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BIFURCATIONS OF DOUBLE HETERODIMENSIONAL CYCLES WITH THREE SADDLE POINTS

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Abstract In this paper, bifurcations of double heterodimensional cycles of an "∞" shape consisting of two saddles of (1,2) type and one saddle of (2,1) type are studied in three dimensional vector field. We discuss the gaps between returning points in transverse sections by establishing a local active coordinate system in the tubular neighborhood of unperturbed double heterodimensional cycles, through which the preservation of "∞"-shape double heterodimensional cycles is proved. We then get the existence of a new heteroclinic cycle consisting of two saddles of (1,2) type and one saddle of (2,1) type, which is composed of one big orbit linking p_1, p_3 and two orbits linking p_3, p_2 and p_2, p_1 respectively, and another heterodimensional cycle consisting of one saddle p_1 of (2,1) type and one saddle p_2 of (1,2) type, which is composed of one orbit starting from p_1 to p_2 and another orbit starting from p_2 to p_1. Moreover, the 1-fold and 2-fold large 1-heteroclinic cycle consisting of two saddles p_1 and p_3 of (1,2) type is also presented. As well as the coexistence of a 1-fold large 1-heteroclinic cycle and the "∞"-shape double heterodimensional cycles and the coexistence conditions are also given in the parameter space.

Keywords Double heterodimensional cycles, heteroclinic bifurcation, bifurcation theory, Poincaré map.

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1. Introduction

Homoclinic and heteroclinic bifurcation is one of the dominant themes in nonlinear dynamical system and has been extensively studied, see [9, 11, 12, 22, 31, 40, 46, 48]. Many effective methods for bifurcation study have been established like the singular perturbation theory [45], the alternative method [37], the Melnikov method [4], the invariant manifold theory [1], the variational method [33] and the blowing up method [38], etc.

However, most of the works about bifurcation of heteroclinic cycles were concerned with the equidimensional cycles. It is known that, in equidimensional heteroclinic cycles with many saddles, the unstable manifolds of all saddles have a same dimension, the bifurcation results can be extended to the case of m-point heteroclinic cycles for $m \geq 3$ from the 3-point case, which has been confirmed by

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Jin and Zhu [15]. Whereas, the heterodimensional cycle with m saddle points for $m \geq 3$, leads to the distributing asymmetry of the codimensions of this cycle. In [1973], Newhouse and Palis originally analyzed heteroclinic cycles with unequal dimension in discrete dynamical systems, which were called heterodimensional cycles, and found that heterodimensional cycles were more general in practical problems contrasted to equidimensional cycles. [8] demonstrated that a heterodimensional cycle could be produced from a heteroclinic cycle connecting non-hyperbolic equilibria when it underwent a transcritical bifurcation. [2] deduced that diffeomorphisms revealing either a homoclinic tangency or a heterodimensional cycle were C^1-dense in the complement of the C^1-closure of a hyperbolic system.

Since then, many scholars have turned to study heterodimensional cycles in continuous systems (see [5, 7, 24, 36]). We know that the unequal dimension of the unstable manifolds of singular orbits may lead to the distributing asymmetry of the codimensions of the cycle, so there are more challenges in studying the bifurcation of heterodimensional cycles than in the equidimensional cycles. Moreover, it is difficult to take the bifurcations of m points heterodimensional cycles with $m > 3$ as a direct extension of the equidimensional case. Zhu et al. [28, 41] discussed heterodimensional cycles bifurcations with two saddle points under orbit-flip and inclination flip, respectively, and derived the conditions for the existence and uniqueness of different orbits types in a four-dimensional system. Liu et al. [23] studied the generic 2-2-1 heterodimensional cycles connecting to three saddles and showed some new bifurcation behaviors different from the well-known equidimensional cycles. After this, they continually considered the bifurcations of heterodimensional cycles containing two saddles in three-dimensional vector fields in [26] and got that the perturbation system did not have any homoclinic orbits coexisting with the persistent heterodimensional cycle and gave an example to show the existence of a heterodimensional cycle. [2] and [25] then analyzed the generic and nongeneric bifurcations of heterodimensional cycles with two saddles of four dimensional nonlinear systems, respectively. Heterodimensional cycles can also be found in solitary wave problems and biology systems, see [29] for example.

Notice that the heterodimensional cycles studied in [23] connected three saddles, two of which had two-dimensional unstable manifolds and one of which had one-dimensional unstable manifold. In fact, there are lots of unsolved problems of bifurcation of heterodimensional cycles, especially in the case of the cycles with m saddles for $m > 2$. To well carry out the research in the paper, we fix the number of saddles $m = 3$ and suppose the heterodimensional cycles are in the shape of "∞", which is called a double heterodimensional cycle.

As to the "∞"-type cycles, Zhang [44] concerned a double homoclinic loops with resonance characteristic roots in a four-dimensional system and got a complete bifurcation diagram under some conditions. Jin et al studied the bifurcation problems of double homoclinic loops with resonant condition for higher dimensional systems and obtained the existence, number and existence regions of the small homoclinic loops, small periodic orbits, and the large homoclinic loops, large periodic orbits, respectively (see [19]). In order to form an "∞"-shaped double heterodimensional cycle, the manifold of the middle saddle point of the three needs to be divided into left and right parts. The left part forms a heterodimensional cycle with the first saddle point, and the right part forms another heterodimensional cycle with the third saddle point. Based on it, we set $e_1^- = -e_4^-$ and $e_3^+ = -e_2^+$ (the symbols are defined in the later paper).

Take a C^r system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \quad (\mathbf{x}, \mu) \in \mathbf{R}^3 \times \mathbf{R}^l, \tag{1.1}$$

when $\mu = 0$, the unperturbed system associated with (1.1) is

$$\dot{\mathbf{x}} = f(\mathbf{x}, 0), \quad \mathbf{x} \in \mathbf{R}^3, \tag{1.2}$$

where $f(\mathbf{x}, \mu)$ is sufficiently smooth with respect to the phase variable $\mathbf{x}$ and the parameter μ, $l \geq 4$, $r \geq 4$, $0 < |\mu| \ll 1$. To further expand our discussion, we make the following five hypotheses.

($\mathbf{A_1}$) System (1.1) has three hyperbolic equilibria $p_i, i = 1, 2, 3$. $W^s_{p_i}$ and $W^u_{p_i}$ are the C^r stable and unstable manifolds of p_i, respectively. Moreover, the spectra of system (1.1) are

$$\begin{aligned}
&\sigma\left(D_{\mathbf{x}} f(p_i, \mu)\right) = \left\{-\rho_i^1(\mu), \lambda_i^1(\mu), \lambda_i^2(\mu)\right\}, \quad i = 1, 3, \\
&\sigma\left(D_{\mathbf{x}} f(p_2, \mu)\right) = \left\{-\rho_2^1(\mu), -\rho_2^2(\mu), \lambda_2^1(\mu)\right\},
\end{aligned}$$

with

$$\begin{aligned}
&-\rho_i^1(\mu) < 0 < \lambda_i^1(\mu) < \lambda_i^2(\mu), \\
&-\rho_2^2(\mu) < -\rho_2^1(\mu) < 0 < \lambda_2^1(\mu), \quad \lambda_k^1(\mu) < \rho_k^1(\mu), (k = 1, 2, 3)
\end{aligned}$$

and for notational convenience, we use $\lambda_i^1 = \lambda_i^1(0)$, $\rho_i^1 = \rho_i^1(0)$, $\lambda_1^2 = \lambda_1^2(0)$, $\lambda_3^2 = \lambda_3^2(0)$, $\rho_2^2 = \rho_2^2(0)$, $\rho_2^j = \rho_2^j(0)$ ($i = 1, 2, 3$; $j = 1, 2$) as the corresponding spectra of the unperturbed system (1.2).

($\mathbf{A_2}$) System (1.2) has a heteroclinic network $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ connecting the saddle equilibrium p_i, where $\Gamma_k = \{\mathbf{x} = \gamma_k(t), t \in \mathbf{R}\}$, $\gamma_1(-\infty) = \gamma_2(+\infty) = p_1$, $\gamma_1(+\infty) = \gamma_2(-\infty) = \gamma_3(-\infty) = \gamma_4(+\infty) = p_2$, $\gamma_3(+\infty) = \gamma_4(-\infty) = p_3$.

($\mathbf{A_3}$) Let $e_i^{\pm} = \lim\limits_{t \to \pm\infty} \dfrac{\dot{\gamma}_i(-t)}{|\dot{\gamma}_i(-t)|}$, then

$$\begin{aligned}
&e_1^+ \in T_{p_1} W_1^u, \quad e_2^+, e_3^+ \in T_{p_2} W_2^u, \quad e_4^+ \in T_{p_3} W_3^u, \\
&e_1^-, e_4^- \in T_{p_2} W_2^s, \quad e_2^- \in T_{p_1} W_1^s, \quad e_3^- \in T_{p_3} W_3^u,
\end{aligned}$$

where $e_i^+, e_i^- (i = 1, 2, 3)$ are unit eigenvectors corresponding to λ_i^1 and $\rho_i^1 (i = 1, 2, 3)$ respectively, where $T_q M$ denotes the tangent space of the manifold M at q. Furthermore they satisfy the equation $e_1^- = -e_4^-$, $e_3^+ = -e_2^+$ (see [44] for details).

