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STANDING WAVE SOLUTIONS FOR THE GENERALIZED MODIFIED CHERN-SIMONS-SCHRÖDINGER SYSTEM

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Abstract In this paper, we study the modified gauged Schrödinger equation under some assumptions on the functions V and f. By using dual approach, Jeanjean's monotone trick and Mountain Pass Theorem, we obtain the standing wave solutions for the generalized modified Chern-Simons-Schrödinger system.

Keywords Chern–Simons–Schrödinger system, dual approach, Nehari–Pohožaev identity, monotone trick, mountain pass theorem.

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1. Introduction

In this paper, we are concerned with the following quasilinear Chern–Simons–Schrödinger equation with general nonlinearity as follows:

$$\begin{aligned}& -\Delta u+V(x)u-\kappa u\Delta(u^2)+q\frac{h^2(|x|)}{|x|^2}(1+\kappa u^2)u,\\ & +q\Big(\int_{|x|}^{+\infty}\frac{h(s)}{s}(2+\kappa u^2(s))u^2(s)\mathrm{d}s\Big)u=f(u)\qquad \text{in }\ \mathbb{R}^2,\end{aligned}\tag{1.1}$$

where $u:\mathbb{R}^2\to\mathbb{R}$ is a radially symmetric function, κ, q are positive constants, $h(l)=\int_0^l u^2(s)s\mathrm{d}s$ $(l\geq 0)$.

If $q=0$, (1.1) reduces to the following quasilinear elliptic problem

$$-\Delta u+V(x)u-\kappa u(\Delta u^2)=f(u)\qquad \text{in }\ \mathbb{R}^2.\tag{1.2}$$

The solutions of (1.2) are related to the solitray wave solutions for the following quasilinear Schrödinger equation

$$i\phi_t+\Delta\phi-W(x)\phi+\kappa\phi\Delta(|\phi|^2)+h(|\phi|^2)\phi=0\qquad \text{in }\ \mathbb{R}^2.\tag{1.3}$$

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where $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^2 \to \mathbb{R}$ is a given potential, $h : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a suitable function. The quasilinear equation of the form (1.3) was used for the superfluid film equation in fluid mechanics by Kurihara [21]. For more physical background, we can refer to [2,29] and references therien. Set $\phi = e^{-iwt}u(x)$, where $w \in \mathbb{R}$ and u is a real function satisfies (1.3) if and only if the function u solves the Eq. (1.2) with $V(x) = W(x) - w$ a new potential.

Compared to the semilinear problem, the quasilinear case ($k \neq 0$) becomes much more complicated since the effect of the non-convex term. A major difficulty of (1.2) here is that the natural functional corresponding to (1.2) is not well defined for all $u \in H_r^1(\mathbb{R}^2)$. In recent years, some ideas and methods have been used to overcome this difficulty. For example, by using a constrained minimization, Ruiz and Siciliano [36] proved for the first time that (1.2) has a ground state solution, by using the change of variables, the problem (1.2) was transformed into a related semilinear problem in [27]. Along this lines, there have been a large number of works about standing wave solutions of problem (1.2), we refer the reader to [5–7, 13, 44, 45, 48, 50, 51] and the references therein.

If $\kappa = 0$, (1.1) turns into the following nonlocal elliptic problem

$$-\Delta u + V(x)u + q\frac{h^2(|x|)}{|x|^2}u + 2q\Big(\int_{|x|}^{+\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s\Big)u = f(u), \tag{1.4}$$

Eq. (1.4) is usually to seek the standing waves of the following nonlinear Chern-Simons-Schrödinger system

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + f(\phi) = 0, \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}(\overline{\phi}D_2\phi), \\ \partial_0 A_2 - \partial_2 A_0 = -\mathrm{Im}(\overline{\phi}D_1\phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2, \end{cases} \tag{1.5}$$

where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar field, $A_j : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_j = \partial_j + iA_j$ is the covariant derivative for $j = 0, 1, 2$. The Chern–Simons–Schrödinger system was first proposed and studied by Jackiw and Pi [16–18], consisting of Schrödinger equation augmented by the gauge field. The two-dimensional Chern–Simons–Schrödinger system is a non-relativistic quantum model describing the dynamics of a large number of particles in the plane, in which these particles interact directly and through the spontaneous electromagnetic field. Furthermore, this feature of the system is important for the study of the high-temperature superconductor, fractional quantum Hall effect and Aharovnov–Bohm scattering. For the further mathematical and physical backgrounds of (1.5), we refer readers to [25,26], and the references therein.

As usual in Chern–Simons theory, (1.5) is invariant under the gauge transofrmation

$$\phi \mapsto \phi e^{i\chi}, \qquad A_j \mapsto A_j - \partial_j\chi,$$

where $\chi : \mathbb{R}^{1+2} \to \mathbb{R}$ is an arbitrary C^∞ function. The existence of standing waves of (1.5) with power function type nonlinearity, that is $f(u) = \lambda|u|^{p-1}u (p > 1, \lambda > 0)$, has been investigated in [3, 4, 12, 23, 24, 34], they look for solutions for (1.5) of type

$$\phi(t, x) = u(|x|)e^{iwt}, \qquad A_0(t, x) = k(|x|),$$

$$A_1(t,x) = \frac{x_2}{|x|^2}h(|x|), \qquad A_2(t,x) = -\frac{x_1}{|x|^2}h(|x|), \tag{1.6}$$

where $w > 0$ is a given frequency, $\lambda > 0$ and $p > 1$, u, k, h are real valued functions depending only on $|x|$. The ansatz (1.6) satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$. Byeon et al. [3] got the following nonlocal semi-linear elliptic equation

$$-\Delta u + wu + \frac{h^2(|x|)}{|x|^2}u + \Big(\int_{|x|}^{+\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s\Big)u = \lambda |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^2. \tag{1.7}$$

Because of the appearance of the Chern-Simons term

$$\Big(\int_{|x|}^{+\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}\Big)u,$$

Eq. (1.7) is not a pointwise identity any more. This nonlocal term cause some mathematical difficulties that make the study of it is rough and particulary interesting. Following [3], (1.7) possesses a variational structure, that is, the standing wave solutions are obtained as critical points of the energy functional associated to (1.7). For the case $p \in (1,3)$, Pomponio and Ruiz [34] have studied the existence and nonexistence of positive solutions for (1.7) under the different range of frequency w. By the method of invariant sets of descending flow and a novel perturbation approach, the authors in [24] studied the existence and multiplicity of sign-changing solutions. For the case $p \in (5, +\infty)$, in [15], Huh studied the existence of infinitely many solutions by using the Mountain Pass Theorem. This result was improved by Seok [38]. After then, the authors in [23] proved the existence of least energy sign-changing radial solutions. In [12], the authors studied the existence of multiple nodal solutions. For the case $p \in (3,5)$, motivated by [35], by using a constraint minimization taking into account the Nehari-Poho*ž*aev manifold, the authors in [3] obtained a positive solution of problem (1.7). For the nice properties of the generalized Nehari manifold, we refer to previous works in [31,32] and references therein. Luo [28] obtained the existence, multiplicity, quantitative property and asymptotic behavior of normalized solutions with prescribed L^2-norm for Eq. (1.7). For the case $p = 3$, the initial value problem, wellposedness, global existence, blow-up and scattering etc. have been considered in [1,14,25,26,30]. Furthermore, many authors study the general nonlinearity case. For example, Wan and Tan [42] studied the existence, non-existence and multiplicty of standing waves for asymptotically 1-linear nonlinearity case. Tang et al. [40] proved the existence and multiplicty of nontrivial solutions which generalized the result in [15]. Ji et al. [20] proved the existence of positive solutions for elliptic equations with the critical exponential growth. Li et al. [22] generalizes the results of [20]. For more related work about the system, we refer to [8,10,33,39,41,47,49,52] and the references therein.

To best of our knowledge, based on the work of [3], there are few articles focused on Chern-Simons term for modified Schrödinger equation, except for [11] which proved the existence and nonexistence of solutions for (1.1) without potential V replacing with a positive constant w and $f(u) = |u|^{p-1}u (p > 1)$ by using constrained minimizatioin, the Pohozaev–Nehari manifold and [21] which founded the ground state solutions of (1.1) involving symmetric variable potential V and $f(u) = |u|^{p-1}u (p > 5)$ by using the change of variables which reduces the quasilinear problem to a semilinear one. Motivated by arguments in [7,51], in the present paper, we also try to consider problem (1.1) with symmetric variable potential by a dual approach, and then establish the existence of ground state solutions and infinitely many solutions by Jeanjean's monotone trick and Mountain Pass Theorem, respectively. In order to state the first result of this paper, we give the following conditions on potential $V \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$:

(V_1) $V(x) = V(|x|)$ and there exists $0 < \beta \leq \gamma$, such that $\beta \leq V(x) \leq \gamma$ for all $x \in \mathbb{R}^2$.

