \beta, \delta > 0$ and $N \geqslant 3,\ p \geqslant \theta > 1$, Hu [17] proved the non-existence of the stable solution.

For the weighted Lane-Emden system, the Liouville type results are less understood for $0 < p < 1$.

In this paper we are going to study the stbale solutions of the weighted Lane-Emden system:

$$\begin{cases} -\Delta u = v^p, u > 0, \text{ in } \mathbb{R}^N, \\ -\Delta v = \rho u^\theta, v > 0, \text{ in } \mathbb{R}^N, \end{cases} \tag{1.9}$$

where $0 < p < 1$ and $\rho : \mathbb{R}^N \to \mathbb{R}$ is a radial continuous function satisfying the following assumption: $(*)$ There exists $\alpha \geqslant 0$ and $A > 0$ such that $\rho(x) \geqslant A\rho_0(x)$ in $\mathbb{R}^N$, where $\rho_0(x) = (1+|x|^2)^{\frac{\alpha}{2}}$.

The main results of this paper are stated as

Theorem 1.1. *If* $0 < p < 1 < p^{-1} < \theta$ *satisfies* (1.5)*, then* (1.9) *has no stable solution.*

2. Proof of the main result

In this section we will prove the stable solution of the weighted

$$\begin{cases} -\Delta u = \rho v^p, u > 0, \text{ in } \mathbb{R}^N, \\ -\Delta v = \rho u^\theta, v > 0, \text{ in } \mathbb{R}^N, \end{cases}$$

we will use the equivalence between $m-$biharmonic weighted equation and the system to prove the stable solution for such system.

To introduce the notion of stability, we consider a general system given by

$$-\Delta u = f(x,v), \quad -\Delta v = g(x,u) \text{ in } \Omega, \text{ a bounded regular domain in } \mathbb{R}^N, \tag{2.1}$$

where $f, g \in C^1(\Omega \times \mathbb{R})$. Following Montenegro [19], a smooth solution (u,v) of (2.1) is said to be stable in Ω if the following eigenvalue problem

$$-\Delta\xi = f_v(x,v)\zeta + \eta\xi, \quad -\Delta\zeta = g_u(x,u)\xi + \eta\zeta, \text{ in } \Omega$$

has a nonnegative eigenvalue η, with a positive smooth eigenfunctions pair(ξ,ζ).

Let (u,v) be a solution of system (1.9) with $\theta > p^{-1} > 1 > p > 0$. Our approach is based on the formal equivalence noticed in [1,6,22], between the weighted Lane-Emden system (1.9) and a fourth order problem, called the m-biharmonic weighted equation. More precisely, let $m := \dfrac{1}{p} + 1 > 2$, as $v = (-\Delta u)^{m-1} > 0$ in $\mathbb{R}^N$, $(-\Delta u)^{m-1} = (-\Delta u)^{m-2}(-\Delta u) > 0$, recall

$$\begin{cases} -\Delta u = v^p > 0, \quad u > 0, \text{ in } \mathbb{R}^N, \\ -\Delta v = \rho u^\theta > 0, \quad v > 0, \text{ in } \mathbb{R}^N. \end{cases}$$

So $0 < (-\Delta u)^{m-2} = |\Delta u|^{m-2}$, we derive that u satisfies

$$\Delta_m^2 u := \Delta(|\Delta u|^{m-2}\Delta u) = \rho u^\theta \qquad \text{in } \mathbb{R}^N.$$

So we are led to consider $\theta > m - 1 > 1$, and

$$\Delta_m^2 u := \Delta(|\Delta u|^{m-2}\Delta u) = \rho |u|^{\theta-1} u. \tag{2.2}$$

We say that $u \in W^{2,m}_{loc}(\mathbb{R}^N) \cap L^{\theta+1}_{loc}(\mathbb{R}^N)$ is a weak solutions of (2.2) in $\mathbb{R}^N$, if for any regular bounded domain Ω, u is a critical point of the followig functional

$$I(v) = \frac{1}{m}\int_\Omega |\Delta v|^m dx - \frac{1}{\theta+1}\int_\Omega \rho |v|^{\theta+1} dx, \quad \forall v \in W^{2,m}(\Omega) \cap L^{\theta+1}(\Omega).$$

Naturally, a weak solution to (2.2) is said stable in $mathbbR^N$, if

$$\Lambda_u(h) := (m-1)\int_\Omega |\Delta v|^{m-2}|\Delta h|^2 dx - \theta \int_\Omega \rho |u|^{\theta-1} h^2 \geqslant 0, \quad \forall h \in C_c^2(\Omega). \tag{2.3}$$