($\mathbf{A_4}$)

$$\begin{aligned}
&\lim_{t \to +\infty} T_{\gamma_1(t)} W_1^u = \operatorname{span}\left\{e_1^-, e_2^+\right\}, \qquad \lim_{t \to -\infty} T_{\gamma_1(t)} W_2^s = \operatorname{span}\left\{e_1^+, e_2^-\right\}; \\
&\lim_{t \to +\infty} T_{\gamma_2(t)} W_2^u = \operatorname{span}\left\{e_2^-\right\}, \qquad \lim_{t \to -\infty} T_{\gamma_2(t)} W_1^s = \operatorname{span}\left\{e_2^+\right\}; \\
&\lim_{t \to +\infty} T_{\gamma_3(t)} W_2^u = \operatorname{span}\left\{e_3^-\right\}, \qquad \lim_{t \to -\infty} T_{\gamma_3(t)} W_3^s = \operatorname{span}\left\{e_3^+\right\}; \\
&\lim_{t \to +\infty} T_{\gamma_4(t)} W_3^u = \operatorname{span}\left\{e_4^-, e_3^+\right\}, \qquad \lim_{t \to -\infty} T_{\gamma_4(t)} W_2^s = \operatorname{span}\left\{e_4^+, e_3^-\right\}.
\end{aligned}$$

($\mathbf{A_5}$)

$$\begin{aligned}
&\dim(T_{\gamma_1(t)} W_1^u \cap T_{\gamma_1(t)} W_2^s) = 1, \qquad \dim(T_{\gamma_2(t)} W_2^u \cap T_{\gamma_2(t)} W_1^s) = 1, \\
&\dim(T_{\gamma_3(t)} W_2^u \cap T_{\gamma_3(t)} W_3^s) = 1, \qquad \dim(T_{\gamma_4(t)} W_3^u \cap T_{\gamma_4(t)} W_2^s) = 1.
\end{aligned}$$

Remark 1.1. Under the assumption (A_1), p_1 and p_3 have a 1-dimensional stable manifold and a 2-dimensional unstable manifold, while p_2 has a 2-dimensional stable manifold and a 1-dimensional unstable manifold, hence Γ is an "∞" shape double heterodimensional cycles with two saddles p_1 and p_3 of (1,2) type and one saddle p_2 of (2,1) type.

Remark 1.2. Hypothesis (A_4) shows that $W^u_{p_i}$ and $W^s_{p_i}$ have strong inclination property. Due to the assumption (A_5), p_1 has a 2-dimensional unstable manifold, p_2 has a 2-dimensional stable manifold, and $\dim(T_{\gamma_1(t)}W^u_1 \cap T_{\gamma_1(t)}W^s_2) = 1$, we can know the codimension of the heteroclinic orbit Γ_1 is 0. In the same way, the codimension of heteroclinic orbit Γ_4 is also 0, and the codimension of the heteroclinic orbits Γ_2 and Γ_3 are both 2. Then the orbits Γ_1 and Γ_4 are transversal, that is, they can be preserved even under small perturbations. (see Figure 1).

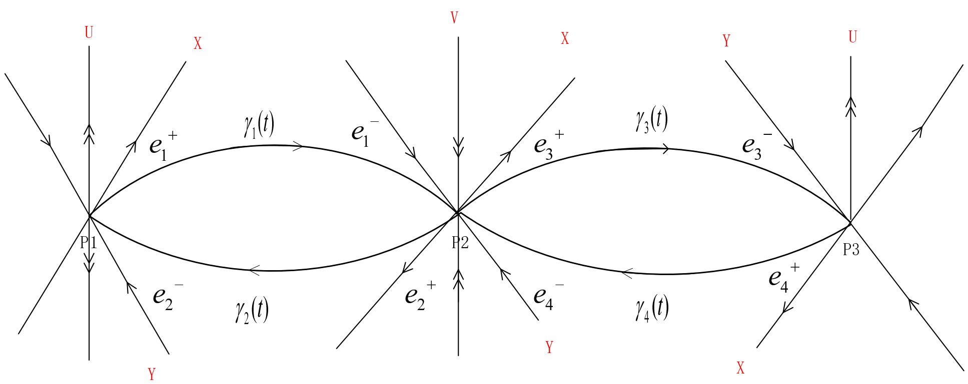

Figure 1. Double heterodimensional cycles of three saddle points p_i with four orbits $\gamma_k(t)$.

Now, take time T_k $(k = 1, 2, 3, 4)$ large enough such that $\gamma_k(\pm T_k)$ are in some small neighborhoods U_i of $p_i (i = 1, 2, 3)$. Then we can take transverse sections vertical to the tangency T_{γ_k} to each orbit γ_k:

$$\begin{aligned}
&S_1^0 : \{\mathbf{x}|\mathbf{x} = \gamma_1(-T_1)\} \subset U_1; \qquad S_1^1 : \{\mathbf{x}|\mathbf{x} = \gamma_1(T_1)\} \subset U_2;\\
&S_2^0 : \{\mathbf{x}|\mathbf{x} = \gamma_2(-T_2)\} \subset U_2; \qquad S_2^1 : \{\mathbf{x}|\mathbf{x} = \gamma_2(T_2)\} \subset U_1;\\
&S_3^0 : \{\mathbf{x}|\mathbf{x} = \gamma_3(-T_3)\} \subset U_2; \qquad S_3^1 : \{\mathbf{x}|\mathbf{x} = \gamma_3(T_3)\} \subset U_3;\\
&S_4^0 : \{\mathbf{x}|\mathbf{x} = \gamma_4(-T_4)\} \subset U_3; \qquad S_4^1 : \{\mathbf{x}|\mathbf{x} = \gamma_4(T_4)\} \subset U_2.
\end{aligned} \tag{1.3}$$

Generally, if the small parameter $\mu \neq 0$, the original double heterodimensional cycles Γ may be broken, then some new orbits $\gamma_k^+(t, \mu)$ (resp. $\gamma_k^-(t, \mu)$) $(k = 1, 2, 3, 4)$ appear from unstable (resp. stable) manifold of the equilibrium $p_i (i = 1, 2, 3)$ of system (1.1) with the following properties

$$\begin{aligned}
&\dot{\gamma}_k^\pm = f(\gamma_k^\pm, \mu);\\
&\gamma_1^+(t, \mu) \in W_1^u(p_1), \quad \gamma_1^-(t, \mu) \in W_2^s(p_2),\\
&\gamma_2^+(t, \mu) \in W_2^u(p_2), \quad \gamma_2^-(t, \mu) \in W_1^s(p_1),\\
&\gamma_3^+(t, \mu) \in W_2^u(p_2), \quad \gamma_3^-(t, \mu) \in W_3^s(p_3),\\
&\gamma_4^+(t, \mu) \in W_3^u(p_3), \quad \gamma_4^-(t, \mu) \in W_2^s(p_2),
\end{aligned} \tag{1.4}$$

$$
\begin{aligned}
&\gamma_k^{\pm}(t,0)=\gamma_k(t),\\
&\gamma_k^{+}(-T_k,\mu)\in S_k^0,\\
&\gamma_k^{+}(-T_k+\tilde{T}_k,\mu),\gamma_k^{-}(T_k,\mu)\in S_k^1,\\
&\left\|\gamma_k^{+}(-T_k+\tilde{T}_k,\mu)-\gamma_k^{-}(T_k,\mu)\right\|=0(k=1,4),\\
&\left\|\gamma_k^{+}(-T_k+\tilde{T}_k,\mu)-\gamma_k^{-}(T_k,\mu)\right\|\ll 1(k=2,3),
\end{aligned}
$$

where $\tilde{T}_k(k=1,2,3,4)$ are the orbit taking time from S_k^0 to S_k^1 and the $W_i^s(p_i)$ and $W_i^u(p_i)$ are the stable and unstable manifolds of the equilibrium p_i, $(i=1,2,3)$. Because the original heteroclinic trajectories $\gamma_1(t)$ and $\gamma_4(t)$ are obtained as two transversal intersections of 2-dimensional manifolds, they are structurally stable, and after a small perturbation, the intersections are preserved. That is, the gaps $\left\|\gamma_k^{+}(-T_k+\tilde{T}_k,\mu)-\gamma_k^{-}(T_k,\mu)\right\|=0$ $(k=1,4)$. As well as, if the gaps $\left\|\gamma_k^{+}(-T_k+\tilde{T}_k,\mu)-\gamma_k^{-}(T_k,\mu)\right\|=0$ $(k=2,3)$ in S_k^1, it mean the original double heterodimensional cycles are kept (see Figure 2).

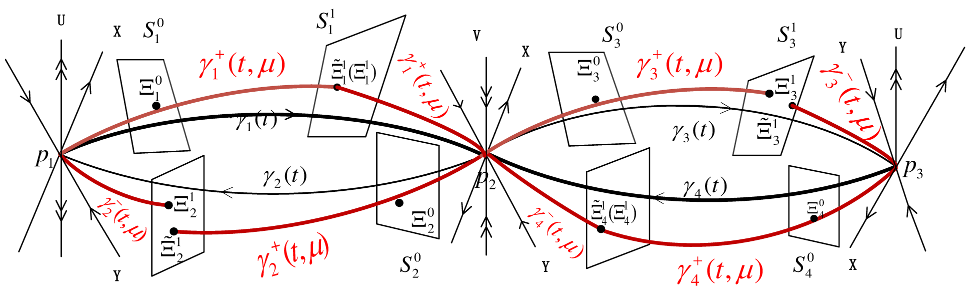

Figure 2. The gap $\left\|\gamma_k^{+}(-T_k+\tilde{T}_k,\mu)-\gamma_k^{-}(T_k,\mu)\right\|\neq 0(k=2,3)$ in the figure, the original double heterodimensional cycles do not exist.

If there is an orbit starting from the section S_1^0 and arriving at the section S_3^1 that passes through the sections S_1^1 and S_3^0 with finite time without orienting to the saddle point p_2, we denote it by $\gamma_1(t,\mu)$. Similarly, we can define $\gamma_4(t,\mu)$ in this way. Set the time of the orbit $\gamma_1(t,\mu)$ from S_1^1 to S_3^0 to be τ_2 and the time of $\gamma_4(t,\mu)$ from S_4^1 to S_2^0 to be τ_4; and from S_k^0 to S_k^1 to be $\tilde{T}_k$, $k=1,2,3,4$, respectively. Moreover, system (1.1) still has solutions $\gamma_j(t,\mu)$, $j=1,4$,

$$
\begin{aligned}
&\dot{\gamma}_j(t,\mu)=f(\gamma_j,\mu);\\
&\gamma_j(-T_j,\mu)\in S_j^0,\gamma_j(-T_j+\tilde{T}_j,\mu)\in S_j^1,\\
&\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu)\in S_3^1,\\
&\gamma_4(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2,\mu)\in S_2^1,\\
&\left\|\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu)-\gamma_3^{-}(T_3,\mu)\right\|\ll 1,\\
&\left\|\gamma_4(-T_4-\tilde{T}_4+\tau_4-\tilde{T}_2,\mu)-\gamma_2^{-}(T_2,\mu)\right\|\ll 1,
\end{aligned}
\tag{1.5}
$$

where $\gamma_k^{\pm}(t,\mu)$ $(k=2,4)$, and $\gamma_3^{-}(t,\mu)$ still meet equation (1.4). Clearly system (1.1) has a heterodimensional cycle composed of one big orbit linking p_1,p_3 and two orbits linking p_3, p_2 and p_2, p_1, respectively, with two saddles of (1,2) type and one saddle of (2,1) type if the gaps $\left\|\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu)-\gamma_3^-(T_3,\mu)\right\| = 0$, $\left\|\gamma_2^-(T_2,\mu) - \gamma_2^+(-T_2+\tilde{T}_2,\mu)\right\| = 0$ (see Figure 3), which is called the second shape heterodimensional cycle, or has a large 1-heteroclinic cycle composed by two big orbits linking p_1, p_3 and p_3, p_1 of (1,2) type respectively, if the gaps $\left\|\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu)-\gamma_3^-(T_3,\mu)\right\| = 0$, $\left\|\gamma_4(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2,\mu)-\gamma_2^-(T_2,\mu)\right\| = 0$ (see Figure 4).