(V_2) The function $\nabla V(x) \cdot x \geq 0$ for all $x \in \mathbb{R}^2$.

(f_1) $\lim\limits_{|s| \to 0} \frac{f(s)}{s} = 0$ and there exist constants $C > 0$ and $q \in (2, +\infty)$ such that

$$|f(s)| \leq C(1 + |s|^{q-1}), \quad \forall s \in \mathbb{R}.$$

(f_2) There exists a constant $p \in (6, +\infty)$ such that $\lim\limits_{|s| \to +\infty} \frac{F(s)}{|s|^p} = +\infty$, where $F(s) = \int_0^s f(t)\mathrm{d}t$.

(f_3) There exists two constants $\epsilon > 8$ and $\xi \in (0, \frac{\epsilon\beta - 2}{2\epsilon}]$ such that

$$\frac{1}{2\epsilon} s f(s) - F(s) + \xi s^2 \geq 0, \qquad \forall s \in \mathbb{R},$$

and $sf(s) \geq 0$ for all $s \in \mathbb{R}^2$.

Let $X = H_r^1(\mathbb{R}^2)$, in general, the problem like (1.1) has an energy functional $I : X \to \mathbb{R}$ of the form

$$\begin{aligned} I(u) =& \frac{1}{2} \int_{\mathbb{R}^2} \Big((1 + 2\kappa u^2)|\nabla u|^2 + V(x)u^2 \Big) \mathrm{d}x + \frac{q}{2} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \Big(\int_0^{|x|} s u^2(s) \mathrm{d}s \Big)^2 \mathrm{d}x \\ &+ \frac{q}{4} \kappa \int_{\mathbb{R}^2} \frac{u^4(x)}{|x|^2} \Big(\int_0^{|x|} s u^2(s) \mathrm{d}s \Big)^2 \mathrm{d}x - \int_{\mathbb{R}^2} F(u) \mathrm{d}x. \end{aligned}$$

It is well known that I is not well defined in general in X and the term $\int_{\mathbb{R}^2} u^2 |\nabla u|^2 \mathrm{d}x$ is not convex. This cause that the usual variational techniques cannot be applied directly to I. To overcome this difficulty, we apply an argument developed by Colin-Jeanjean [9] and Liu et al [27]. We make use of a change of $u = g(v)$, where $g'(t) = 1/\sqrt{1 + 2g^2(t)}$ on $[0, +\infty)$ and $g(-t) = -g(t)$ on $(-\infty, 0]$, then define an associated equation that we shall call dual. If v is a weak solution of

$$\begin{aligned} &- \Delta v + V(x) g(v) g'(v) + q \frac{\hat{h}^2[g(v(|x|))]}{|x|^2} (1 + \kappa g^2(v)) g(v) g'(v) \\ &+ q \Big(\int_{|x|}^{+\infty} \frac{\hat{h}[g(v(s))]}{s} (2 + \kappa g^2(v(s)) g^2(v(s)) \mathrm{d}s \Big) g(v) g'(v) = f(g(v)) g'(v), \end{aligned} \tag{1.8}$$

where $\hat{h}^2[g(v(|x|))] := \Big(\int_0^{|x|} g^2(v(s)) s \mathrm{d}s \Big)^2$, then $u = g(v) \in X$ is a weak solution of (1.1). Therefore, after the change of variables, the energy functional on X associated to (1.8) in form can be transformed into

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^2} \Big(|\nabla v|^2 + V(x) g^2(v) \Big) \mathrm{d}x + \frac{q}{2} \mathcal{C}(g(v)) + \frac{q}{4} \kappa \mathcal{D}(g(v)) - \int_{\mathbb{R}^2} F(g(v)) \mathrm{d}x, \tag{1.9}$$

where

$$\mathcal{C}(g(v)) := \int_{\mathbb{R}^2} \frac{g^2(v(x))}{|x|^2}\Big(\int_0^{|x|} sg^2(v(s))\mathrm{d}s\Big)^2\mathrm{d}x,$$
$$\mathcal{D}(g(v)) := \int_{\mathbb{R}^2} \frac{g^4(v(x))}{|x|^2}\Big(\int_0^{|x|} sg^2(v)(s)\mathrm{d}s\Big)^2\mathrm{d}x.$$

We note from the Cauchy inequality that for some $C_0 > 0$,

$$\hat{h}^2[gv((|x|))] := \Big(\int_0^{|x|} sg^2(v(s))\mathrm{d}s\Big)^2 = \Big(\int_{B_{|x|}} \frac{1}{2\pi} g^2(v(y))\mathrm{d}y\Big)^2 \le C_0|x|^2\|g(v)\|_{L^4}^4.$$

Then for $v \in X$, we have

$$\mathcal{C}(g(v)) \le C_0\|g(v)\|_{L^4}^4\|g(v)\|_{L^2}^2, \tag{1.10}$$
$$\mathcal{D}(g(v)) \le C_0\|g(v)\|_{L^4}^8. \tag{1.11}$$

Our first result is as follows:

Theorem 1.1. *Assume that the conditions* (V_1)-(V_2) *and* (f_1)-(f_3) *are satisfied. Then problem* (1.1) *has a ground state solution.*

Another purpose of the present is to establish the infinitely many nontrivial solutions for problem (1.1). For this purpose, besides (V_1), we introduce the following assumptions on f:

(f_1') there exist constants c_1', $c_2' > 0$ and $2 < j < 8, j < r < +\infty$ such that

$$|f(s)| \le c_1'|s|^{j-1} + c_2'|s|^{r-1}, \quad \forall s \in \mathbb{R};$$

(f_2') $\lim\limits_{|s|\to+\infty} \frac{F(s)}{|s|^8} = +\infty$ and there exists $r_0 \ge 0$ such that $F(s) \ge 0$ for all $s \in \mathbb{R}$ and $|s| \ge r_0$;

(f_3') $\bar{F}(s) := \frac{1}{8}f(s)s - F(s) \ge 0$ and there exists $C_1 > 0$ and $\tau > 1$ such that

$$|F(s)|^\tau \le C_1|s|^{2\tau}\bar{F}(s)$$

for all $s \in \mathbb{R}$ with s large enough;

(f_4) $f(s) = -f(-s)$ for all $s \in \mathbb{R}$.

Our second result is as follows:

Theorem 1.2. *Assume that the conditions* (V_1), (f_1')-(f_3') *and* (f_4) *are satisfied. Then problem* (1.1) *possess infinitely many solutions* $\{u_n\}$ *such that* $\|u_n\| \to +\infty$ *and* $I(u_n) \to \infty$.

Remark 1.1. The authors in [6] considered the existence of ground state solutions for quasilinear equation, which improved the main results obtained in [45]. Very recently, using the similar agruments in [46], the authors generalize the results from quasilinear schrödinger equation [5] to modified Chern-Simons-Schrödinger equation and improved the main results in [11]. Furthermore, our results in Theorem 1.1-Theorem 1.2 extend some results for quasilinear schrödinger equation [6, 51] to the generalized modified Chern-Simons-Schrödinger systems.

An outline of this paper is as follows: In section 2, we give some preliminaries. In section 3, we complete the proof of Theorem 1.1 Consequently, the proof of Theorem 1.2 is given in section 4.

Notations. Throughout this paper, we make use of the following notations:

- $C, c_i, C_i (i = 0, 1, 2.....)$ possibly denote positive constants, not necessarily the same one;
- $L^r(\mathbb{R}^2)$ denotes the Lebesgue space with norm $\|v\|_{L^r} = \left(\int_{\mathbb{R}^2} |v|^r \mathrm{d}x\right)^{1/r}$, where $1 \le r < +\infty$;
- $H^1(\mathbb{R}^2)$ denotes a sobolev space with norm $\|v\| = \left(\int_{\mathbb{R}^2}(v^2 + |\nabla v|^2)\mathrm{d}x\right)^{1/2}$;
- $H^1_r(\mathbb{R}^2)\colon = \{v \in H^1(\mathbb{R}^2)\colon v(x) = v(|x|)\}$;
- " $\rightharpoonup$ " and " $\to$ " denote weak and strong convergence, respectively.