Next, we will use the relationship between the stability for the $m-$biharmonic weighted equation and the stability for the system (1.9) to handle the case $0 < p < 1$. In fact, a direct calculation yields that if $p\theta > 1$ (or equivalently $\theta > m - 1$),

$$(1.6) \Leftrightarrow N < \frac{2m(\theta+1)}{\theta - (m-1)} \Leftrightarrow \theta < \frac{N(m-1) + 2m}{N - 2m}. \tag{2.4}$$

It means that the range of pairs (p, θ) satisfying (1.5) and $p\theta > 1$ corresponds exactly to the subcritical case of the $m-$biharmonic weighted equarion (2.2).

Lemma 2.1. *Let (u, v) be a solution of system (1.9) with $\theta > \dfrac{1}{p} := m - 1 > 1$. Suppose that (u, v) is stable in a regular bounded domain Ω , then u is a stable solution of equation (2.2) .*

Proof. By the definition of stability, there exist smooth positive functions ξ, ζ and $\eta \geq 0$ such that

$$-\Delta \xi = p v^{p-1}\zeta + \eta\xi, \qquad -\Delta\zeta = \theta\rho u^{\theta-1} + \eta\zeta. \quad \text{in } \Omega$$

Using (ξ, ζ) as super-solution, $(\min_{\overline{\Omega}} \xi, \ \min_{\overline{\Omega}} \zeta)$ as sub-solution, and the standard monotone iterations, we can claim that there exist positive smooth functions φ, χ verifying

$$-\Delta\varphi = p v^{p-1}\chi, \qquad -\Delta\chi = \theta\rho u^{\theta-1}\varphi \quad \text{in } \ \Omega.$$

Therefore, we have

$$\theta\rho u^{\theta-1}\varphi = \Delta\Big(\frac{v^{1-p}}{p}\Delta\varphi\Big) \quad \text{in } \ \Omega.$$

Let $\gamma \in C_c^2(\Omega)$. Multiplying the above equation by $\gamma^2\varphi^{-1}$ and integrating by parts, there holds

$$\begin{aligned}
\int_\Omega \theta\rho u^{\theta-1}\gamma^2 dx &= \int_\Omega \gamma^2\varphi^{-1}\Delta(\frac{1}{p}v^{1-p}\Delta\varphi) \\
&= -\int_\Omega \nabla(\gamma^2\varphi^{-1})\nabla(\frac{1}{p}v^{1-p}\Delta\varphi) + \int_{\partial\Omega}\gamma^2\varphi^{-1}\frac{\partial(\frac{1}{p}v^{1-p}\Delta\varphi)}{\partial \boldsymbol{n}}dS(x) \\
&= \frac{1}{p}\int_\Omega v^{1-p}\Delta\varphi\Delta(\gamma^2\varphi^{-1})dx - \int_{\partial\Omega}\frac{1}{p}v^{1-p}\Delta\varphi\frac{\partial(\gamma^2\varphi^{-1})}{\partial \boldsymbol{n}}dS(x) \\
&\quad + \int_{\partial\Omega}\gamma^2\varphi^{-1}\frac{\partial(\frac{1}{p}v^{1-p}\Delta\varphi)}{\partial \boldsymbol{n}}dS(x) \\
&= \frac{1}{p}\int_\Omega v^{1-p}\Delta\varphi\Delta(\gamma^2\varphi^{-1})dx \\
&= \frac{1}{p}\int_\Omega v^{1-p}\Delta\varphi\nabla(2\gamma\varphi^{-1}\nabla\gamma - \gamma^2\varphi^{-2}\nabla\varphi)dx \\
&= \frac{1}{p}\int_\Omega -4v^{1-p}\Delta\varphi\gamma\varphi^{-2}\nabla\varphi\cdot\nabla\gamma dx + \frac{1}{p}\int_\Omega 2v^{1-p}\Delta\varphi\varphi^{-1}|\nabla\gamma|^2dx \\
&\quad + \frac{1}{p}\int_\Omega 2v^{1-p}\Delta\varphi\gamma\varphi^{-1}\Delta\gamma dx + \frac{1}{p}\int_\Omega 2v^{1-p}\Delta\varphi\gamma^2\varphi^{-3}|\nabla\varphi|^2dx \\
&\quad - \frac{1}{p}\int_\Omega v^{1-p}\Delta\varphi\gamma^2\varphi^{-2}\Delta\varphi dx. \qquad (2.5)
\end{aligned}$$