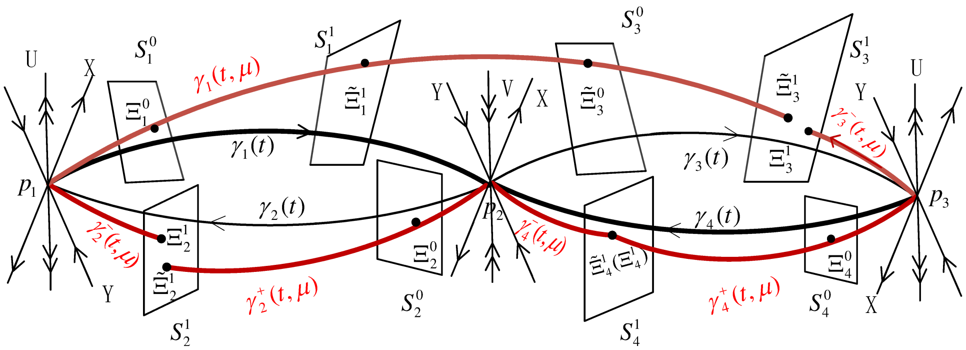

Figure 3. The gap $||\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu)-\gamma_3^-(T_3,\mu)|| \neq 0$, $\left\|\gamma_2^-(T_2,\mu)-\gamma_2^+(-T_2+\tilde{T}_2,\mu)\right\| \neq 0$ in the figure, there is not the second heterodimensional cycle in general, which consists of two saddles of (1,2) type and one saddle of (2,1) type and is composed of one big orbit linking p_1, p_3 and two orbits linking p_3, p_2 and p_2, p_1 respectively.

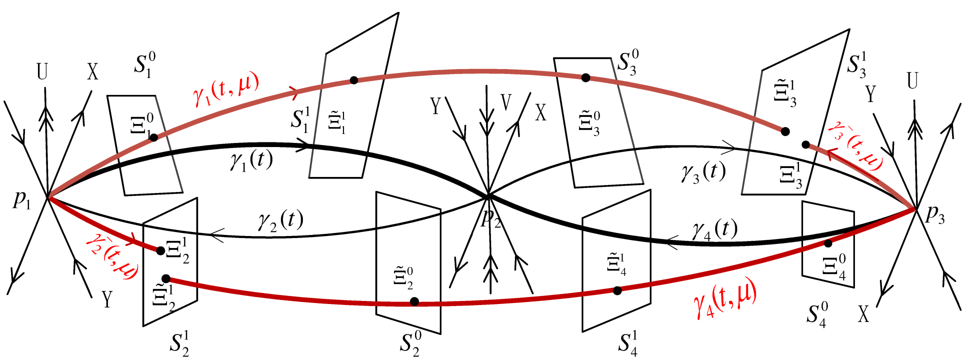

Figure 4. The gap $\left\|\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu)-\gamma_3^-(T_3,\mu)\right\| \neq 0$, $\left\|\gamma_4(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2,\mu)-\gamma_2^-(T_2,\mu)\right\| \neq 0$ in the figure, there is not a large 1-heteroclinic cycle in general.

Remark 1.3. In fact, system (1.1) has another second heterodimensional cycle composed of one big orbit linking p_3, p_1 and two orbits linking p_1, p_2 and p_2, p_3 respectively, if the gaps $\left\|\gamma_4(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2,\mu)-\gamma_2^-(T_2,\mu)\right\| = 0$ and $||\gamma_3^-(T_3,\mu)-\gamma_3^+(-T_3+\tilde{T}_3,\mu)|| = 0$ hold. The corresponding image is similar to Figure 3 by turning Figure 3 upside down.

Next, we regularize the normal form of system (1.1). As a direct application of the stable (unstable) manifold theorem and the strong stable (unstable) manifold theorem, one may find two successive C^r and C^{r-1} transformations in some neighborhood U_i $(i = 1, 2, 3)$ of $z = 0$ to straighten the invariant manifolds such that for $j = 1, 3$

$$\begin{aligned} W^u_{p_j} &= \{(x, y, u) : y = u = 0\}, & W^s_{p_j} &= \{(x, y, u) : x = u = 0\},\\ W^{uu}_{p_j} &= \{(x, y, u) : x = y = 0\}, & W^s_{p_2} &= \{(x, y, v) : x = v = 0\},\\ W^u_{p_2} &= \{(x, y, v) : y = v = 0\}, & W^{ss}_{p_2} &= \{(x, y, v) : x = y = 0\}. \end{aligned}$$

Also, we can straighten the orbit segments $\Gamma_k \bigcap U_i$ $(k = 1, 2, 3, 4;\ i = 1, 2, 3)$.

After that system (1.1) has the following form in the small neighborhood U_i $(i = 1, 3)$ of p_i:

$$\begin{aligned} \dot{x} &= [{\lambda_i}^1(\mu) + a(\mu)xu + o(|xu|)]x + \mathrm{O}(\mu)[\mathrm{O}(x^2u) + \mathrm{O}(y)],\\ \dot{y} &= [-{\rho_i}^1(\mu) + b(\mu)xu + o(|xu|)]y + \mathrm{O}(\mu)[\mathrm{O}(xyu) + \mathrm{O}(\mu)],\\ \dot{u} &= [{\lambda_i}^2(\mu) + c(\mu)xu + o(|xu|)]u + y^2H_i(x, y, u), \end{aligned} \tag{1.6}$$

and has C^k normal form in U_2 of p_2 as:

$$\begin{aligned} \dot{x} &= [{\lambda_2}^1(\mu) + a(\mu)xv + o(|xv|)]x + \mathrm{O}(\mu)[\mathrm{O}(x^2v) + \mathrm{O}(y)],\\ \dot{y} &= [-{\rho_2}^1(\mu) + b(\mu)xv + o(|xv|)]y + \mathrm{O}(\mu)[\mathrm{O}(xyv) + \mathrm{O}(\mu)],\\ \dot{v} &= [{\lambda_2}^2(\mu) + c(\mu)xv + o(|xv|)]v + y^2H_2(x, y, v), \end{aligned} \tag{1.7}$$

where $H_1(x, 0, 0) = 0$, $H_2(0, y, 0) = 0$. System (1.6)-(1.7) are at least C^k, where $k = \min\left\{r - 2, [\frac{\lambda_i^2}{\lambda_i^1}] - 1,\ [\frac{\rho_2^2}{\rho_2^1}] - 1\right\} \geq 2$, which is owing to that the weak stable manifold of p_i and the weak unstable manifold of p_2 are approximately $C^{[\frac{\lambda_i^2}{\lambda_i^1}]}, C^{[\frac{\rho_2^2}{\rho_2^1}]}$, respectively (see [35] P.56). Of course, the same kind of change of variable can be achieved by using the theory of exponential dichotomies and weighted exponential dichotomies. But by [35], the extra conditions $\rho_1^2 \geq 3\rho_1^1$ and $\lambda_j^2 \geq \lambda_j^1 (j = 1, 3)$ ensure that such change of coordinates are possible, so that systems (1.6)-(1.7) are smooth.

The rest of the paper is structured as follows. In section 2, we firstly establish a local moving frame system near the unperturbed heterodimensional cycle, then we define a Poincaré map to give the successor function and the bifurcation equations by using the implicit function theorem. Section 3 shows the bifurcation results on different parameter regions by analyzing the bifurcation equations.

2. Main Method

From the above discussion, we find that the gap in the transverse section S_k^1 of some orbits is crucial to study bifurcations of system (1.1). So in this section we try to quantizate the gap size by the method mentioned in [46,47]. That is, we firstly need to take fundamental solutions of linear variational equation (see equation (2.1) as below) and use them as an active coordinate system along the heteroclinic orbits. Then using the new coordinates, we construct the global map spanned by the flow of (2.1) between the sections along the orbits. Next, we set up local maps near equilibria. Finally the whole Poincaré map can be obtained by composing these maps and the implicit function theorem reveals the bifurcation equation.

The linear variational system of (1.2) is:

$$\dot{Y}_k = D_{\mathbf{x}} f\left(\gamma_k(t), 0\right) Y_k. \tag{2.1}$$

Based on the hypotheses (A_3) and (A_4), system (2.1) has a fundamental solution matrix $Y_k(t) = (\varphi_k^1, \varphi_k^2, \varphi_k^3)$ with $\varphi_k^1(t) \in (T_{\gamma_k(t)} W_k^u)^c \cap T_{\gamma_k(t)} W_2^s$, $\varphi_k^2(t) = \dot{\gamma}_k(t)/\left|\dot{\gamma}_k(-T_k)\right| \in T_{\gamma_k(t)} W_k^u \cap T_{\gamma_k(t)} W_2^s$, $\varphi_k^3(t) \in T_{\gamma_k(t)} W_k^u \cap (T_{\gamma_k(t)} W_2^s)^c$, for $k = 1, 4$, and it satisfies

$$Y_k(-T_k) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \bar{w}_k^{13} & 0 & 1 \end{pmatrix}, \quad Y_k(T_k) = \begin{pmatrix} 0 & 0 & w_k^{31} \\ 0 & w_k^{22} & 0 \\ w_k^{13} & 0 & w_k^{33} \end{pmatrix}, \tag{2.2}$$

where $w_k^{13} > 0$, $w_k^{33} w_k^{31} > 0$, $w_1^{22} < 0$, $\left|(w_k^{13})^{-1}\right| \ll 1$, $\left|w_k^{33} \cdot (w_k^{31})^{-1}\right| \ll 1$, $W_4^u = W_3^u$. The notation $(W)^c$ means subspace complementary to W.