2. Variational framework and preliminaries

In this section, we show the variational framework and some preliminary lemmas which are crucial for proving our results. Let us recall some properties of the change of variables: $g : \mathbb{R} \to \mathbb{R}$, which are proved in [9, 27, 48] as follows:

Lemma 2.1 ([9, 27, 48]). *The function $g(t)$ and its derivative satisfy the following properties: (g_1) g is uniquely defined, C^∞ and invertible;*
(g_2) $\frac{g(t)}{t} \to 1$ *as* $t \to 0$;
(g_3) $|g(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
(g_4) $g(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to +\infty$;
(g_5) $|g(t)| \le 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$;
(g_6) $g(t)/2 \le tg'(t) \le g(t)$ *for all* $t > 0$;
(g_7) $g^2(t)/2 \le tf(t)g'(t) \le g^2(t)$ *for all* $t \in \mathbb{R}$;
(g_8) *there exists a positive constant C such that*

$$|g(t)| \ge \begin{cases} C|t|, & \text{if } |t| \le 1, \\ C|t|^{1/2}, & \text{if } |t| \ge 1; \end{cases}$$

(g_9) $|g'(t)| \le 1$ *for all* $t \in \mathbb{R}$;
(g_{10}) $|g(t)g'(t)| \le 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$;
(g_{11}) *for all each $\alpha > 0$, there exists a positive constant $C(\alpha)$ such that*

$$|g(\alpha t)|^2 \le C(\alpha)|g(t)|^2.$$

Arguing as in [3, 46] standard computation show that

Proposition 2.1. *The functional J is continuously differentiable on X and its critical point v is a weak solution of* (1.8).

Moreover, following Proposition2.1, for any $\psi \in X$,

$$\begin{aligned}\langle J'(v), \psi\rangle = &\int_{\mathbb{R}^2} \nabla v \nabla \psi \mathrm{d}x + \int_{\mathbb{R}^2} V(x)g(v)g'(v)\psi \mathrm{d}x + q\int_{\mathbb{R}^2} \Big\{\frac{\hat{h}^2[g(v(|x|))]}{|x|^2}[1+\kappa g^2(v)] \\ &+ \int_{|x|}^{+\infty} \frac{\hat{h}[g(v(s))]}{s}(2+\kappa g^2(v(s)))g^2(v(s))\mathrm{d}s\Big\} g(v)g'(v)\psi \mathrm{d}x \qquad (2.1) \\ &- \int_{\mathbb{R}^2} f(g(v))g'(v)\psi \mathrm{d}x.\end{aligned}$$

In particular, for $\varsigma = 2$ or $\varsigma = 4$, by using the integrate by parts, we have

$$\int_{\mathbb{R}^2} \frac{\hat{h}^2[g(v(|x|))]}{|x|^2} g^\varsigma(v)\mathrm{d}x = \int_{\mathbb{R}^2} \Big(\int_{|x|}^{+\infty} \frac{g^\varsigma(v(s))\hat{h}[g(v(s))]}{s} \mathrm{d}s \Big) g^2(v)\mathrm{d}x. \tag{2.2}$$

By a standard argument as in [3, 11, 45], we establish the following identities for a solution of (1.8).

Lemma 2.2. *Any weak solution of* (1.8) *satisfies Nehari identity* $N(v) = 0$ *and the Pohožaev identity* $P(v) = 0$, *where*

$$\begin{aligned} N(v) = &\int_{\mathbb{R}^2} \Big(|\nabla v|^2 + V(x)g(v)g'(v)v + q\frac{\hat{h}^2[g(v(|x|)]}{|x|^2}(1+\kappa g^2(v))g'(v)v \Big) dx \\ &+ q\Big(\int_{\mathbb{R}^2} \Big(\int_{|x|}^{+\infty} \frac{\hat{h}[g(v(s))]}{s}(2+\kappa g^2(v(s)))g^2(v(s))\, ds \Big) g(v)g'(v)v\, dx \Big) \\ &- \int_{\mathbb{R}^2} f(g(v))g'(v)v\, dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} P(v) = &\int_{\mathbb{R}^2} \Big(V(x)g^2(v) + \frac{1}{2}\nabla V(x)\cdot x g^2(v) \Big) dx + 2qC(g(v)) + q\kappa D(g(v) \\ &- 2\int_{\mathbb{R}^2} F(g(v))\, dx. \end{aligned} \tag{2.4}$$

Next, the following convergence lemma is necessary for proving the compactness:

Lemma 2.3 ([11]). *Suppose that a sequence* $\{v_n\}$ *converges weakly to a function* v *in* X *as* $n \to +\infty$. *Then for each* $\psi \in X$, $\mathcal{C}(v_n)$, $\mathcal{C}'(v_n)\psi$ *and* $\mathcal{C}'(v_n)v_n$, $\mathcal{D}(v_n)$ *and* $\mathcal{D}'(v_n)\psi$, $\mathcal{D}'(v_n)v_n$ *converges up to a subsequence to* $\mathcal{C}(v)$, $\mathcal{C}'(v)\psi$ *and* $\mathcal{C}'(v)v$, $\mathcal{D}(v)$ *and* $\mathcal{D}'(v)\psi$, $\mathcal{D}'(v)v$, *respectively, as* $n \to +\infty$.

Finally, we note that the following inequality holds only for the functions in X.

Proposition 2.2 ([3]). *For* $v \in X$, *the following inequality holds*

$$\int_{\mathbb{R}^2} |v|^4\, dx \le 2\Big(\int_{\mathbb{R}^2} |\nabla v|^2\, dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^2} \frac{v^2}{|x|^2} \Big(\int_0^{|x|} s v^2(s)\, ds \Big)^2 dx \Big)^{\frac{1}{2}}.$$

3. Existence of ground state solutions

To complete the proof of Theorem 1.1, we will use the following critical point theorem.

Theorem 3.1 ([19]). *Let* $(E, \|\cdot\|)$ *be a Banach space and let* $T \subset \mathbb{R}^+$ *be an interval. Consider a family* Φ_η *of* C^1 *functional on* E *of the form*

$$\Phi_\eta(v) = A(v) - \eta B(v), \qquad \forall \eta \in T,$$

where $B(v) \ge 0$ *and either* $A(v) \to +\infty$ *or* $B(v) \to +\infty$ *as* $\|v\| \to +\infty$. *Assume that there are two points* v_1, v_2 *such that*

$$c_\eta = \inf_{\gamma\in\Gamma_\eta} \max_{t\in[0,1]} \Phi_\eta(\gamma(t)) > \max\Big\{ \Phi_\eta(v_1), \Phi_\eta(v_2) \Big\}, \qquad \forall \eta \in T,$$

where $\Gamma_\eta = \Big\{ \gamma \in \mathcal{C}([0,1], E) : \gamma(0) = v_1, \gamma(1) = v_2 \Big\}$. *Then for almost every* $\eta \in T$, *there is a sequence* $\{v_n\} \subset E$ *such that* *(i)* $\{v_n\}$ *is bounded;*

(ii) $\Phi_\eta(v_n) \mapsto c_\eta$*;*

(iii) $\Phi'_\eta(v_n) \mapsto 0$ *in the dual* E^{-1} *of* E*.*

Moreover, the map $\eta \mapsto c_\eta$ *is nonincreasing and continuous from the left.*

Letting $T = [\delta, 1]$, where $\delta \in (0,1)$ is a positive constant, we define the following energy functional by

$$J_\eta(v) = \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(x)g^2(v)\Big)\mathrm{d}x + \frac{q}{2}C(g(v)) + \frac{q}{4}\kappa D(g(v) - \eta\int_{\mathbb{R}^2}F(g(v))\mathrm{d}x$$

for all $v \in X$. Similarly, for each $\eta \in [\delta, 1]$, the functional J_η possesses the mountain-pass geometry. Define the mountain pass level

$$c_\eta = \inf_{\gamma\in\Gamma_\eta}\max_{t\in[0,1]} J_\eta(\gamma(t)),$$

where $\Gamma_\eta = \Big\{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0, J_\eta(\gamma(1)) < 0\Big\}$. Clearly $c_1 \le c_\eta \le c_\delta$ for each $\eta \in T$.

