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CAPACITY SOLUTION AND NUMERICAL APPROXIMATION TO A NONLINEAR COUPLED ELLIPTIC SYSTEM IN ANISOTROPIC SOBOLEV SPACES*

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Abstract In this paper, we analyze the existence and the numerical simulation of a capacity solution to a coupled nonlinear elliptic system, whose unknowns are the temperature inside a semiconductor material u, and the electric potential φ. The model problem we refer to is

$$\begin{cases} -\Delta_{\vec{p}}\, u = \rho(u)|\nabla\varphi|^2 & \text{in } \Omega \\ \operatorname{div}(\rho(u)\nabla\varphi) = 0 & \text{in } \Omega \\ \varphi = \varphi_0 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 2$ and $\Delta_{\vec{p}}\, u = \sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2}\partial_i u\right)$, is the $\vec{p}$–Laplacian operator. We consider the case of a nonuniformly elliptic problem.

Keywords Anisotropic Sobolev spaces, capacity solutions, weak solution, nonlinear elliptic equation, thermistor problem.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $N \geq 2$ an integer and $\vec{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N$, with $p_j \geq 2$, for all $j = 1, \ldots, N$. Without loss of generality, we will assume that

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$2 \leq p_1 \leq p_2 \leq \ldots \leq p_N < \infty$. We consider the following nonlinear coupled elliptic system

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) = \rho(u)|\nabla\varphi|^2 & \text{in } \Omega \\ \operatorname{div}(\rho(u)\nabla\varphi) = 0 & \text{in } \Omega \\ \varphi = \varphi_0 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where ∂_i stands for the i-th partial derivative operator, that is, $\partial_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq N$; the function φ_0 is given, and $\rho \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is such that $\rho(s) > 0$, for all $s \in \mathbb{R}$.

In this framework, the domain Ω represents the spatial region occupied by a semiconductor device, ρ is the electric conductivity, u is the temperature and φ is the electric potential. In this situation, Ω is called **thermistor** and for $\vec{p} = (2, \ldots, 2)$ the system (1.1) becomes the classical thermistor problem for an isotropic material.

Notice that we are not assuming that ρ is bounded below far from zero. In fact, in many practical cases it is $\rho(s) \to 0$ as $s \to +\infty$. This means that we are dealing with a nonuniformly elliptic problem and, consequently, the search for weak solutions to problem (1.1) is not suitable in this context. Indeed, if u is unbounded in Ω, the equation for φ becomes degenerate, so that no a priori estimates for $\nabla\varphi$ will be available. In order to circumvent this difficulty, we will consider the function $\Phi = \rho(u)\nabla\varphi$ as a whole, and then show that it belongs to $L^2(\Omega)^N$. This means that a new formulation of the system (1.1) is possible and the solution to this new formulation will be called **capacity solution**.

The concept of capacity solution was first introduced by Xu in [20] in the analysis of a modified version of the evolution thermistor problem. This author adapted this concept to more general settings by assuming weaker assumptions [19] or with mixed boundary conditions [21, 22].

The existence of a weak solution of the thermistor problem associated with (1.1) in the case where the first elliptic equation is of the form $-\operatorname{div}(a(x, \nabla u)) + g(x, u)$, where the operator a is of Leray-Lions type, and g satisfies the sign condition but without any restriction on its growth, in the classical Sobolev spaces has been proved in [4]. An existence result of a capacity solution to the parabolic-elliptic equation in the classical Sobolev spaces is given by González Montesinos and Ortegón Gallego in [11]. Also, Moussa, Ortegón Gallego and Rhoudaf have studied this problem in the setting of the Orlicz-Sobolev spaces [17]. Other similar situations have been considered in this direction, including the evolution case. See for instance [2, 3, 7, 10, 12, 13, 16, 18].

In the present work we show an existence result of a capacity solution (Definition 2.1) to the thermistor problem (1.1). The thesis of this paper generalizes the one given by Xu in [20] and by González Montesinos and Ortegón Gallego in [11] both in the evolution case. In fact, our functional setting is different since we study the existence of a temperature u in the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$, $p_j \geq 2$ for all $j = 1, \ldots, N$, which have never been considered in previous works on the thermistor problem. We have also carried out some 2D numerical simulations in order to approximate a solution of problem (1.1). To do so, we have developed a fixed-point like iterative algorithm, which is then discretized by the finite element method (FEM) with mesh adapting. These numerical experiments have revealed that this algorithm seems to be convergent for all values of the exponents p_1 and p_2 in the interval $(2,5)$. The rate of convergence is very rapid for p_1 and p_2 near 3, but it becomes lower and lower as one of the exponent tends to 2 or 5.