Using Cauchy-Schwarz's inequality and the fact that $-\Delta\varphi > 0$, we get

$$\begin{aligned}
|-\frac{4}{p}\int_\Omega v^{1-p}\gamma\varphi^{-2}\Delta\varphi\nabla\varphi\cdot\nabla\gamma dx| &\le -\frac{2}{p}\int_\Omega v^{1-p}\Delta\varphi|\nabla\gamma|^2\varphi^{-1}dx \\
&\quad - \frac{2}{p}\int_\Omega v^{1-p}\Delta\varphi\gamma^2|\nabla\varphi|^2\varphi^{-3}dx.
\end{aligned} \qquad (2.6)$$

Combining (2.5) and (2.6), one obtains, using again the Cauchy-Schwarz's inequality,

$$\begin{aligned}
\int_\Omega \theta\rho u^{\theta-1}\gamma^2 &\le \frac{2}{p}\int_\Omega v^{1-p}\Delta\varphi\gamma\Delta\gamma\varphi^{-1}dx - \frac{1}{p}\int_\Omega v^{1-p}(\Delta\varphi)^2\varphi^{-2}\gamma^2dx \\
&\le \frac{1}{p}\int_\Omega v^{1-p}\Delta\varphi(\Delta\varphi)^2\varphi^{-2}\gamma^2dx + \frac{1}{p}\int_\Omega v^{1-p}(\Delta\gamma)^2dx \\
&\quad - \frac{1}{p}\int_\Omega v^{1-p}\Delta\varphi(\Delta\varphi)^2\varphi^{-2}\gamma^2dx \\
&= \frac{1}{p}\int_\Omega v^{1-p}(\Delta\gamma)^2dx.
\end{aligned}$$

Recall that $p = \frac{1}{m-1}$ and $(-\Delta u)^{\frac{1}{p}} = v$, we obtain the desired result (2.3). □

According to the above lemma, we know that system (1.9) is equivalent to the equation (2.2). Therefore, to prove them 1.1 in the case $p \in (0,1)$ and $p\theta > 1$, we need only to prove

Lemma 2.2. *Let $\theta > m-1 > 1$, if u is a weak stable solution to the equation* (2.2) *in $\mathbb{R}^N$ with N verifying* (2.4)*, then $u \equiv 0$.*

To lemma 2.2, we use first the stability condition (2.3) to get the following crucial lemma which provides an important integral estimate for, u and Δu.

Lemma 2.3. *Let $u \in W^{2,m}_{loc}(\Omega) \cap L^{\theta+1}_{loc}(\Omega)$ be a weak stable solution of (2.2) in Ω, with $\theta > m-1 > 1$. Then, for any integer*

$$k \geqslant \max\left(m, \frac{m(\theta+1)}{2(\theta+1-m)}\right),$$

there exists a positive constant $C = C(N,\epsilon,m,k)$such that for any $\zeta \in C^2_c(\Omega)$ satisfying $0 \leqslant \zeta \leqslant 1$,

$$\int_\Omega |\Delta u|^m \zeta^{4k} dx + \int_\Omega \rho |u|^{\theta+1}\zeta^{4k} dx \leqslant C\left[\int_\Omega (|\Delta\zeta|^m + |\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)^{\frac{\theta+1}{\theta-(m-1)}} dx\right]. \tag{2.7}$$

Proof. For any $\epsilon \in (0,1)$ and, $\eta \in C^2(\Omega)$, there holds

$$\begin{aligned}
&\int_\Omega |\Delta u|^{m-2}[\Delta(u\eta)]^2 dx \\
=&\int_\Omega |\Delta u|^{m-2}(u\Delta\eta + 2\nabla u \nabla\eta + \eta\Delta u)^2 dx \\
\leqslant&(1+\epsilon)\int_\Omega |\Delta u|^m \eta^2 dx + \frac{C}{\epsilon}\int_\Omega |\Delta u|^{m-2}(u^2|\Delta\eta|^2 + |\nabla u|^2|\nabla\eta|^2) dx.
\end{aligned} \tag{2.8}$$