Lemma 3.1. *Assume that* (f_1)*-*(f_2) *and* (V_1) *are satisfied. Then there holds:*

(i) there exists $v \in X\backslash\{0\}$ *such that* $J_\eta(v) < 0$ *for all* $\eta \in T$*;*

(ii) $c_\eta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\eta(\gamma(t)) > \max\Big\{J_\eta(0), J_\eta(v)\Big\}$ *for all* $\eta \in T$*, where* $\Gamma = \Big\{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0, \gamma(1) = v\Big\}$*;*

(iii) for any $v \in X\backslash\{0\}$*, there exists a constant* $C > 0$ *independent of* η *such that* $c_\eta \le C$ *for all* $\eta \in T$*.*

Proof. (i) Let $v \in X\backslash\{0\}$ be fixed. For any $\eta \in T$, by (f_2), we infer that

$$\begin{aligned} J_\eta(v) \le J_\delta(v) = &\frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(x)g^2(v)\Big)\mathrm{d}x + \frac{q}{2}C(g(v)) + \frac{q}{4}\kappa D(g(v) \\ &- \delta\int_{\mathbb{R}^2}F(g(v))\mathrm{d}x, \end{aligned}$$

we hereafter denote v_t by $v_t(x) = (v)_t(x) = g^{-1}(t^\alpha g(v(tx)))$ for some $\alpha > 0$, satisfying $1 < \alpha < 2/8 - p$ if $p \in (6,8)$ and $\alpha > 1$ arbitrary for $p \ge 8$, then by direct calculations, we have $\mathcal{C}(g(v_t)) = t^{6\alpha-4}\mathcal{C}(g(v))$ and $\mathcal{D}(g(v_t)) = t^{8\alpha-4}\mathcal{D}(g(v))$. Thus we get

$$\begin{aligned} J_\delta(v_t) = &\frac{t^{2\alpha}}{2}\int_{\mathbb{R}^2}\frac{1+2t^{2\alpha}g^2(v)}{1+2g^2(v)}|\nabla v|^2\mathrm{d}x + \frac{t^{2\alpha-2}}{2}\int_{\mathbb{R}^2}V(t^{-1}x)g^2(v)\mathrm{d}x \\ &+ \frac{t^{6\alpha-4}}{2}q\mathcal{C}(g(v)) + \frac{t^{8\alpha-4}}{4}q\kappa\mathcal{D}(g(v)) - \delta t^{-2}\int_{\mathbb{R}^2}F(t^\alpha g(v))\mathrm{d}x, \qquad \forall v \in X. \end{aligned}$$

From (f_1) and (f_2), for every $\theta > 0$, there exists $C_\theta > 0$ such that

$$F(\varrho) \ge \theta|\varrho|^p - C_\theta\varrho^2, \quad \forall \varrho \in \mathbb{R}. \tag{3.1}$$

Which implies that $J_\delta(v_t) \to -\infty$ as $t \to +\infty$. Thus we can take a function $v = g^{-1}(t_0^\alpha v(t_0 x)) \in X\backslash\{0\}$ with sufficiently large $t_0 > 0$ to satisfy $J_\eta(v) \le 0$ for all $\eta \in T$.

(ii) Let $H(s) := -\frac{1}{2}g^2(s) + F(g(s))$, then by (f_1) and (g_2), (g_4), we have

$$\lim_{s\to 0}\frac{H(s)}{s^2} = \lim_{s\to 0}\Big(-\frac{1}{2}\Big(\frac{g(s)}{s}\Big)^2 + \frac{F(g(s))}{s^2}\Big) = -\frac{1}{2}, \tag{3.2}$$

$$\lim_{s\to +\infty}\frac{H(s)}{|s|^q} = \lim_{s\to +\infty}\Big(-\frac{1}{2}\Big(\frac{g(s)}{\sqrt{s}}\Big)^2\Big(\frac{1}{|s|^{q-1}}\Big) + \frac{F(g(s))}{|s|^q}\Big) \le C, \tag{3.3}$$

for all $q \in (2, +\infty)$. It follows from (3.1)-(3.3) that for any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that

$$H(s) \le -\frac{1}{2}s^2 + \varepsilon s^2 + C_\varepsilon |s|^q.$$

Thus by (g_3) and (V_1), we have

$$\begin{aligned} J_\eta(v) \ge& \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla v|^2 + \beta v^2)\mathrm{d}x - \eta\int_{\mathbb{R}^2}(\varepsilon|v|^2 + C_\varepsilon|v|^q)\mathrm{d}x \\ \ge& \min\{\frac{1}{4}, \frac{\beta}{4}\}\|v\|^2 - C\|v\|^q, \end{aligned}$$

where ε is small enough. Since $q > 2$, we deduce that J_η has a strict local minimum at 0 and hence $c_\eta > 0$.

(iii) By point (i) and (ii), define

$$\gamma_0(t) = \begin{cases} 0, & \text{if } t = 0, \\ g^{-1}((tt_0)^\alpha g(u(tt_0 x))), & \text{if } t > 0. \end{cases}$$

Clearly, $\gamma_0(t) \in \Gamma$ and for any $v \in X\backslash\{0\}$, $c_\eta \le \max_{t>0} J_\eta(\gamma_0(t)) \le \max_{t>0} J_\delta(\gamma_0(t))$ for all $\eta \in T$. Thus we can choose $C > \max_{t>0} J_\delta(\gamma_0(t)) \ge 0$ such that $c_\eta \le C$. This completes the proof. □

By Theorem 3.1, it is easy to know that for any a.e. $\eta \in T$, there exists a bounded sequence $\{v_n\} \subset X$ such that $J_\eta(v_n) \to c_\eta$ and $J'_\eta(v_n) \to 0$, which is called $(PS)_{c_\eta}$ sequence.

Lemma 3.2. *Let $\eta \in [\delta, 1]$ be fixed. Assume that $\{v_n\} \subset X$ is a sequence of obtain above. Then there exists $v_\eta \in X\backslash\{0\}$, such that $J_\eta(v_\eta) = c_\eta$ and $J'_\eta(v_\eta) = 0$.*

Proof. Since $\{v_n\}$ is bounded in X, up to a subsequence, there exists $v_\eta \in X\backslash\{0\}$ such that $v_n \rightharpoonup v_\eta$ in X, $v_n \to v_\eta$ in $L^q(\mathbb{R}^2)$ for all $q > 2$ and $v_n \to v_\eta$ a.e. in $\mathbb{R}^2$. By Lebesgue dominated convergence Theorem shows that v_η is a critical point of J_η. Let $\widetilde{F}(x,s) = \frac{1}{2}V(x)s^2 - \frac{1}{2}V(x)g^2(s) + \eta F(g(s))$, where $\widetilde{F}(x,s) = \int_0^s \widetilde{f}(x,t)\mathrm{d}x$. By (f_1) and (g_2)-(g_3), we have

$$\lim_{s\to 0}\frac{\widetilde{F}(x,s)}{s^2} = \lim_{s\to 0}\Big(\frac{1}{2}V(x) - \frac{1}{2}V(x)\Big(\frac{g(s)}{s}\Big)^2 + \eta\frac{F(g(s))}{s^2}\Big) = 0, \tag{3.4}$$

and

$$\lim_{|s|\to +\infty}\frac{\widetilde{F}(x,s)}{|s|^q} \le C. \tag{3.5}$$

By (3.4) and (3.5), we have that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|\widetilde{F}(x,s)| \le \varepsilon s^2 + C_\varepsilon |s|^q, \quad \forall s \in \mathbb{R}.$$

It follows from the above facts

$$\int_{\mathbb{R}^2} (\widetilde{f}(x,v_n) - \widetilde{f}(x,v_\eta))(v_n - v_\eta)\mathrm{d}x \to 0, \qquad \text{as} \quad n \to +\infty.$$

Moreover, by virtue of Lemma 2.4 and (g_7), we have

$$\begin{aligned} o(1) = \langle J'_\eta(v_n) - J'_\eta(v_\eta), v_n - v_\eta\rangle =& \int_{\mathbb{R}^2} |\nabla(v_n - v_\eta)|^2\mathrm{d}x + V(x)|v_n - v_\eta|^2\mathrm{d}x \\ &+ q\langle \mathcal{C}'(g(v_n)) - \mathcal{C}'(g(v_\eta)), v_n - v_\eta\rangle \\ &+ q\kappa\langle \mathcal{D}'(g(v_n)) - \mathcal{D}'(g(v_\eta)), v_n - v_\eta\rangle \\ &- \int_{\mathbb{R}^2} (\widetilde{f}(x,v_n) - \widetilde{f}(x,v_\eta))(v_n - v_\eta)\mathrm{d}x \\ \ge& \min\{1,\beta\}\|v_n - v_\eta\|^2 + o(1), \end{aligned}$$

which implies that $v_n \to v_\eta$ in X. Thus v_η is a nontrivial critical point of J_η with $J_\eta(v_\eta) = c_\eta$. By the strong maximum principle, we know that v_η is positive. This completes the proof. □

Proof of Theorem 1.1. The proof of Theorem 1.1 consist of the three steps.