As for $k = 2, 3$, the fundamental solution matrix of system (2.1) satisfies $\varphi_k^1(t)$, $\varphi_k^3(t) \in (T_{\gamma_k(t)} W_2^u)^c$, $\varphi_k^2(t) = \dot{\gamma}_k(t)/\left|\dot{\gamma}_k(-T_\mathrm{k})\right| \in T_{\gamma_k(t)} W_2^u \cap T_{\gamma_k(t)} W_{k-(-1)^k}^s$, $W_4^s = W_3^s$, such that

$$Y_k(-T_k) = \begin{pmatrix} 0 & w_k^{21} & 0 \\ \bar{w}_k^{12} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y_k(T_k) = \begin{pmatrix} w_k^{11} & 0 & w_k^{31} \\ w_k^{12} & 1 & w_k^{32} \\ w_k^{13} & 0 & w_k^{33} \end{pmatrix}, \tag{2.3}$$

where $w_2^{11} < 0, w_k = \begin{vmatrix} w_k^{11} & w_k^{31} \\ w_k^{13} & w_k^{33} \end{vmatrix} \neq 0, \left|w_k^{i2} \cdot w_k^{-1}\right| \ll 1, i = 1, 3.$

Remark 2.1. For the elements of fundamental solution matrix, one can refer to [30, 44] or [42, 43] for details.

Now we select $(\varphi_k^1, 0, \varphi_k^3)$ as a local coordinate system along Γ_k. Take a coordinate transformation

$$\mathbf{X}(t) = \gamma_k(t) + Y_k(t)\Xi_k(t) = \gamma_k(t) + \varphi_k^1(t)\xi_k^1 + \varphi_k^3(t)\xi_k^3, \quad t \in [-T_k, T_k], \tag{2.4}$$

where $\Xi_k = (\xi_k^1, 0, \xi_k^3)^*$ is the coordinate decomposition of system (1.1) and the sign "*" stands for transposition. Let $\gamma_k(t) = (\gamma_k^x(t), \gamma_k^y(t), \gamma_k^u(t))^*$ in the small neighborhood U_i of p_i, $(i = 1, 3)$, and $\gamma_k(t) = (\gamma_k^x(t), \gamma_k^y(t), \gamma_k^v(t))^*$ in the small neighborhood U_2 of p_2. Since $T_k > 0$ $(k = 1, 2, 3, 4)$ is large enough so that $\gamma_1(-T_1), \gamma_2(T_2) \in U_1$, $\gamma_3(T_3), \gamma_4(-T_4) \in U_3$, $\gamma_1(T_1), \gamma_2(-T_2), \gamma_3(-T_3), \gamma_4(T_4) \in U_2$ and for $k = 1, 3, 4$, $\gamma_k(-T_k) = (\delta, 0, 0)^*$, $\gamma_2(-T_2) = (-\delta, 0, 0)^*$, for $k = 2, 3, 4$, $\gamma_k(T_k) = (0, \delta, 0), \gamma_1(T_1) = (0, -\delta, 0)$, where $\delta > 0$ is small enough. Then system (1.1) can be rewritten in the new variable Ξ, namely, for $k = 1, 2, 3, 4$

$$\dot{\Xi} = Y_k^{-1} D_\mu f(\gamma_k(t), 0)\mu + Y_k^{-1} D_{\mathbf{x}\mu}^2 f(\gamma_k(t), 0) Y \Xi \mu + \mathrm{O}(|Y_k| \, |\Xi|^2) + \mathrm{O}(|Y_k|^{-1} \, |\mu|^2). \tag{2.5}$$

To integrate (2.5), we get

$$\int_{-T_k}^{T_k} \dot{\Xi}\, dt = \int_{-T_k}^{T_k} Y_k^{-1} D_\mu f(\gamma_k(t), 0)\mu\, dt + h.o.t., \tag{2.6}$$

that is,

$$\Xi(T_k) = \Xi(-T_k) + M_k\mu + h.o.t., \tag{2.7}$$

where $M_k = (M_k^1, 0, M_k^3)^* = \int_{-T_k}^{T_k} Y_k^{-1} D_\mu f(\gamma_k(t), 0)\, dt$ and Y^{-1} is the fundamental solution matrix of the adjoint system for system (2.1).

Notice that in (2.4), $\Xi_k = (\xi_k^1, 0, \xi_k^3)^*$ represents the coordinate decomposition of (1.1) in the new local coordinate system corresponding to $\varphi_k^1(t)$ and $\varphi_k^3(t)$, so $\Xi(-T_k) \in S_k^0$ and $\Xi(T_k) \in S_k^1$, scilicet, (2.7) maps a point in S_k^0 to a point in S_k^1, which produces the global map from points in S_k^0 to points in S_k^1 in some small subset of U_i.

On the other hand, there is a local linearization of system (1.1) as follow due to (A_1) and (1.6),(1.7) for $i = 1, 3$

$$\begin{aligned} D_{\mathbf{x}} f(p_i, \mu) &= \lambda_i^1(\mu) x \frac{\partial \mathbf{x}}{\partial x} - \rho_i^1(\mu) u \frac{\partial \mathbf{x}}{\partial y} + \lambda_i^2(\mu) y \frac{\partial \mathbf{x}}{\partial u}, \\ D_{\mathbf{x}} f(p_2, \mu) &= \lambda_2^1(\mu) x \frac{\partial \mathbf{x}}{\partial x} - \rho_2^1(\mu) y \frac{\partial \mathbf{x}}{\partial y} - \rho_2^2(\mu) v \frac{\partial \mathbf{x}}{\partial v}, \end{aligned} \tag{2.8}$$

where $\mathbf{x} = (x, y, u)$, $(x_k^0, y_k^0, u_k^0) \in S_k^0$ or $(x_j^1, y_j^1, u_j^1) \in S_j^1$ for $k = 1, 4, j = 2, 3$ and $(x_k^0, y_k^0, v_k^0) \in S_k^0$ or $(x_j^1, y_j^1, v_j^1) \in S_j^1$ for $k = 2, 3, j = 1, 4$.

Let $\Xi_k(T_k) = \Xi_k^1 = (\xi_k^{1,1}, 0, \xi_k^{1,3})$ and $\Xi_k(-T_k) = \Xi_k^0 = (\xi_k^{0,1}, 0, \xi_k^{0,3})$. Suppose $\lambda_i^1 < \rho_i^1$ $(i = 1, 2, 3)$, we take $s_k = e^{-\lambda_k^1(\mu)\tau_k}$, $k = 1, 2, 3, 4$, where τ_1, τ_3 are the time spent from S_2^1 to S_1^0 and from S_3^1 to S_4^0 respectively, and $\lambda_4^1 = \lambda_2^1$; For the convenience of calculation, we denote by $\lambda_i^j(\mu) = \lambda_i^j$, $\rho_i^1(\mu) = \rho_i^1$, $\lambda_2^1(\mu) = \lambda_2^1$, $\rho_2^j(\mu) = \rho_2^j$, $i = 1, 3$; $j = 1, 2$.

Based on the linear approximation solutions of equation (2.8), we have

$$\begin{aligned} \mathbf{x}_k^0 &= e^{\lambda_k^1 \tau_k} x_k^1 \frac{\partial \mathbf{x}}{\partial x} + e^{-\rho_k^1 \tau_k} y_k^1 \frac{\partial \mathbf{x}}{\partial y} + e^{\lambda_k^2 \tau_k} u_k^1 \frac{\partial \mathbf{x}}{\partial u}\ (k = 1, 3), \\ \mathbf{x}_k^0 &= e^{\lambda_k^1 \tau_k} x_k^1 \frac{\partial \mathbf{x}}{\partial x} + e^{-\rho_k^1 \tau_k} y_k^1 \frac{\partial \mathbf{x}}{\partial y} + e^{-\rho_k^2 \tau_k} v_k^1 \frac{\partial \mathbf{x}}{\partial v}\ (k = 2, 4). \end{aligned} \tag{2.9}$$

By equation (2.9), it is easy to obtain the local map from points in S_k^1 to points in S_k^0, where the $\beta_h = \rho_h^1/\lambda_h^1, (h = 1, 2, 3)$.

$$\begin{aligned} &x_2^1 = x(T_2) \approx x_1^0 s_1, \quad y_1^0 = y(T_2 + \tau_1) \approx y_2^1 s_1^{\beta_1}, \quad u_2^1 = u(T_2) \approx u_1^0 s_1^{\frac{\lambda_1^2}{\lambda_1^1}}; \\ &x_1^1 = x(T_1) \approx x_3^0 s_2, \quad y_3^0 = y(T_1 + \tau_2) \approx y_1^1 s_2^{\beta_2}, \quad v_3^0 = v(T_2 + \tau_2) \approx v_1^1 s_2^{\frac{\rho_2^2}{\lambda_2^1}}; \\ &x_3^1 = x(T_3) \approx x_4^0 s_3, \quad y_4^0 = y(T_3 + \tau_3) \approx y_3^1 s_3{}^{\beta_3}, \quad u_3^1 = u(T_3) \approx u_4^0 s_3^{\frac{\lambda_3^2}{\lambda_3^1}}; \\ &x_4^1 = x(T_4) \approx x_2^0 s_4, \quad y_2^0 = y(T_4 + \tau_4) \approx y_4^1 s_4^{\beta_2}, \quad v_2^0 = x(T_4 + \tau_4) \approx v_4^1 s_4^{\frac{\rho_2^2}{\lambda_2^1}}. \end{aligned} \tag{2.10}$$

From (2.1)-(2.4) and replacing $\mathbf{x}$ by Ξ, we establish the relationship between the old coordinates and their new coordinates

$$\begin{aligned}
&\xi_k^{0,1} = y_k^0, \quad \xi_k^{0,3} = u_k^0 - \bar{w}_k^{13} y_k^0, \quad x_k^0 = \delta,\\
&\xi_k^{1,1} = (w_k^{13})^{-1} v_k^1 - (w_k^{13})^{-1} w_k^{33} (w_k^{31})^{-1} x_k^1,\\
&y_k^1 = (-1)^k \delta, \quad \xi_k^{1,3} = (w_k^{31})^{-1} x_k^1, \ (k = 1, 4).\\
&\xi_i^{0,1} = v_i^0, \quad \xi_i^{0,3} = y_i^0 - \bar{w}_k^{12} v_i^0, \quad x_i^0 = (-1)^{i-1}\delta,\\
&\xi_i^{1,1} = w_i^{-1}(w_i^{33} x_i^1 - w_i^{31} u_i^1), \quad \xi_i^{1,3} = w_i^{-1}(w_i^{11} u_i^1 - w_i^{13} x_i^1),\\
&y_i^1 = \delta + {w_i}^{-1}(w_i^{12} w_i^{33} - w_i^{32} w_i^{13}) x_i^1 + w_i^{-1}(w_i^{32} w_i^{11} - w_i^{12} w_i^{31}) u_i^1 \approx \delta, \ (i = 2, 3).
\end{aligned} \tag{2.11}$$