Take $\eta = \zeta^{2k}$, $\zeta \in C^2_c(\Omega)$, $0 \leqslant \zeta \leqslant 1, k \geqslant m > 2$. Apply Young's inequality, we get

$$\begin{aligned}
&\int_\Omega |u|^2|\Delta u|^{m-2}|\Delta(\zeta^{2k})|^2 dx \\
\leqslant& C_k\int_\Omega |u|^2|\Delta u|^{m-2}(|\Delta\zeta|^2 + |\nabla\zeta|^4)\zeta^{4k-4} dx \\
\leqslant&\epsilon^2\int_\Omega |\Delta u|^m\zeta^{4k} dx + C_{\epsilon,k,m}\int_\Omega |u|^m(|\Delta\zeta|^2 + |\nabla\zeta|^4)\zeta^{4k-2m} dx.
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\Delta u|^{m-2}|\nabla u|^2|\nabla(\zeta^{2k})|^2 dx =& 4k^2\int_\Omega |\Delta u|^{m-2}|\nabla u|^2|\nabla\zeta|^2\zeta^{4k-2} dx \\
\leqslant& \epsilon^2\int_\Omega |\Delta u|^m\zeta^{4k} dx + \frac{C_{m,k}}{\epsilon^{m-2}}\int_\Omega |\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m} dx.
\end{aligned}$$

Inserting the two above estimates into (2.8), we arrive at

$$\begin{aligned}
\int_\Omega |\Delta u|^{m-2}|\Delta(u\zeta^{2k})|^2 dx \leqslant& (1+C_\epsilon)\int_\Omega |\Delta u|^m\zeta^{4k} dx + \frac{C_{m,k}}{\epsilon^{m-2}}\int_\Omega |\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m} dx \\
&+ C_{\epsilon,k,m}\int_\Omega |u|^m(|\Delta\zeta|^2 + |\nabla\zeta|^4)^{\frac{m}{2}}\zeta^{4k-2m} dx.
\end{aligned} \tag{2.9}$$

We need also the following technical lemma [18].

Lemma 2.4. *Let $k \geqslant \dfrac{m}{2} > 1$ and $\epsilon > 0$, there exists $C_{N,\epsilon,m,k} > 0$ such that for any $u \in W^{2,m}_{loc}(\Omega)$ verifying (2.3), $\zeta \in C^\infty_c(\Omega)$ with $0 \leqslant \zeta \leqslant 1$, there holds*

$$\int_\Omega \frac{|\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m}}{\rho} dx \leqslant \epsilon\int_\Omega \frac{|\Delta u|^m\zeta^{4k}}{\rho} dx \tag{2.10}$$

$$+ C_{N,\epsilon,m,k}\int_{\Omega}\frac{|u|^m(|\nabla\zeta|^{2m}+|\nabla^2\zeta|^m)\zeta^{4k-2m}}{\rho}dx. \quad (2.11)$$

Proof. A direct calculation gives

$$\begin{aligned}\int_{\Omega}\frac{|\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx =&\int_{\Omega}\frac{\nabla u\cdot\nabla u|\nabla u|^{m-2}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx\\ =&-\int_{\Omega}div(\nabla u|\nabla u|^{m-2})u\frac{|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx\\ &-\int_{\Omega}u|\nabla u|^{m-2}\nabla u\cdot\nabla(\frac{|\nabla\zeta|^m\zeta^{4k-m}}{\rho})dx\\ :=&I_1+I_2.\end{aligned} \quad (2.12)$$

The integral I_1 can be estimated as

$$\begin{aligned}I_1=&-(m-2)\int_{\Omega}\frac{u|\nabla u|^{m-4}|\nabla\zeta|^m\nabla^2u(\nabla u,\nabla u)\zeta^{4k-m}}{\rho}dx\\ &-\int_{\Omega}\frac{u\Delta u|\nabla u|^{m-2}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx-\int_{\Omega}\frac{u\nabla u|\nabla u|^{m-2}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx\\ \leqslant&C_m\int_{\Omega}\frac{|u||\nabla^2u||\nabla u|^{m-2}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx+\int_{\Omega}\frac{|u||\Delta u||\nabla u|^{m-2}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx\\ &+\int_{\Omega}\frac{|u||\nabla u|^{m-1}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx.\end{aligned} \quad (2.13)$$