Step 1: In view of Theorem 3.1, for a.e. $\eta \in T$, there exists $v_\eta \in X$ such that $v_n \rightharpoonup v_\eta \ne 0$ in X, $J_\eta(v_n) \to c_\eta$ and $J'_\eta(v_n) \to 0$. By Lemma 3.2, we have $J_\eta(v_\eta) = c_\eta$ and $J'_\eta(v_\eta) = 0$. Thus take $\{\eta_n\} \subset T$ such that $\eta_n \to 1$, $v_{\eta_n} \in X$, $J'_{\eta_n}(v_{\eta_n}) = 0$ and $J_{\eta_n}(v_{\eta_n}) = c_{\eta_n}$. Next, we prove that $\{v_{\eta_n}\}$ is bounded in X. Similar to Lemma 2.3, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \Big(|\nabla v_{\eta_n}|^2 + V(x)g(v_{\eta_n})g'(v_{\eta_n})v_{\eta_n} \\ &+ q\frac{\hat{h}^2(g(v_{\eta_n}(|x|))}{|x|^2}(1+\kappa g^2(v_{\eta_n}))g(v_{\eta_n})g'(v_{\eta_n})v_{\eta_n}\Big)\mathrm{d}x \\ &+ q\int_{\mathbb{R}^2}\Big(\int_{|x|}^{+\infty}\frac{\hat{h}[g(v_{\eta_n}(s))]}{s}(2+\kappa g^2(v_{\eta_n}(s)))g^2(v_{\eta_n}(s))\mathrm{d}s\Big)g(v_{\eta_n})g'(v_{\eta_n})v_{\eta_n}\mathrm{d}x \\ =&\eta\int_{\mathbb{R}^2} f(g(v_{\eta_n}))g'(v_{\eta_n})v_{\eta_n}\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^2}\Big(V(x)g^2(v_{\eta_n}) + \frac{1}{2}\nabla V(x)\cdot x g^2(v_{\eta_n}) + q\frac{\hat{h}^2[g(v_{\eta_n}(|x|))]}{|x|^2}(2+\kappa g^2(v_{\eta_n}))g^2(v_{\eta_n})\Big)\mathrm{d}x \\ =&2\eta\int_{\mathbb{R}^2} F(g(v_{\eta_n}))\mathrm{d}x. \end{aligned}$$

Hence, we infer that, by(f_3), (V_2) and (g_7),

$$\begin{aligned} &J_{\eta_n}(v_{\eta_n}) \\ \ge&\frac{\epsilon}{\epsilon-2}\int_{\mathbb{R}^2}\eta\Big(\frac{1}{\epsilon}f(g(v_{\eta_n}))g'(v_{\eta_n})v_{\eta_n} - F(g(v_{\eta_n}))\Big)\mathrm{d}x + \frac{\frac{\epsilon}{2}-2}{\epsilon-2}\int_{\mathbb{R}^2}|\nabla v_{\eta_n}|^2\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{\frac{1}{2}}{\epsilon-2}\int_{\mathbb{R}^2}\nabla V(x)\cdot xg^2(v_{\eta_n})\mathrm{d}x+\frac{\frac{\epsilon}{2}}{\epsilon-2}\int_{\mathbb{R}^2}V(x)g^2(v_{\eta_n})\mathrm{d}x\\
&-\frac{1}{\epsilon-2}\int_{\mathbb{R}^2}g(v_{\eta_n})g'(v_{\eta_n})v_{\eta_n}\mathrm{d}x+\frac{\frac{\epsilon}{8}-1}{\epsilon-2}q\int_{\mathbb{R}^2}\Big\{\frac{\hat{h}^2(g(v_{\eta_n}(|x|))}{|x|^2}(1+\kappa g^2(v_{\eta_n}))g(v_{\eta_n})\\
&+\int_{|x|}^{+\infty}\frac{\hat{h}(g(v_{\eta_n}(s)))}{s}(2+\kappa g^2(v_{\eta_n}(s)))g^2(v_{\eta_n}(s))\mathrm{d}s\Big\}g(v_{\eta_n})g'(v_{\eta_n})v_{\eta_n}\mathrm{d}x\\
\geq&\frac{\epsilon}{\epsilon-2}\int_{\mathbb{R}^2}\eta\Big(\frac{1}{2\epsilon}f(g(v_{\eta_n}))g(v_{\eta_n})-F(g(v_{\eta_n}))+\xi g^2(v_{\eta_n})\Big)\mathrm{d}x+\frac{\frac{\epsilon}{2}-2}{\epsilon-2}\int_{\mathbb{R}^2}|\nabla v_{\eta_n}|^2\mathrm{d}x\\
&+\frac{\frac{\epsilon}{2}}{\epsilon-2}\int_{\mathbb{R}^2}V(x)g^2(v_{\eta_n})\mathrm{d}x-\frac{1}{\epsilon-2}\int_{\mathbb{R}^2}g^2(v_{\eta_n})\mathrm{d}x-\frac{\epsilon\eta\xi}{\epsilon-2}\int_{\mathbb{R}^2}g^2(v_{\eta_n})\mathrm{d}x\\
\geq&\frac{\frac{\epsilon}{2}-2}{\epsilon-2}\int_{\mathbb{R}^2}|\nabla v_{\eta_n}|^2\mathrm{d}x+\frac{\frac{1}{2}\epsilon\beta-1-\epsilon\eta\xi}{\epsilon-2}\int_{\mathbb{R}^2}g^2(v_{\eta_n})\mathrm{d}x\\
\geq&C\Big(\int_{\mathbb{R}^2}(|\nabla v_{\eta_n}|^2\mathrm{d}x+g^2(v_{\eta_n})\mathrm{d}x\Big).
\end{aligned}
\tag{3.6}
$$

Since $c_1\leq J_{\eta_n}(v_{\eta_n})=c_{\eta_n}\leq c_\delta$, we conclude the boundedness of sequence $\{\|\nabla v_{\eta_n}\|_{L^2}\}$. From (g_1), (g_3) and (g_8), it holds

$$
\begin{aligned}
\int_{\mathbb{R}^2}|v_{\eta_n}|^2\mathrm{d}x&=\int_{|v_{\eta_n}|>1}|v_{\eta_n}|^2\mathrm{d}x+\int_{|v_{\eta_n}|\leq1}|v_{\eta_n}|^2\mathrm{d}x\\
&\leq C\Big(\int_{\mathbb{R}^2}|g(v_{\eta_n})|^4\mathrm{d}x+\int_{\mathbb{R}^2}|g(v_{\eta_n})|^2\mathrm{d}x\Big).
\end{aligned}
$$

Then by (1.10), Proposition 2.2 and (3.6), we infer that there exists $C>0$ such that $\int_{\mathbb{R}^2}|v_{\eta_n}|^2\mathrm{d}x\leq C$.

Step 2: Next, we will prove that there exists a nontrivial critical point of J. Since $J_{\eta_n}(v_{\eta_n})=c_{\eta_n}\leq c_\delta$, $\{v_{\eta_n}\}$ is bounded in X by step 1. Then by Theorem 3.1, we get $c_{\eta_n}\to c_1$. Therefore,

$$
J(v_{\eta_n})=J_{\eta_n}(v_{\eta_n})+(\eta_n-1)\int_{\mathbb{R}^2}F(g(v_{\eta_n}))\mathrm{d}x=c_{\eta_n}+o(1)=c_1,
$$

and for any $\varphi\in X\backslash\{0\}$, there holds

$$
\begin{aligned}
\langle J'(v_{\eta_n}),\varphi\rangle=&\langle J'_{\eta_n}(v_{\eta_n}),\varphi\rangle+(\eta_n-1)\int_{\mathbb{R}^2}f(g(v_{\eta_n}))g'(v_{\eta_n})\varphi\mathrm{d}x\\
=&o(1).
\end{aligned}
$$

Then $\{v_{\eta_n}\}$ is a bounded $(PS)_{c_1}$ sequence of J. This implies that J has a critical point $v\in X$ satisfying $J(v)=c_1$, $J'(v)=0$.