Now take points $\Xi_1^1 \in S_1^1$, $\Xi_2^1 \in S_2^1$, $\Xi_3^1 \in S_3^1$, $\Xi_4^1 \in S_4^1$, system (1.1) has orbits $\gamma_1(t,\mu)$, $\gamma_2(t,\mu)$, $\gamma_3(t,\mu)$, $\gamma_4(t,\mu)$ starting from Ξ_2^1, Ξ_1^1, Ξ_3^1, Ξ_4^1 and intersecting S_1^1, S_3^1, S_4^1, S_2^1 at points $\tilde{\Xi}_2^1$, $\tilde{\Xi}_1^1$, $\tilde{\Xi}_4^1$, $\tilde{\Xi}_3^1$, respectively. From (2.4), (2.7) and (2.11), Poincaré map can be defined as $\Psi = \tilde{\Xi}_k^1 - \Xi_k^1 = (\Psi_1, \Psi_2, \Psi_3, \Psi_4) = \Psi(\Psi_1^1, \Psi_1^3, \Psi_2^1, \Psi_2^3, \Psi_3^1, \Psi_3^3, \Psi_4^1, \Psi_4^3) = \Psi(s_1, s_2, s_3, s_4, u_1^0,\ u_4^0, v_1^1, v_4^1)$, where

$$\begin{aligned}
&\Psi_1^1 = \delta s_1^{\beta_1} - (w_1^{13})^{-1} v_1^1 + (w_1^{13})^{-1} w_1^{33} (w_1^{31})^{-1} \delta s_2 + M_1^1 \mu + h.o.t.,\\
&\Psi_1^3 = u_1^0 - \bar{w}_1^{13} \delta s_1^{\beta_1} - (w_1^{31})^{-1} \delta s_2 + M_1^3 \mu + h.o.t.,\\
&\Psi_2^1 = v_4^1 s_4^{\frac{\rho_2^2}{\lambda_2^1}} + {w_2}^{-1} w_2^{31} u_1^0 s_1^{\frac{\lambda_1^2}{\lambda_1^1}} - {w_2}^{-1} w_2^{33} \delta s_1 + M_2^1 \mu + h.o.t.,\\
&\Psi_2^3 = \delta s_4^{\beta_2} + {w_2}^{-1} w_2^{13} \delta s_1 - {w_2}^{-1} w_2^{11} u_1^0 s_1^{\frac{\lambda_1^2}{\lambda_1^1}} + \bar{w}_2^{12} v_1^4 s_4^{\frac{\rho_2^2}{\lambda_2^1}} + M_2^3 \mu + h.o.t.,\\
&\Psi_3^1 = v_1^1 s_2^{\frac{\rho_2^2}{\lambda_2^1}} - {w_3}^{-1} w_3^{33} \delta s_3 + {w_3}^{-1} w_3^{31} u_4^0 s_3^{\frac{\lambda_3^2}{\lambda_3^1}} + M_3^1 \mu + h.o.t.,\\
&\Psi_3^3 = -\delta s_2^{\beta_2} + {w_3}^{-1} w_3^{13} \delta s_3 - {w_3}^{-1} w_3^{11} u_4^0 s_3^{\frac{\lambda_3^2}{\lambda_3^1}} - \bar{w}_3^{12} v_1^1 s_2^{\frac{\rho_2^2}{\lambda_2^1}} + M_3^3 \mu + h.o.t.,\\
&\Psi_4^1 = \delta s_3^{\beta_3} - (w_4^{13})^{-1} v_4^1 - (w_4^{13})^{-1} w_4^{33} (w_4^{31})^{-1} \delta s_4 + M_4^1 \mu + h.o.t.,\\
&\Psi_4^3 = u_4^0 - \bar{w}_4^{13} \delta {s_3}^{\beta_3} + (w_4^{31})^{-1} \delta s_4 + M_4^3 \mu + h.o.t..
\end{aligned} \tag{2.12}$$

By the implicit function theorem, equations $(\Psi_1, \Psi_4) = 0$ can give solutions of v_1^1, u_1^0, v_4^1 and u_4^0. Putting them into $(\Psi_2, \Psi_3) = 0$, and fixing $s_1 = s_3 = 0$, we can get the bifurcation equation

$$\begin{cases}
w_1^{33} (w_1^{31})^{-1} \delta {s_2}^{\frac{\rho_2^2}{\lambda_2^1}+1} + w_1^{13} M_1^1 \mu {s_2}^{\frac{\rho_2^2}{\lambda_2^1}} + M_3^1 \mu + h.o.t. = 0,\\
\delta s_2^{\beta_2} + \bar{w}_3^{12} M_1^1 \mu w_1^{13} s_2^{\frac{\rho_2^2}{\lambda_2^1}} - M_3^3 \mu + h.o.t. = 0,\\
- w_4^{33} (w_4^{31})^{-1} \delta s_4^{\frac{\rho_2^2}{\lambda_2^1}+1} + w_4^{13} M_4^1 \mu s_4^{\frac{\rho_2^2}{\lambda_2^1}} + M_2^1 \mu + h.o.t. = 0,\\
\delta s_4^{\beta_2} - \bar{w}_3^{12} M_4^1 \mu w_4^{13} s_4^{\frac{\rho_2^2}{\lambda_2^1}} + M_2^3 \mu + h.o.t. = 0.
\end{cases} \tag{2.13}$$

Remark 2.2. Generally, in two dimensional plane system, when we study bifurcations of singular cycle, Poincaré mapping can only be established on one side of the singular cycle. Therefore, there are no other types of orbits except the one with infinite approaching to saddle point on the left side of p_1 and the right side of p_3. However, in high-dimensional system, it remains to be verified whether other types of orbits can bypass different surfaces for connection. To make the study go on, we assume that $s_1 = s_3 = 0$, that is, the orbit starting from S_2^1 to S_1^0 just be a singular orbit which is infinitely approaching p_1 when $t \to \pm\infty$; for the orbit starting from S_3^1 to S_4^0 is similar near p_3.

3. Heterodimensional cycle bifurcation of "∞" type

In this section, we analyze the bifurcation of system (1.1) under hypotheses (A_1)-(A_5) about the existence of double heterodimensional cycles ("∞"), the second heterodimensional cycle and large 1-heteroclinic cycle. Clearly if $||\gamma_k^-(T_k, \mu) - \gamma_k^+(-T_k + \tilde{T}_k, \mu)|| = 0$ $(k = 2, 3)$, the double heterodimensional cycle ("∞") of system (1.1) is persistent; if $\left\|\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3, \mu) - \gamma_3^-(T_3, \mu)\right\| = 0$ and $\left\|\gamma_2^-(T_2, \mu) - \gamma_2^+(-T_2 + \tilde{T}_2, \mu)\right\| = 0$ or $||\gamma_4(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2, \mu) - \gamma_2^-(T_2, \mu)|| = 0$ and $||\gamma_3^-(T_3, \mu) - \gamma_3^+(-T_3 + \tilde{T}_3, \mu)|| = 0$, that is, $s_4 = 0$, $s_2 > 0$ or $s_2 = 0$, $s_4 > 0$, system (1.1) has the second shape heterodimensional cycle; if $\left\|\gamma_1(-T_1 + \tilde{T}_1 + \tau_2 + \tilde{T}_3, \mu) - \gamma_3^-(T_3, \mu)\right\| = 0$, $\left\|\gamma_4(-T_4 + \tilde{T}_4 + \tau_4 + \tilde{T}_2, \mu) - \gamma_2^-(T_2, \mu)\right\| = 0$, that is, $s_2 > 0$ and $s_4 > 0$, system (1.1) has the large 1-heteroclinic cycle. What is noteworthy is that we find if $0 < ||\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3, \mu) - \gamma_3^-(T_3, \mu)|| \ll 1$, $\left\|\gamma_2^+(-T_2+\tilde{T}_2, \mu) - \gamma_2^-(T_2, \mu)\right\| = 0$ (or $0 < \left\|\gamma_4(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2, \mu) - \gamma_2^-(T_2, \mu)\right\| \ll 1$, $\left\|\gamma_3^+(-T_3+\tilde{T}_3, \mu) - \gamma_3^-(T_3, \mu)\right\| = 0$), that is, the conditions make $s_2 > 0$ (or $s_4 > 0$) untenable and $s_4 = 0$ (or $s_2 = 0$) be tenable, system (1.1) has the third heterodimensional cycle consisting of one saddle p_1 of (2.1) type and one saddle p_2 of (1,2) type and composed of one orbit starting from p_1 (or p_2) to p_2 (or p_3) and another orbit starting from p_2 (or p_3) to p_1 (or p_2) under the assumption (A_5). So in the following, we need to consider solutions s_2 and s_4 of the bifurcation equation (2.13).

Theorem 3.1. *Suppose that (A_1)-(A_5) hold and Rank $(M_2^1, M_2^3, M_3^1, M_3^3) = 4$, there is an $(l-4)$-dimensional surface*

$$L_{23} = \left\{\mu : M_2^1\mu + h.o.t. = 0; M_2^3\mu + h.o.t. = 0;\ M_3^1\mu + h.o.t. = 0; M_3^3\mu + h.o.t. = 0\right\},$$

with a normal plane $\Sigma_{23} = span\left\{M_2^1, M_2^3, M_3^1, M_3^3\right\}$, such that system (1.1) has a unique double heteroclinic loop ("∞") connecting p_1, p_2 and p_3 in the tubular neighborhood of Γ as $\mu \in L_{23}$, $0 < |\mu| \leq 1$.