Applying Young's inequality, there holds, for any $\epsilon>0$,

$$\begin{aligned}&\int_{\Omega}\frac{|u||\Delta u||\nabla u|^{m-2}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx\\ \leqslant&C_{\epsilon,m}\int_{\Omega}\frac{|u|^{\frac{m}{2}}|\Delta u|^{\frac{m}{2}}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx+\epsilon\int_{\Omega}\frac{|\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx\\ \leqslant&C_{\epsilon,m}\int_{\Omega}\frac{|u|^m|\nabla\zeta|^{2m}\zeta^{4k-2m}}{\rho}dx+\epsilon\int_{\Omega}\frac{|\Delta u|^m\zeta^{4k}}{\rho}dx+\epsilon\int_{\Omega}\frac{|\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx.\end{aligned} \quad (2.14)$$

On the other hand,

$$\begin{aligned}&\int_{\Omega}\frac{|u||\nabla^2u||\nabla u|^{m-2}|\nabla\zeta|^{m-2+2}\zeta^{4k-m}}{\rho}dx\\ \leqslant&C_{\epsilon,m}\int_{\Omega}\frac{|u|^{\frac{m}{2}}|\nabla^2u|^{\frac{m}{2}}|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx+\epsilon\int_{\Omega}\frac{|\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx\\ \leqslant&C_{\epsilon,m}\int_{\Omega}v\frac{|u|^m|\nabla\zeta|^{2m}\zeta^{4k-2m}}{\rho}dx+\epsilon\int_{\Omega}\frac{|\nabla^2u|^m\zeta^{4k}}{\rho}dx+\epsilon\int_{\Omega}\frac{|\nabla u|^m|\nabla\zeta|^m\zeta^{4k-m}}{\rho}dx.\end{aligned} \quad (2.15)$$

Now we shall estimate the integral

$$\int_{\Omega}\frac{|\nabla^2u|^m\zeta^{4k}}{\rho}dx.$$

Remark that there exists $C_0N, m > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla^2 \varphi|^m dx \leqslant C_0N, m \int_{\mathbb{R}^N} |\Delta\varphi|^m dx, \quad \forall \varphi \in W^{2,m}(\mathbb{R}^N). \tag{2.16}$$

$$\begin{aligned}\int_\Omega \frac{|\nabla^2(u\zeta^{\frac{4k}{m}})|^m}{\rho} dx \leqslant C_0(N,m) \int_\Omega \frac{|\Delta(u\zeta^{\frac{4k}{m}})|^m}{\rho} dx \\ \leqslant C \int_\Omega \frac{|\Delta u|^m \zeta^{4k}}{\rho} dx + C \int_\Omega \frac{|\nabla u|^m |\nabla\zeta|^m \zeta^{4k-m}}{\rho} dx \\ + C \int_\Omega \frac{|u|^m(|\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)\zeta^{4k-2m}}{\rho} dx.\end{aligned} \tag{2.17}$$

So

$$\begin{aligned} I_1 \leqslant & C \int_\Omega \frac{|\Delta u|^m \zeta^{4k}}{\rho} dx + C \int_\Omega \frac{|\nabla u|^m |\nabla\zeta|^m \zeta^{4k-m}}{\rho} dx \\ & + C\int_\Omega \frac{|u|^m(|\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)\zeta^{4k-2m}}{\rho} dx + \int_\Omega \frac{|u||\nabla u|^{m-1}|\nabla\zeta|^m \zeta^{4k-m}}{\rho} dx. \end{aligned} \tag{2.18}$$

$$I_2 = -m \tag{2.19}$$

□

Using lemma 2.4 with ϵ^m and (2.9), we see that

$$\begin{aligned}\int_\Omega |\Delta u|^{m-2} ||\Delta(u\zeta^{2k})|^2 dx \leqslant & C_{N,\epsilon,m,k} \int_\Omega |u|^m (|\Delta\zeta|^m + |\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)\zeta^{4k-2m} dx \\ & + (1 + C_{m,k,\epsilon}) \int_\Omega |\Delta u|^m \zeta^{4k} dx.\end{aligned} \tag{2.20}$$

Thanks to the approximation argument, the stability property (2.3) holds true with $u\zeta^{2k}$. We deduce then, for any $\epsilon > 0$, there exists $C_{N,\epsilon,m,k} > 0$ such that

$$\begin{aligned} & \theta \int_\Omega \rho |u|^{\theta+1} \zeta^{4k} dx - (m-1)(1 + C_{m,k,\epsilon}) \int_\Omega |\Delta u|^m \zeta^{4k} dx \\ \leqslant & C_{N,\epsilon,m,k} \int_\Omega |u|^m (|\Delta\zeta|^m + |\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)\zeta^{4k-2m} dx. \end{aligned} \tag{2.21}$$