Step 3: To seek ground state solutions, we need to define $\varpi=\inf\{J(v):v\neq 0,J'(v)=0\}$. By step 1, we can deduce that $\varpi\geq0$. Let $\{\overline{v}_n\}$ be a sequence such that $J(\overline{v}_n)\to\varpi$, $J'(\overline{v}_n)=0$. Similar argument in step 1, 2, we can show that $\{\overline{v}_n\}$ is a bounded $(PS)_\varpi$ sequence of I. Similar arguments in Lemma 3.2, we can prove that there exists a function $\bar{v}\in X$ such that $J(\bar{v})=\varpi$, $J'(\bar{v})=0$ which shows that $\bar{u}=g(\bar{v})$ is a ground state solution of (1.1). This completes the proof. □

4. Existence of infinitely many solutions

Recall that a sequence $\{v_n\} \subset X$ is said to be a $(C)_c$-sequence if $J(v_n) \to c$ and $(1 + \|v_n\|)J'(v_n) \to 0$. X is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergence subsequence. To prove Theorem 1.2, we state the symmetric Mountain Pass Theorem of Ranbinowitz (see [37], Theorem 9.12).

Proposition 4.1. *Let $\mathbb{X}$ be an infinite dimensional Banach space, $\mathbb{X} = \mathbb{Y} \oplus \mathbb{Z}$, where $\mathbb{Y}$ is finite dimensional. If $\varphi \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$ and*
(i) $\varphi(0) = 0$, $\varphi(-u) = \varphi(u)$ for all $u \in \mathbb{X}$;
(ii) there exist constants ρ, $\varrho > 0$ such that $\varphi|_{\partial B_\rho \cap Z} \geq \varrho$;
(iii) for any finite dimensional subspace $\tilde{\mathbb{X}} \subset \mathbb{X}$, there is $R = R(\tilde{\mathbb{X}}) > 0$ such that $\varphi(u) \leq 0$ on $\tilde{\mathbb{X}} \setminus B_R$;
then φ possesses an unbounded sequence of critical values.

Lemma 4.1. *Assume that (V_1), (f_1')-(f_3') are satisfied. Then J satisfies $(C)_c$-condition.*

Proof. Let $\{v_n\} \subset X$ be such that

$$J(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|)J'(v_n) \to 0. \tag{4.1}$$

Then, there is a constant $C > 0$ such that we have

$$J(v_n) - \frac{1}{8}\langle J'(v_n), v_n\rangle \leq C. \tag{4.2}$$

We first prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \Big(|\nabla v_n|^2 + V(x)g^2(v_n)\Big)\mathrm{d}x \leq C.$$

Suppose to the contrary that

$$A_n^2 := \int_{\mathbb{R}^2} \Big(|\nabla v_n|^2 + V(x)g^2(v_n)\Big)\mathrm{d}x \to +\infty.$$

Setting $\bar{g}(v_n) := g(v_n)/A_n$, then $\|\bar{g}(v_n)\| \leq 1$. Up to a subsequence, we may assume that $\bar{g}(v_n) \rightharpoonup w$ in X, $\bar{g}(v_n) \to w$ in $L^s(\mathbb{R}^2)$, $2 < s < +\infty$, $\bar{g}(v_n) \to w$ in $L^s_{loc}(\mathbb{R}^2)$, $2 \leq s < +\infty$ and $\bar{g}(v_n) \to w$ a.e. on $\mathbb{R}^2$. Set $\varphi_n = \frac{g(v_n)}{g'(v_n)}$, then there is a constant $c_3 > 0$ such that $\|\varphi_n\| \leq c_3\|v_n\|$. Since $\{v_n\}$ is a $(C)_c$-sequence of J, then from (4.2), we obtain

$$\begin{aligned} C \geq& J(v_n) - \frac{1}{8}\langle J'(v_n), \varphi_n\rangle \\ =& \frac{1}{8}\int_{\mathbb{R}^2} (2 + (g'(v_n))^2)|\nabla v_n|^2 \mathrm{d}x + \frac{3}{8}\int_{\mathbb{R}^2} V(x)g^2(v_n)\mathrm{d}x + \frac{1}{8}q\mathcal{C}(g(v_n)) \\ &+ \int_{\mathbb{R}^2} [\frac{1}{8}f(g(v_n))g(v_n) - F(g(v_n))]\mathrm{d}x, \end{aligned} \tag{4.3}$$

which implies that

$$C \geq \int_{\mathbb{R}^2} \bar{F}(g(v_n))\mathrm{d}x. \tag{4.4}$$

It follows from (1.9), Proposition 2.2 and (4.1), (4.3), (g_6)-(g_7) that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^2}\frac{|F(g(v_n))|}{A_n^2}\mathrm{d}x = 1/2. \tag{4.5}$$

For $0 \le a < b$, let

$$\Omega_n(a,b) = \{x \in \mathbb{R}^2 : a \le |g(v_n(x))| < b\}.$$

If $w = 0$, then $\bar{g}(v_n) \to 0$ in $L^s_{loc}(\mathbb{R}^2)$, $2 \le s < +\infty$, $\bar{g}(v_n) \to 0$ in $L^s(\mathbb{R}^2)$, $2 < s < +\infty$. For any $0 < \varepsilon < \frac{1}{16}$, there exist large r_1, $N > 0$ such that

$$\begin{aligned}\int_{\Omega_n(0,r_1)}\frac{|F(g(v_n))|}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x &\le \int_{\Omega_n(0,r_1)}\frac{c_1'|g(v_n)|^j + c_2'|g(v_n)|^r}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x\\ &\le (c_1' r_1^{j-2} + c_2' r_1^{r-2})\int_{\Omega_n(0,r_1)}|\bar{g}(v_n)|^2 < \varepsilon\end{aligned} \tag{4.6}$$

for all $n > N$. Set $\tau' = \frac{\tau}{\tau-1}$. Since $\tau > 1$, one see that $2\tau' \in (2,+\infty)$. Hence, it follows from(f_3') that

$$\begin{aligned}&\int_{\Omega_n(r_1,+\infty)}\frac{|F(g(v_n))|}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x\\ &\le \Big(\int_{\Omega_n(r_1,+\infty)}\Big(\frac{|F(g(v_n))|}{|g(v_n)|^2}\Big)^\tau\mathrm{d}x\Big)^{\frac{1}{\tau}}\Big(\int_{\Omega_n(r_1,+\infty)}|\bar{g}(v_n)|^{2\tau'}\mathrm{d}x\Big)^{\frac{1}{\tau'}}\\ &\le C_1^{\frac{1}{\tau}}\Big(\int_{\Omega_n(r_1,+\infty)}\bar{F}(g(v_n))\mathrm{d}x\Big)^{\frac{1}{\tau}}\Big(\int_{\Omega_n(r_1,+\infty)}|\bar{g}(v_n)|^{2\tau'}\Big)\mathrm{d}x)^{\frac{1}{\tau'}}\\ &\le C_2\Big(\int_{\Omega_n(r_1,+\infty)}|\bar{g}(v_n)|^{2\tau'}\mathrm{d}x\Big)^{\frac{1}{\tau'}}\\ &\le C_2\Big(\int_{\mathbb{R}^2}|\bar{g}(v_n)|^{2\tau'}\mathrm{d}x\Big)^{\frac{1}{\tau'}} < \varepsilon\end{aligned} \tag{4.7}$$

for all $n > N$. Combining (4.6) with (4.7), we have

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{|F(g(v_n))|}{A_n^2}\mathrm{d}x =& \int_{\Omega_n(0,r_1)}\frac{|F(g(v_n))|}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x\\ &+ \int_{\Omega_n(r_1,+\infty)}\frac{|F(g(v_n))|}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x < 2\varepsilon < \frac{1}{8},\end{aligned}$$

for all $n > N$, which contradicts (4.5).