Proof. As we explained above, $s_2 = 0$ in equation (2.13) means the flying time of an orbit starting from S_1^1 to S_3^0 is infinite, that is, the orbit must go into the equilibrium p_2, which corresponds to a heteroclinic orbit; and for $s_4 = 0$, it is similar. Hence, set $s_2 = s_4 = 0$ in equation (2.13), we have

$$\begin{cases} M_2^1\mu + h.o.t. = 0, \\ M_2^3\mu + h.o.t. = 0, \\ M_3^1\mu + h.o.t. = 0, \\ M_3^3\mu + h.o.t. = 0. \end{cases}$$

If $\mathrm{Rank}(M_2^1, M_2^3, M_3^1, M_3^3) = 4$, they can define a codimension-4 surface $L_{23}(\mu)$. When $\mu \in L_{23}$, system (1.1) has four heteroclinic orbits connecting the equilibriums p_i, $i = 1, 2, 3$, and they form an "∞"-type double heterodimensional cycle, or it says that the original heterodimensional loop is preserved. □

Define eight regions:

$$\begin{aligned} B_+^+ &= \{\mu | w_1^{13} M\ {}_1^1\mu > 0, M_3^1\mu > 0\}, & B_+^- &= \{\mu \,| w_1^{13} M_1^1\mu > 0, M_3^1\mu < 0\}; \\ B_-^+ &= \{\mu | \, w_1^{13} M_1^1\mu < 0, M_3^1\mu > 0\}, & B_-^- &= \{\mu | \, w_1^{13} M_1^1\mu < 0, M_3^1\mu < 0\}; \\ N_+^+ &= \{\mu \,| w_4^{13} M_4^1\mu > 0, M_2^1\mu > 0\}, & N_+^- &= \{\mu \,| w_4^{13} M_4^1\mu > 0, M_2^1\mu < 0\}, \\ N_-^+ &= \{\mu | \, w_4^{13} M_4^1\mu < 0, M_2^1\mu > 0\}, & N_-^- &= \{\mu | \, w_4^{13} M_4^1\mu < 0, M_2^1\mu < 0\}. \end{aligned}$$

From the discussion of Theorem 3.1, we know that if one of s_2 and s_4 is 0, the second heterodimensional cycle will appear; if $s_2 > 0$ and $s_4 > 0$, a large 1-heteroclinic cycle connecting with p_1 and p_3 will exist. As well as, if $s_2 > 0$ and $s_4 = 0$ or $s_4 > 0$ and $s_2 = 0$, the third heterodimensioonal cycle will arise.

Since the first two equations of equation (2.13) have the same structure as the last two, we only analyze the first and second equations as following

$$\begin{cases} w_1^{33}(w_1^{31})^{-1}\delta s_2^{\frac{\rho_2^2}{\lambda_2^1}+1} + w_1^{13}M_1^1\mu s_2^{\frac{\rho_2^2}{\lambda_2^1}} + M_3^1\mu + h.o.t. = 0, \\ \delta s_2^{\beta_2} + \bar{w}_3^{12} w_1^{13} M_1^1 \mu s_2^{\frac{\rho_2^2}{\lambda_2^1}} - M_3^3\mu + h.o.t. = 0. \end{cases} \tag{3.1}$$

Set $t_2 = s_2^{\frac{\rho_2^2}{\lambda_2^1}}, \alpha = \frac{\rho_2^2 + \lambda_2^1}{\rho_2^2}$ and rewrite the first equation of (3.1) as

$$w_1^{13}M_1^1\mu t_2 + M_3^1\mu + w_1^{33}(w_1^{31})^{-1}\delta t_2^{\alpha} + h.o.t. \triangleq L(t_2,\mu) - N(t_2,\mu) = 0, \tag{3.2}$$

where

$$L(t_2,\mu) = w_1^{13}M_1^1\mu t_2 + M_3^1\mu + h.o.t., \quad N(t_2,\mu) = -w_1^{33}(w_1^{31})^{-1}\delta t_2^{\alpha} + h.o.t..$$

Then we have

$$\begin{aligned} &L(0,\mu) - N(0,\mu) = M_3^1\mu + h.o.t., \\ &{L'}_{t_2}(t_2,\mu) - {N'}_{t_2}(t_2,\mu) = w_1^{13}M_1^1\mu + w_1^{33}(w_1^{31})^{-1}\alpha\delta t_2^{\alpha-1} + h.o.t.. \end{aligned}$$

If $w_1^{13}w_1^{33}w_1^{31}M_1^1\mu < 0$, equation ${L'}_{t_2}(t_2,\mu) - {N'}_{t_2}(t_2,\mu) = 0$ has a unique small positive solution $\tilde{t}_2 = (-(\alpha\delta w_1^{33})^{-1} w_1^{13} w_1^{31} M_1^1\mu)^{\frac{1}{\alpha-1}} + h.o.t..$

(1) If $w_1^{33}w_1^{31} < 0$, $\mu \in B_-^-$ or $w_1^{33}w_1^{31} > 0$, $\mu \in B_+^+$, the straight line L and the curve N can not intersect in the half plane for $t_2 > 0$, so equation (3.2) has

not any positive solutions, which means that system (1.1) only has the transversal heteroclinic orbit $\gamma_1(t)$ in the region $\Gamma_1 \cup \Gamma_2$.

(2) If $w_1^{33}w_1^{31} < 0$, $\mu \in B_-^+$ or $w_1^{33}w_1^{31} > 0$, $\mu \in B_+^-$, the straight line $L_2(t_2, \mu)$ and the curve $N_2(t_2, \mu)$ intersect at one positive point, then (3.2) has one positive solution.

Without loss of generality, we discuss the case $w_1^{33}w_1^{31} < 0$, $\mu \in B_-^+$. There are

$$L(0,\mu) > N(0,\mu), {L'}_{t_2}(t_2,\mu) < {N'}_{t_2}(t_2,\mu), L(\bar{t}_2,\mu) - N(\bar{t}_2,\mu) = w_1^{33}(w_1^{31})^{-1}\delta\bar{t}_2^{\alpha} < 0,$$

where $\bar{t}_2 = -\frac{M_3^1\mu}{w_1^{13}M_1^1\mu} + h.o.t.$.

When $\left|M_3^1\mu\right| = O(\left|M_1^1\mu\right|)$, $0 < \bar{t_2} \ll 1$, (3.2) has a unique solution t_2^{1*} satisfying $0 < t_2^{1*} < \bar{t_2} \ll 1$. Putting it into the second equation of (3.1) yields $(t_2^{1*})^{\frac{\rho_2^1}{\rho_2^2}}\delta + \bar{w}_3^{12}M_1^1\mu w_1^{13}(t_2^{1*}) - M_3^3\mu + h.o.t. = 0$, it defines a surface

$$L_2^1(\mu) = \left\{\mu : \delta(M_3^1\mu)^{\frac{\rho_2^1}{\rho_2^2}} = (-w_1^{13}M_1^1\mu)^{\frac{\rho_2^1}{\rho_2^2}}M_3^3\mu + h.o.t.\right\},$$

with a normal surface $\Sigma = span\{M_3^3\}$ at $\mu = 0$ for $M_3^3\mu > 0$. That is to say, system (1.1) has a heteroclinic orbit $\gamma_1(t)$ consisting of p_1 and p_3 near $\Gamma_1 \cup \Gamma_3$ for $\mu \in L_2^1(\mu)$.

(3) If $w_1^{33}w_1^{31} < 0$, $\mu \in B_+^+$ or $w_1^{33}w_1^{31} > 0$, $\mu \in B_-^-$, there are two special cases:

(i) As $\left|M_1^1\mu\right| t_2 \ll \max\{t_2^{\alpha},\ \left|M_3^1\mu\right|\}$, equation (3.2) can be simplified to be

$$w_1^{33}(w_1^{31})^{-1}\delta t_2^{\alpha} + M_3^1\mu + h.o.t. = 0. \tag{3.3}$$

It has a solution $t_2^{2*} = \left(\frac{-w_1^{31}M_3^1\mu}{w_1^{33}\delta}\right)^{\frac{1}{\alpha}} + h.o.t.$ or $s_2^{2*} = \left(\frac{-w_1^{31}M_3^1\mu}{w_1^{33}\delta}\right)^{\frac{\lambda_2^1}{\rho_2^2+\lambda_2^1}} + h.o.t.$. Substituting s_2^{2*} into the second equation of (2.13), we can get a surface

$$L_2^2(\mu) = \left\{\mu : \delta(M_3^1\mu)^{\frac{\lambda_2^1}{\rho_2^2+\lambda_2^1}} = (-w_1^{33}(w_1^{31})^{-1}\delta)^{\frac{\lambda_2^1}{\rho_2^2+\lambda_2^1}}M_3^3\mu + h.o.t.\right\}$$

tangent to $L_{23}(\mu)$ for $M_3^3\mu > 0$. So system (1.1) has a heteroclinic orbit consisting of p_1 and p_3 near $\Gamma_1 \cup \Gamma_3$ for $\mu \in L_2^2(\mu)$. Next putting t_2^{2*} into the verification condition, it is equivalently $\left|M_1^1\mu\right| \ll \left|M_3^1\mu\right|^{1-\frac{1}{\alpha}}$.

(ii) As $\max\{\left|M_1^1\mu\right| t_2,\ t_2^{\alpha}\} \gg \left|M_3^1\mu\right|$, equation (3.2) is then

$$w_1^{33}(w_1^{31})^{-1}\delta t_2^{\alpha} + w_1^{13}M_1^1\mu t_2 + h.o.t. = 0, \tag{3.4}$$

there is a small positive solution $t_2^{3*} = \left(-\frac{w_1^{13}w_1^{31}M_1^1\mu}{w_1^{33}\delta}\right)^{\frac{1}{\alpha-1}} + h.o.t.$ or $s_2^{3*} = -\frac{w_1^{13}w_1^{31}M_1^1\mu}{w_1^{33}\delta} + h.o.t.$. In the same way, we can get the surface $L_2^3(\mu)$ which is tangent to L_{23} with the condition $\left|M_1^1\mu\right| \gg \left|M_3^1\mu\right|^{1-\frac{1}{\alpha}}$, where

$$L_2^3(\mu) = \left\{\mu : \delta(w_1^{13}M_1^1\mu)^{\beta_2} = (-\delta w_1^{33}(w_1^{31})^{-1})^{\beta_2}M_3^3\mu + h.o.t.\right\}.$$

So system (1.1) has a heteroclinic orbit consisting of p_1 and p_3 in the region $\Gamma_1 \cup \Gamma_3$ for $\mu \in L_2^3(\mu)$.