Moreover, multiplying the equation (2.2) by $u\zeta^{4k}$ and integrating by parts, there holds

$$\begin{aligned} & \int_\Omega |\Delta u|^m \zeta^{4k} dx - \int_\Omega \rho(x) u^{\theta+1} \zeta^{4k} \\ \leqslant & \int_\Omega |u||\Delta u|^{m-1} |\Delta(\zeta^{4k})| dx + C \int_\Omega |\Delta u|^{m-1} |\nabla u||\nabla(\zeta)^{4k}| dx. \end{aligned}$$

Using Young's inequality and applying again lemma 2.4, we can deduce that for any $\epsilon > 0$, there exists $C_{N,\epsilon,m,k} > 0$ such that

$$\begin{aligned} & (1 - C_{m,k,\epsilon}) \int_\Omega |\Delta u|^m \zeta^{4k} dx - \int_\Omega \rho |u|^{\theta+1} \zeta^{4k} dx \\ \leqslant & C_{N,\epsilon,k,m} \int_\Omega |u|^m (|\Delta\zeta|^m + |\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)\zeta^{4k-2m} dx. \end{aligned} \tag{2.22}$$

Taking $\epsilon > 0$ but small enough, multiplying (2.22) by $\dfrac{(m-1)(1+2C_{m,k,\epsilon})}{1-C_{m,k,\epsilon}}$, adding it with (2.21), we get

$$(m-1)C_{m,k,\epsilon}\int_\Omega |\Delta u|^m\zeta^{4k}dx + \big[\theta - \frac{(m-1)(1+2C_{m,k,\epsilon})}{1-C_{m,k,\epsilon}}\big]\int_\Omega \rho|u|^{\theta+1}\zeta^{4k}dx$$
$$\leqslant C_{N,\epsilon,k,m}\int_\Omega |u|^m(|\Delta\zeta|^m + |\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)\zeta^{4k-2m}dx.$$

As $\theta > m-1 > 1$, using $\epsilon > 0$ small enough, we have

$$\int_\Omega |\Delta u|^m\zeta^{4k}dx + \int_\Omega \rho|u|^{\theta+1}\zeta^{4k}dx \leqslant C\int_\Omega |u|^m(|\Delta\zeta|^m + v|\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)\zeta^{4k-2m}dx. \tag{2.23}$$

For $k \geqslant \dfrac{m(\theta+1)}{2(\theta+1-m)}$ so that $4km \leqslant (4k-2m)(\theta+1)$, applying Hölder inequality, we conclude then

$$\int_\Omega |\Delta u|^m\zeta^{4k}dx + \int_\Omega \rho|u|^{\theta+1}\zeta^{4k}dx$$
$$\leqslant C\Big[\int_\Omega (|\Delta\zeta|^m + |\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)^{\frac{\theta+1}{\theta-(m-1)}}dx\Big]^{\frac{\theta-(m-1)}{\theta+1}}\Big(\int_\Omega \rho|u|^{\theta+1}\zeta^{\frac{(4k-2m)(\theta+1)}{m}}dx\Big)^{\frac{m}{\theta+1}}$$
$$\leqslant C\Big[\int_\Omega (|\Delta\zeta|^m + |\nabla\zeta|^{2m} + |\nabla^2\zeta|^m)^{\frac{\theta+1}{\theta-(m-1)}}dx\Big]^{\frac{\theta-(m-1)}{\theta+1}}\Big(\int_\Omega \rho|u|^{\theta+1}\zeta^{4k}dx\Big)^{\frac{m}{\theta+1}}.$$

We get readily the estimate (2.7). □

Now we choose ϕ_0 a cut-off function in $C_c^\infty(B_2)$ verifying $0 \leqslant \phi_0 \leqslant 1$, and $\phi_0 \equiv 1$ in B_1. Applying (2.7) with $\zeta = \phi_0(R^{-1}x)$ for $R > 0$, there holds

$$\int_{B_R} \rho|u|^{\theta+1}dx \leqslant \int_{R^N} \rho|u|^{\theta+1}\zeta^{4k}dx \leqslant CR^{N-\frac{2m(\theta+1)}{\theta-(m-1)}}.$$

Under the assumption (2.4), tending $R \to \infty$, we obtain $u \equiv 0$, we prove then lemma 2.2, hence the case $\theta p > 1 > p > 0$ for theorem 1.1. □

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