If $w \ne 0$, then meas$(\Omega) > 0$, where $\Omega := \{x \in \mathbb{R}^2 : w \ne 0\}$. For $x \in \Omega$, we have $|g(v_n)| \to +\infty$ as $n \to +\infty$. Hence $\Omega \subset \Omega_n(r_0,+\infty)$ for large $n \in N$, where r_0 is given in (f_2'). By (f_2'), we have

$$\frac{F(g(v_n))}{|g(v_n)|^8} \to +\infty \quad \text{as} \quad n \to +\infty.$$

Hence, using Fatou's Lemma, we have

$$\int_\Omega \frac{F(g(v_n))}{|g(v_n)|^8}\mathrm{d}x \to +\infty \quad \text{as} \quad n \to +\infty. \tag{4.8}$$

It follows from (4.1) and (4.8) that

$$\begin{aligned}
0 =& \lim_{n\to+\infty} \frac{c+o(1)}{A_n^2} = \lim_{n\to+\infty} \frac{J(v_n)}{A_n^2} \\
=& \lim_{n\to+\infty} \frac{1}{A_n^2}\Big(\frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v_n|^2 + V(x)g^2(v_n)\Big)\mathrm{d}x + \frac{q}{2}\mathcal{C}(g(v_n)) \\
&+ \frac{q}{4}k\mathcal{D}(g(v_n)) - \int_{\mathbb{R}^2} F(g(v_n))\mathrm{d}x\Big) \\
=& \lim_{n\to+\infty}\Big(\frac{1}{2} + \frac{q}{2}\frac{\mathcal{C}(g(v_n))}{A_n^2} + \frac{qk}{4}\frac{\mathcal{D}(g(v_n))}{A_n^2} - \int_{\Omega_n(0,r_0)} \frac{|F(g(v_n))|}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x \\
&- \int_{\Omega_n(r_0,+\infty)} \frac{|F(g(v_n))|}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x\Big) \\
\leq& \frac{1}{2} + \limsup_{n\to+\infty}\Big((c_1' r_0^{j-2} + c_2' r_0^{r-2})\int_{\mathbb{R}^2}|\bar{g}(v_n)|^2\mathrm{d}x \\
&- \int_{\Omega_n(r_0,+\infty)} \frac{|F(g(v_n))|}{|g(v_n)|^2}|\bar{g}(v_n)|^2\mathrm{d}x\Big) \\
\leq& C - \liminf_{n\to+\infty}\int_{\Omega}\frac{|F(g(v_n))|}{|g(v_n)|^8}|g(v_n)|^2|\bar{g}(v_n)|^8\mathrm{d}x \\
=& -\infty,
\end{aligned}$$

which is a contradiction. Thus, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2}\Big(|\nabla v_n|^2 + V(x)g^2(v_n)\Big)\mathrm{d}x \leq C.$$

Next, we prove that $\{v_n\}$ is bounded in X, we claim that there exists $C > 0$ such that

$$A_n^2 := \int_{\mathbb{R}^2}\Big(|\nabla v_n|^2 + V(x)g^2(v_n)\Big)\mathrm{d}x \geq C\|v_n\|^2. \tag{4.9}$$

In fact, we may assume that $v_n \neq 0$ (if not, the conclusion is trivial). If this conclusion is not true, up to a subsequence, we have $\frac{A_n^2}{\|v_n\|^2} \to 0$. Set $w_n = \frac{v_n}{\|v_n\|}$ and $l_n = \frac{g^2(v_n)}{\|v_n\|^2}$. Then

$$\int_{\mathbb{R}^2}\Big(|\nabla w_n|^2 + V(x)l_n(x)\Big)\mathrm{d}x \to 0.$$

Hence,

$$\int_{\mathbb{R}^2}|\nabla w_n|^2\mathrm{d}x \to 0, \qquad \int_{\mathbb{R}^2}V(x)l_n(x)\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2}w_n^2\mathrm{d}x \to 1,$$

as $n \to +\infty$. Similar to the idea of [44], we claim that for each $\mu > 0$, there exists $C > 0$ (independent of n) such that $\mathrm{meas}(\Omega_n) < \mu$, where $\Omega_n := \{x \in \mathbb{R}^2 : |v_n(x)| \geq C\}$. Otherwise, there is an $\mu_0 > 0$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that for any positive integer k,

$$\mathrm{meas}(\{x \in \mathbb{R}^2 : |v_{n_k}(x)| \geq k\}) \geq \mu_0 > 0.$$

Set $\Omega_{n_k} := \{x \in \mathbb{R}^2 : |v_{n_k}(x)| \geq k\}$. By (g_8), we have

$$A_{n_k}^2 \geq \int_{\mathbb{R}^2} V(x) g^2(v_{n_k}) \mathrm{d}x \geq \int_{\Omega_{n_k}} V(x) g^2(v_{n_k}) \mathrm{d}x \geq Ck\mu_0 \to +\infty,$$

as $k \to +\infty$, a contradiction. Hence the claim is true. Notice that as $|v_n(x)| \leq C_3$, by (g_8) and (g_{11}), we have

$$\frac{C}{C_3^3} {v_n}^2 \leq g^2\Big(\frac{v_n}{C_3}\Big) \leq C_4 g^2(v_n).$$

Thus,

$$\int_{\mathbb{R}^2 \setminus \Omega_n} w_n^2 \mathrm{d}x \leq C_5 \int_{\mathbb{R}^2 \setminus \Omega_n} \frac{g^2(v_n)}{\|v_n\|^2} \mathrm{d}x \leq C_5 \int_{\mathbb{R}^2} l_n(x) \mathrm{d}x \to 0. \tag{4.10}$$

Besides, by virtue of the integral absolutely continuity, there exists $\mu > 0$ such that whenever $\Omega' \subset \mathbb{R}^2$ and $\operatorname{meas}(\Omega') < \mu$,

$$\int_{\Omega'} w_n^2 \mathrm{d}x \leq \frac{1}{2}. \tag{4.11}$$

Combining (4.10) with (4.11), we have

$$\int_{\mathbb{R}^2} w_n^2 \mathrm{d}x = \int_{\mathbb{R}^2 \setminus \Omega_n} w_n^2 \mathrm{d}x + \int_{\Omega_n} w_n^2 \mathrm{d}x \leq \frac{1}{2} + o(1),$$

which implies that $1 \leq \frac{1}{2}$, a contradiction. This implies that (4.9) holds. Hence $\{v_n\}$ is bounded in X. Finally, we prove that $\{v_n\}$ has a convergence subsequence in X. We claim that there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} |\nabla(v_n - v)|^2 + V(x)(g(v_n)g'(v_n) - g(v)g'(v))(v_n - v) \mathrm{d}x \geq C\|v_n - v\|^2. \tag{4.12}$$

Indeed, we may assume that $v_n \neq v$ (if not, the conclusion is trivial). Set

$$w_n = \frac{v_n - v}{\|v_n - v\|} \quad \text{and} \quad l_n = \frac{g(v_n)g'(v_n) - g(v)g'(v)}{v_n - v}.$$

We argue by contradiction and assume that

$$\int_{\mathbb{R}^2} \Big(|\nabla w_n|^2 + V(x) l_n(x) w_n^2\Big) \mathrm{d}x \to 0.$$

Since

$$\frac{d}{dt}(g(t)g'(t)) = (g'(t))^2 + g(t)g'(t) = \frac{1}{(1 + 2g^2(t))^2} > 0,$$

$g(t)g'(t)$ is strictly increasing and for each $C > 0$ there is $\varrho_1 > 0$ such that

$$\frac{d}{dt}(g(t)g'(t)) \geq \varrho_1,$$

as $|t| \le C$. From this, we see that $l_n(x)$ is positive. Hence

$$\int_{\mathbb{R}^2} |\nabla w_n|^2 \mathrm{d}x \to 0, \qquad \int_{\mathbb{R}^2} V(x) l_n(x) w_n^2 \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} w_n^2 \mathrm{d}x \to 1,$$

as $n \to +\infty$. By a similar argument as (4.10)-(4.11), we can conclude a contradiction.