(4) If $w_1^{33}w_1^{31} < 0$, $\mu \in B_+^-$ or $w_1^{33}w_1^{31} > 0$, $\mu \in B_-^+$, we have $L(0,\mu) < N(0,\mu)$, $L''_{t_2}(t_2,\mu) - N''_{t_2}(t_2,\mu) = w_1^{33}(w_1^{31})^{-1}\alpha(\alpha-1)\delta t_2^{\alpha-2} + h.o.t. < 0$.

Set $L(\tilde{t}_2,\mu) - N(\tilde{t}_2,\mu) = H_2^1(\mu)$, where $\tilde{t}_2 = (-\frac{w_1^{31}w_1^{13}M_1^1\mu}{w_1^{33}\alpha\delta})^{\frac{1}{\alpha-1}} + h.o.t.$ is the solution of $L'(t_2,\mu) - N'(t_2,\mu) = 0$ and

$$H_2^1(\mu) = w_1^{13}M_1^1\mu(1-\alpha^{-1})\Big(-w_1^{31}w_1^{13}(w_1^{33}\delta\alpha)^{-1}M_1^1\mu\Big)^{\frac{1}{\alpha-1}} + M_3^1\mu + h.o.t..$$

When $H_2^1(\mu) > 0$, the straight line $L(t_2,\mu)$ intersects the curve $N(t_2,\mu)$ exactly at two points $0 < t_2^{4*} < \tilde{t}_2 < t_2^{5*}$, which means equation (3.2) has two positive solutions. Therefore, system (1.1) has two heteroclinic orbits connecting p_1 and p_3 near $\Gamma_1 \cup \Gamma_3$.

When $H_2^1(\mu) = 0$, the equations ${L'}_{t_2}(t_2,\mu) = {N'}_{t_2}(t_2,\mu)$ and $L(t_2,\mu) = N(t_2,\mu)$ have the solution $\tilde{t}_2$, therefore the straight line $L(t_2,\mu)$ must be tangent to the curve $N(t_2,\mu)$ at the point $\tilde{t}_2$. Putting it into the second equation of (3.1) yields a surface $L_2^4(\mu)$ with a normal surface $\Sigma = span\{M_3^3\}$ at $\mu = 0$, where

$$L_2^4(\mu) = \big\{\mu : \delta(w_1^{13}M_1^1\mu)^{\beta_2} = (-\alpha\delta w_1^{33}(w_1^{31})^{-1})^{\beta_2}M_3^3\mu + h.o.t.\big\}$$

for $M_3^3\mu > 0$. Then, system (1.1) has a 2-fold heteroclinic orbit connecting p_1 and p_3 near $\Gamma_1 \cup \Gamma_3$.

When $H_2^1(\mu) < 0$, the straight line $L_2(t_2,\mu)$ does not intersect the curve $N_2(t_2,\mu)$ in the half plane, then there is only the transversal heteroclnic orbit $\gamma_1(t)$ connecting p_1 and p_2 near $\Gamma_1 \cup \Gamma_2$.

(5) If $\mu \in \{\mu : M_3^1\mu + h.o.t. = M_3^3\mu + h.o.t. = 0\}$, equation (3.1) is

$$\begin{cases} w_1^{33}(w_1^{31})^{-1}\delta s_2^{\frac{\rho_2^2}{\lambda_2^1}+1} + w_1^{13}M_1^1\mu s_2^{\frac{\rho_2^2}{\lambda_2^1}} + h.o.t. = 0, \\ \delta s_2^{\beta_2} + \bar{w}_3^{12}w_1^{13}M_1^1\mu s_2^{\frac{\rho_2^2}{\lambda_2^1}} + h.o.t. = 0. \end{cases} \tag{3.5}$$

To solve the first equation of (3.5), there is

$$(w_1^{33}(w_1^{31})^{-1}\delta s_2 + w_1^{13}M_1^1\mu)s_2^{\frac{\rho_2^2}{\lambda_2^1}} + h.o.t. = 0,$$

we can get two solutions $s_2^{'} = 0$ and $s_2^{''} = -\frac{w_1^{31}w_1^{13}M_1^1\mu}{w_1^{33}\delta} + h.o.t.$ for $(w_1^{13}M_1^1\mu)\cdot(w_1^{33}w_1^{31}) < 0$. While for $(w_1^{13}M_1^1\mu)\cdot(w_1^{33}w_1^{31}) \geq 0$, there is only a zero solution. Equation (3.5) finally defines a surface

$$\bar{L}_2^3(\mu) = \big\{\mu : M_3^1\mu + h.o.t. = M_3^3\mu + h.o.t. = 0, (w_1^{13}M_1^1\mu)\cdot(w_1^{33}w_1^{31}) < 0\big\}.$$

Putting the expression $s_2^{''}$ into the second equation of (3.5) obtains the set of μ as $\{\mu|\delta(-\frac{w_1^{13}w_1^{31}M_1^1\mu}{w_1^{33}\delta})^{\beta_2} + \bar{w}_3^{12}M_1^1\mu w_1^{13}(-\frac{w_1^{13}w_1^{31}M_1^1\mu}{w_1^{33}\delta})^{\frac{\rho_2^2}{\lambda_2^1}} = 0, M_1^1\mu \neq 0\}$, which means system (1.1) has two types of heteroclinic orbits: a large 1-heteroclinc orbit connecting with p_1 and p_3 and two heteroclinic orbits connecting with p_1 and p_2 and with p_2 and p_3 respectively in the region $\Gamma_1 \cup \Gamma_3$ as $\mu \in \overline{L}_2^3(\mu)$.

The analysis of the third and the fourth equations of (2.13) is similar to that of the first and the second equations of (2.13), we omit the details and give the main results in the following.

(1) If $w_4^{33}w_4^{31} > 0$, $\mu \in N_-^-$ or $w_4^{33}w_4^{31} < 0$, $\mu \in N_+^+$, system (1.1) only has the transversal heteroclinic orbit $\gamma_4(t)$ in the region $\Gamma_2 \cup \Gamma_3$.

(2) If $w_4^{33}w_4^{31} > 0$, $\mu \in N_-^+$ or $w_4^{33}w_4^{31} < 0$, $\mu \in N_+^-$, system (1.1) has a heteroclinic orbit connecting p_3 and p_1 in the region $\Gamma_1 \cup \Gamma_3$ for $\mu \in L_4^1(\mu)$ and $\left|M_2^1\mu\right| = o(\left|M_4^1\mu\right|)$, where

$$L_4^1(\mu) = \left\{\mu : \delta(M_2^1\mu)^{\frac{\rho_2^1}{\rho_2^2}} + (-w_4^{13}M_4^1\mu)^{\frac{\rho_2^1}{\rho_2^2}} M_2^3\mu + h.o.t. = 0\right\}.$$

(3) If $w_4^{33}w_4^{31} > 0$, $\mu \in N_+^+$ or $w_4^{33}w_4^{31} < 0$, $\mu \in N_-^-$,

(i) as $\left|M_4^1\mu\right| \ll \left|M_2^1\mu\right|^{1-\frac{1}{\alpha}}$, system (1.1) has a heteroclinic orbit connecting p_3 and p_1 near $\Gamma_1 \cup \Gamma_3$ for $\mu \in L_4^2(\mu)$, where

$$L_4^2(\mu) = \left\{\mu : \delta\big(M_2^1\mu\big)^{\frac{\rho_2^1}{\rho_2^2+\lambda_2^1}} + ((w_4^{31})^{-1}w_4^{33}\delta)^{\frac{\rho_2^1}{\rho_2^2+\lambda_2^1}} M_2^3\mu + h.o.t.\right\};$$

(ii) as $\left|M_4^1\mu\right| \gg \left|M_2^1\mu\right|^{1-\frac{1}{\alpha}}$, system (1.1) has a heteroclinic orbit connecting p_3 and p_1 in the region $\Gamma_1 \cup \Gamma_3$ for $\mu \in L_4^3(\mu)$, where

$$L_4^3(\mu) = \left\{\mu : \delta(M_4^1\mu)^{\beta_2} + ((w_4^{31}w_4^{13})^{-1}w_4^{33}\delta)^{\beta_2} M_2^3\mu + h.o.t. = 0\right\}.$$

(4) If $w_4^{33}w_4^{31} > 0$, $\mu \in N_+^-$ or $w_4^{33}w_4^{31} < 0$, $\mu \in N_-^+$ and $H_4^1(\mu) > 0$, system (1.1) has two heteroclinic orbits connecting p_3 and p_1 near $\Gamma_1 \bigcup \Gamma_3$; when $H_4^1(\mu) = 0$, system (1.1) has a 2-fold heteroclinic orbit connecting p_3 and p_1 near $\Gamma_1 \cup \Gamma_3$ for $\mu \in L_4^4(\mu)$; when $H_4^1(\mu) < 0$, there is no heteroclinic orbit connecting p_3 and p_1 near $\Gamma_1 \cup \Gamma_3$, where

$$H_4^1(\mu) = w_4^{13}M_4^1\mu(1 - \frac{1}{\alpha})\Big(w_4^{13}w_4^{31}(w_4^{33}\delta\alpha)^{-1}M_4^1\mu\Big)^{\frac{1}{\alpha-1}} + M_2^1\mu + h.o.t.$$

and

$$L_4^4(\mu) = \left\{\mu : \delta(w_4^{13}w_4^{31}M_4^1\mu)^{\beta_2} + (\alpha\delta w_4^{33})^{\beta_2} M_2^3\mu + h.o.t. = 0\right\}.$$

(5) If $\mu \in \{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0\}$, system (1.1) has two types of heteroclinic orbits: a large 1-heteroclinc orbit connecting p_3 and p_1 and two heteroclinic orbits connecting p_3 and p_2 and p_2 and p_1 respectively in the region $\Gamma_1 \cup \Gamma_3$ as $\mu \in \overline{L}_4^3(\mu)$, where

$$\bar{L}_4^3(\mu) = \left\{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0, (w_4^{13}M_4^1\mu) \cdot (w_4^{33}w_4^{31}) > 0\right\}.$$

As well as, when $w_4^{13}w_4^{33}w_4^{31}M_4^1\mu \leq 0$, the third equation of (2.13) only has zero solution.