On the other hand, by (f_1'), (g_1), (g_3) and (g_9)-(g_{10}), we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^2} (f(g(v_n))g'(v_n) - f(g(v))g'(v))(v_n - v)\mathrm{d}x \Big| \\
\le & \int_{\mathbb{R}^2} C(|v_n|^{j-1} + |v_n|^{r-1} + |v|^{j-1} + |v|^{r-1})|v_n - v|\mathrm{d}x \qquad (4.13)\\
\le & C\Big((\|v_n\|_{L^j}^{j-1} + \|v_n\|_{L^j}^{j-1})\|v_n - v\|_{L^j} + (\|v\|_{L^r}^{r-1} + \|v\|_{L^r}^{r-1})\|v_n - v\|_{L^r} \Big) \\
= & o(1).
\end{aligned}$$

Therefore, by (4.12), (4.13), we have

$$\begin{aligned}
o(1) =& \langle J'(v_n) - J'(v), v_n - v\rangle \\
=& \int_{\mathbb{R}^2} |\nabla(v_n - v)|^2 \mathrm{d}x + (g(v_n)g'(v_n) - g(v)g'(v))(v_n - v)\mathrm{d}x \\
&+ q\langle \mathcal{C}'(g(v_n)) - \mathcal{C}'(g(v)), v_n - v\rangle \\
&+ q\kappa\langle \mathcal{D}'(g(v_n)) - \mathcal{D}'(g(v)), v_n - v\rangle \\
&- \int_{\mathbb{R}^2} (f(g(v_n))g'(v_n) - f(g(v))g'(v))(v_n - v)\mathrm{d}x \\
\ge & C\|v_n - v\|^2 + o(1).
\end{aligned}$$

This implies that $\|v_n - v\| \to 0$. This completes the proof. □

Let $\{e_j\}$ is a total orthonormal basis of X and define $X_j = \mathbb{R}_{e_j}$,

$$Y_k = \bigoplus_{j=1}^{k} X_j, Z_k = \overline{\bigoplus_{j=k+1}^{\infty} X_j}, \quad k \in Z,$$

and Y_k is finite-dimensional.

Proposition 4.2. *Assume that* (V_1) *and* (f_1') *are satisfied. Then there exist constant* $m, \rho, \varrho > 0$ *such that* $J|_{S_\rho \cap Z_m} \ge \varrho$.

Proof. From Lemma 3.8 in [43], we know that for any $1 \le s < +\infty$, $\beta_k(s) := \sup\limits_{v \in Z_k, \|v\|=1} \|v\|_{L^s} \to 0$, as $k \to +\infty$. Thus, we can choose an integer $m > 1$ such that

$$\|v\|_{L^j}^j \le \frac{C}{8c_1'}\|v\|^j, \qquad \|v\|_{L^r}^r \le \frac{C}{8c_2'}\|v\|^r, \quad \forall v \in Z_m. \qquad (4.14)$$

By a similar argument as (4.9), we can prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \Big(|\nabla v|^2 + V(x)g^2(v)\Big)\mathrm{d}x \ge C\|v\|^2, \qquad \forall v \in S_\rho, \qquad (4.15)$$

where $S_\rho := \{v \in X | \|v\| = \rho\}$. For any $v \in Z_m$ with $\|v\| = \rho < 1$, by (g_3) and (4.14)-(4.15), we have

$$\begin{aligned} J(v) &= \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(x)g^2(v)\Big)\mathrm{d}x + \frac{q}{2}\mathcal{C}(g(v)) + \frac{q}{4}k\mathcal{D}(g(v)) - \int_{\mathbb{R}^2} F(g(v))\mathrm{d}x \\ &\geq \frac{C}{2}\|v\|^2 - \int_{\mathbb{R}^2}\Big(c_1'|g(v)|^j + c_2'|g(v)|^r\Big)\mathrm{d}x \\ &\geq \frac{C}{2}\|v\|^2 - \int_{\mathbb{R}^2}\Big(c_1'|v|^j + c_2'|v|^r\Big)\mathrm{d}x \\ &\geq \frac{C}{2}\|v\|^2 - \frac{C}{8}\|v\|^j - \frac{C}{8}\|v\|^r \\ &= \frac{C}{4}\|v\|^2(1 - \|v\|^{j-2}) > 0, \end{aligned}$$

since $2 < j < 8$, $j < r < +\infty$. This completes the proof. □

Lemma 4.2. *Assume that* (V_1), (f_1') *and* (f_2')*are satisfied. Then for any finite dimensional subspace* $\tilde{X} \subset X$*, there is* $R = R(\tilde{X}) > 0$ *such that*

$$J(v) \leq 0, \quad \forall v \in \tilde{X} \setminus B_R.$$

Proof. For any finite dimensional subspace $\tilde{X} \subset X$, there is a positive integral number m such that $\tilde{X} \subset X_m$. Suppose to the contrary that there is a sequence $\{v_n\} \subset \tilde{X}$ such that $\|v_n\| \to +\infty$ and $J(v_n) > 0$. Hence

$$\frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v_n|^2 + V(x)g^2(v_n)\Big)\mathrm{d}x + \frac{q}{2}\mathcal{C}(g(v_n)) + \frac{q}{4}k\mathcal{D}(g(v_n)) > \int_{\mathbb{R}^2} F(g(v_n))\mathrm{d}x. \tag{4.16}$$

Set $w_n = \frac{v_n}{\|v_n\|}$. Then, up to a subsequence, we can assume that $w_n \rightharpoonup w$ in X, $w_n \to w$ in $L^s(\mathbb{R}^2)$ for all $2 < s < +\infty$ and $w_n \to w$ a.e. on $\mathbb{R}^2$. Set $\Omega_1 := \{x \in \mathbb{R}^2 : w(x) \neq 0\}$ and $\Omega_2 := \{x \in \mathbb{R}^2 : w(x) = 0\}$. If $\mathrm{meas}(\Omega_1) > 0$, then by (f_2'), (g_5) and Fatou's Lemma, we have

$$\int_{\Omega_1} \frac{F(g(v_n))}{\|v_n\|^8}\mathrm{d}x = \int_{\Omega_1} \frac{F(g(v_n))}{g^8(v_n)}\frac{g^8(v_n)}{v_n^4}w_n^4\mathrm{d}x \to +\infty.$$

By (f_1') and (f_2'), there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$F(t) \geq -\lambda_1|t|^8 - \lambda_2|t|^j, \quad \forall t \in \mathbb{R}.$$

Hence

$$\begin{aligned} \int_{\Omega_2} \frac{F(g(v_n))}{\|v_n\|^8}\mathrm{d}x &\geq -\lambda_1\int_{\Omega_2}\frac{g^8(v_n)}{\|v_n\|^8}\mathrm{d}x - \lambda_2\int_{\Omega_2}\frac{g^j(v_n)}{\|v_n\|^8}\mathrm{d}x \\ &\geq -\lambda_1\int_{\Omega_2}|w_n|^8\mathrm{d}x - \lambda_2\int_{\Omega_2}|w_n|^j\frac{1}{\|v_n\|^{8-j}}\mathrm{d}x. \end{aligned}$$

Since $w_n \to w$ in $L^s(\mathbb{R}^2)(2 < s < +\infty)$, it is clear that

$$\liminf_{n\to+\infty}\int_{\Omega_2}\frac{F(g(v_n))}{\|v_n\|^8}\mathrm{d}x \geq 0.$$

Consequently,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^2}\frac{F(g(v_n))}{\|v_n\|^8}\mathrm{d}x=+\infty.$$

Then by (1.10)-(1.11), (4.16), we obtain $C>+\infty$, a contradiction. This shows $\mathrm{meas}(\Omega_1)=0$, ie. $w(x)=0$ a.e. on $\mathbb{R}^2$. By the equivalency of all norms in $\tilde{X}$, there exists $\nu>0$ such that

$$\|v\|_q^q\geq\nu\|v\|^2,\quad\forall v\in\tilde{X}.$$

Hence

$$0=\lim_{n\to+\infty}\|w_n\|_q^q\geq\nu\|w_n\|^2=\nu,$$

a contradiction. This completes the proof. □

Proof of Theorem 1.2. Let $\mathbb{X}=X$, $\mathbb{Y}=Y_m$ and $\mathbb{Z}=Z_m$. Obviously, $J(0)=0$ and (f_4) implies that J is even. By Lemma 4.1, Proposition 4.2 and Lemma 4.2, all conditions of Propositoion 4.1 are satisfied. Thus problem (1.8) possesses infinitely many nontrivial solutions sequence $\{v_n\}$ such that $J(v_n)\to\infty$ as $n\to+\infty$. In other words, problem (1.1) also possesses infintely many nontrivial solutions sequence $\{u_n\}$ such that $I(u_n)\to\infty$ as $n\to+\infty$. This completes the proof. □

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