With the above analysis, we can get the following theorems about existence of the second and the third shape heterodimensional cycle and the large 1-heteroclinic cycle under small perturbation.

Theorem 3.2. *Under* (A_1)*-*(A_5) *and Rank*$(M_2^1, M_2^3, M_3^1, M_3^3) \geq 3$*, as well as* $\mu \in \{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0\}$.

(1) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, \mu \in B_-^-\}$ *or* $\mu \in \{\mu : w_1^{33}w_1^{31} > 0, \mu \in B_+^+\}$, *system* (1.1) *has the third shape heterodimensional cycle in the* $(l-2)$*-dimensional surface* $L_1^1(\mu)$ *with normal vector span*$\{M_2^1, M_2^3\}$ *at* $\mu = 0$, *where*

$$L_1^1(\mu) = \left\{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0, w_4^{13}w_4^{33}w_4^{31}M_4^1\mu \le 0\right\}.$$

(2) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, \mu \in B_+^-\}$ *or* $\mu \in \{\mu : w_1^{33}w_1^{31} > 0, \mu \in B_-^+\}$, *system* (1.1) *has the third shape heterodimensional cycle near* Γ *as* $\mu \in \tilde{L}_2^4$, *where*

$$\tilde{L}_2^4 = \left\{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0, w_4^{13}w_4^{33}w_4^{31}M_4^1\mu \le 0, H_2^1 < 0\right\}.$$

(3) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, \mu \in B_-^+\}$ *or* $\mu \in \{\mu : w_1^{33}w_1^{31} > 0, \mu \in B_+^-\}$, *system* (1.1) *has the second shape hetrodimensional cycle near* Γ *in an* $(l-1)$*-dimensional surface*

$$\hat{L}_2^1 = \left\{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0, w_4^{13}w_4^{33}w_4^{31}M_4^1\mu \le 0\right\} \cap L_2^1(\mu)$$

with normal vector span$\{M_2^1, M_2^3, M_3^3\}$ *at* $\mu = 0$ *for* $\left|M_3^1\mu\right| = o(\left|M_1^1\mu\right|)$, *which is tangent to the surface* $L_{23}(\mu)$ *at* $\mu = 0$.

(4) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, \mu \in B_+^+\}$ *or* $\mu \in \{\mu : w_1^{33}w_1^{31} > 0, \mu \in B_-^-\}$, *there exists two* $(l-1)$*-dimensional surfaces*

$$\hat{L}_2^2 = \left\{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0, w_4^{13}w_4^{33}w_4^{31}M_4^1\mu \le 0\right\} \cap L_2^2(\mu)$$

and

$$\hat{L}_2^3 = \left\{\mu : M_2^1\mu + h.o.t. = M_2^3\mu + h.o.t. = 0, w_4^{13}w_4^{33}w_4^{31}M_4^1\mu \le 0\right\} \cap L_2^3(\mu)$$

for $\left|M_1^1\mu\right| \ll \left|M_3^1\mu\right|^{1-\frac{1}{\alpha}}$ *and* $\left|M_1^1\mu\right| \gg \left|M_3^1\mu\right|^{1-\frac{1}{\alpha}}$ *respectively, such that system* (1.1) *has the second shape heterodimensional cycle near* Γ *as* $\mu \in \hat{L}_2^2$, $\mu \in \hat{L}_2^2$, *respectively, and* $0 < |\mu| \ll 1$.

Remark 3.1. The second heterodimensional cycle consists of two saddles of (1,2) type and one saddle of (2,1) type and is composed of one big orbit linking p_1, p_3 and two orbits linking p_3, p_2 and p_2, p_1 respectively (see Figure 3).

Remark 3.2. The existence of another second shape heterdimensional cycle composed of one big orbit linking p_3, p_1 and two orbits linking p_1, p_2 and p_2, p_3 respectively is analogous to Theorem 3.2, we do not repeat here.

Theorem 3.3. *Suppose* (A_1)*-*(A_5) *are valid and Rank* $(M_2^1, M_2^3, M_3^1, M_3^3) \ge 3$.

(1) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, w_4^{33}w_4^{31} > 0, \mu \in B_-^+ \cup N_-^+\}$ *or* $\mu \in \{\mu : w_1^{33}w_1^{31} > 0, w_4^{33}w_4^{31} < 0, \mu \in B_+^- \cup N_+^-\}$, *there exists an* $(l-2)$*-dimensional surface*

$$H_{24}^1(\mu) = \left\{\mu : \left|M_3^1\mu\right| = o(\left|M_1^1\mu\right|), \left|M_2^1\mu\right| = o(\left|M_4^1\mu\right|), \mu \in L_2^1(\mu) \cap L_4^1(\mu)\right\}$$

with normal vector span$\{M_2^1, M_3^1\}$ *at* $\mu = 0$, *which is tangent to the surface* $L_{23}(\mu)$, *system* (1.1) *has a 1-fold large 1-heteroclinic cycle near* Γ *as* $\mu \in H_{24}^1(\mu)$ *and* $0 < |\mu| \ll 1$.

(2) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, w_4^{33}w_4^{31} > 0, \mu \in B_+^+ \cup N_+^+\}$ *or* $\mu \in \{\mu : w_1^{33}w_1^{31} > 0, w_4^{33}w_4^{31} < 0, \mu \in B_-^- \cup N_-^-\}$, *there exists two* $(l-2)$*-dimensional surface*

$$H_{24}^2(\mu) = \left\{\mu : \left|M_1^1\mu\right| \ll \left|M_3^1\mu\right|^{1-\frac{1}{\alpha}}, \left|M_4^1\mu\right| \ll \left|M_2^1\mu\right|^{1-\frac{1}{\alpha}}, \mu \in L_2^2(\mu) \cap L_4^2(\mu)\right\}$$

and

$$H_{24}^3(\mu) = \left\{\mu : \left|M_1^1\mu\right| \ll \left|M_3^1\mu\right|^{1-\frac{1}{\alpha}}, \left|M_4^1\mu\right| \ll \left|M_2^1\mu\right|^{1-\frac{1}{\alpha}}, \mu \in L_2^3(\mu) \cap L_4^3(\mu)\right\}$$

both with normal vector span$\{M_2^1, M_3^1\}$ *at* $\mu = 0$ *and tangent to the surface* $L_{23}(\mu)$, *system* (1.1) *has a 1-fold large 1-heteroclinic cycle near* Γ *as* $\mu \in H_{24}^2(\mu)$ *and* $\mu \in H_{24}^3(\mu)$, *respectively, and* $0 < |\mu| \ll 1$.

(3) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, w_4^{33}w_4^{31} > 0, \mu \in B_+^- \cup N_+^-\}$, *and* $H_2^1 > 0$, $H_4^1 > 0$, *system* (1.1) *has two 1-fold large 1-heteroclinic cycles near* Γ.

(4) *If* $\mu \in \{\mu : w_1^{33}w_1^{31} > 0, w_4^{33}w_4^{31} < 0, \mu \in B_-^+ \cup N_-^+\}$, *there exists an* $(l-2)$-*dimensional surface* H_{24}^4 *with normal vector span*$\{M_2^1, M_3^1\}$ *and tangent to the surface* $L_{23}(\mu)$ *at* $\mu = 0$, *where*

$$H_{24}^4(\mu) = \left\{\mu : H_2^1 = 0, H_4^1 = 0, \mu \in L_2^4 \cap L_4^4\right\},$$

system (1.1) *has one 2-fold large 1-heteroclinic cycles near* Γ *for* $\mu \in H_{24}^4$.

Finally the coexistence of the large 1-heteroclinc cycle, the second shape heterodimensional cycle and double heterodimensional cycles are concluded in the the last theorem.

Theorem 3.4. *Suppose* (A_1)-(A_5) *are valid, Rank* $(M_2^1, M_2^3, M_3^1, M_3^3) \geq 3$ *and* $0 < |\mu| \ll 1$.

(1) *System* (1.1) *does not have any types of heteroclinic cycles coexisting with the persistent heterodimensional cycle* Γ *as* $\mu \in \{\mu : \mu \in L_{23}(\mu), w_1^{13}w_1^{33}w_1^{31}M_1^1\mu > 0, w_4^{13}w_4^{33}w_4^{31}M_4^1\mu < 0\}$.

(2) *System* (1.1) *has exactly the second shape heterodimensional cycle coexisting with the 1-fold large 1-heteroclinic cycle near* Γ *if* $\mu \in \{\mu : w_4^{33}w_4^{31} > 0, \mu \in N_+^-, H_4^1(\mu) > 0, \mu \in \overline{L}_2^3(\mu)\}$.

(3) *System* (1.1) *has exactly the third heterdimensional cycle coexisting with the another large 1-heteroclinic cycle near* Γ *if* $\mu \in \{\mu : w_1^{33}w_1^{31} < 0, \mu \in B_+^-, H_2^1(\mu) = 0, \mu \in L_2^4 \cap \overline{L}_4^3(\mu)\}$.

(4) *System* (1.1) *has exactly the double heterodimensional cycles coexisting with the another large 1-heteroclinic cycle near* Γ *if* $\mu \in L_{23}(\mu)$ *and* $\mu \in \overline{L}_2^3(\mu) \cap \overline{L}_4^3(\mu)$.

Remark 3.3. "Another large 1-heteroclinic cycle" in the third and the fourth conclusions of Theorem 3.4 means the 2 fold heteroclinc orbit connecting p_1 and p_3 and the 1 fold heteroclinic orbit connecting p_3 and p_1.

4. Conclusion

In the paper, we obtain the coexistence conditions of the large 1-heteroclinic orbits and the persistent ∞-shape double heterodimensional cycles for the first time. As well as, the coexistence conditions of the large 1-heteroclinic cycle and the heteroclinic cycle composed of three orbits connecting three saddle points, and two heteroclinic orbits, respectively. These results are an effective supplement of ∞-shape double heterodimensional cycles bifurcation with three saddles points and have profound theoretical significance.

Since the problem we studied has multiple equilibria and the dimension of the stable (unstable) manifolds at each equilibrium point is different, the analysis is rarely difficult, especially for giving expressions of bifurcation equations. In fact, there are eight successor functions and four bifurcation equations. Under this circumstance, we only study some specific orbit bifurcations and obtain some bifurcation results as much as possible, such as the large 1-heteroclinic cycle bifurcation. For large n-heteroclinic cycle bifurcation and the other cases, we leave it for future maybe with computer assistance.

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