The remaining part of this paper is organized as follows. In Section 2 we introduce the anisotropic Sobolev spaces $W^{1,\vec{p}}(\Omega)$ and recall some useful results concerning these spaces. In Section 3 we first study the existence of a weak solution to problem (1.1) under a stronger assumption on ρ (uniform ellipticity). Then, we show the existence of a capacity solution to problem (1.1) in several steps: introduction of approximate problems, setting of a priori estimates and passing to the limit. We show that the sequence of solutions to these approximate problems converge (up to a subsequence) to a capacity solution to the system (1.1). Section 4 is devoted to the description of an iterative algorithm leading to the approximate solution of the system (1.1). We have implemented this algorithm and run it in the bidimensional case. We discuss the behavior of this algorithm for different values of the exponents p_1 and p_2 in the range $[2,5]$ and show some graphs of the numerical temperature distribution u.

2. Preliminaries and definitions

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$. We begin by recalling the definition of the anisotropic Sobolev spaces, and giving some of their properties.

We denote by $\vec{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N$. For a distribution u in Ω, $\partial_i u$ is the i-th partial derivative operator, that is, $\partial_i u = \partial u/\partial x_i$, $i = 1, \ldots, N$ in the sense of distributions. Without loss of generality, we shall assume that the components of the vector $\vec{p}$ are ordered as follows

$$1 \leq p_1 \leq p_2 \leq \ldots \leq p_{N-1} \leq p_N < \infty.$$

We introduce the anisotropic Sobolev space of exponent $\vec{p}$, $W^{1,\vec{p}}(\Omega)$, defined as (notice that, since Ω is bounded, its measure is finite and thus $L^{p_i}(\Omega) \subset L^{p_1}(\Omega)$ for all $i = 1, \ldots, N$)

$$W^{1,\vec{p}}(\Omega) = \Big\{ u \in L^{p_1}(\Omega),\ \partial_i u \in L^{p_i}(\Omega),\ \text{for all } i = 1, \ldots, N \Big\}.$$

The space $W^{1,\vec{p}}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{L^{p_1}(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)} \tag{2.1}$$

where $\|\cdot\|_{L^{p_i}(\Omega)}$ is the usual norm in the Lebesgue space $L^{p_i}(\Omega)$. We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $C_c^\infty(\Omega) = \{v \in C^\infty(\Omega)\ /\ \operatorname{supp} v \text{ is compact in } \Omega\}$ in $W^{1,\vec{p}}(\Omega)$, i.e.

$$W_0^{1,\vec{p}}(\Omega) = \Big\{ u \in W_0^{1,p_1}(\Omega)\ /\ \partial_i u \in L^{p_i}(\Omega),\ \text{for all } i = 1, \ldots, N \Big\},$$

and, thanks to the Poincaré inequality, we can equip this space with the following norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}.$$

The dual of $W_0^{1,\vec{p}}(\Omega)$ is denoted by $W^{-1,\vec{p'}}(\Omega)$, where $\vec{p'} := (p'_1, \ldots, p'_N)$, $p'_i \in \mathbb{R} \cup \{+\infty\}$ being the conjugate of p_i, i.e. $1/p'_i + 1/p_i = 1$ for all $i = 1, \ldots, N$.

Remark 2.1. Assume that $p_i \geq 2$ for all $i = 1, \ldots, N$ then, since Ω is bounded,

$$L^{p_i}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{p'_i}(\Omega), \text{ for all } i = 1, \ldots, N. \tag{2.2}$$

In particular,

$$W_0^{1,\vec{p}}(\Omega) \hookrightarrow H_0^1(\Omega), \text{ and } H^{-1}(\Omega) \hookrightarrow W^{-1,\vec{p'}}(\Omega).$$

We will make use of the following anisotropic Sobolev embedding result.

Lemma 2.1. *Let Ω be a bounded open set of $\mathbb{R}^N$. Then, the natural injection $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_1}(\Omega)$ is compact.*

The proof of this lemma follows immediately from the classical embedding theorems of Sobolev spaces and the fact that $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,p_1}(\Omega)$ with continuous injection (see [9]).

Now we state the definition of a capacity solution to problem (1.1).

Definition 2.1. A triplet (u, φ, Φ) is called a capacity solution to problem (1.1) if the following conditions are fulfilled:
(C_1) $u \in W_0^{1,\vec{p}}(\Omega)$, $\varphi \in L^\infty(\Omega)$ and $\Phi \in L^2(\Omega)^N$.
(C_2) (u, φ, Φ) satisfies the system of differential equations

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u\right) = \operatorname{div}(\varphi \Phi) \quad \text{in } \Omega, \\ \operatorname{div} \Phi = 0 \quad \text{in } \Omega. \end{cases}$$

(C_3) For every $S \in C_c^1(\mathbb{R}) = \{v \in C^1(\mathbb{R}) \ / \ \operatorname{supp} v \ \text{ is compact}\}$, one has

$$\begin{aligned} &S(u)\varphi - S(0)\varphi_0 \in H_0^1(\Omega) \quad \text{and} \\ &S(u)\Phi = \rho(u)[\nabla(S(u)\varphi) - \varphi \nabla S(u)]. \end{aligned} \tag{2.3}$$

Remark 2.2. The notion of capacity solution requires a triplet (u, φ, Φ) whereas the problem (1.1) refers only to two unknowns (u, φ). Evidently, the third component appearing in the triplet (u, φ, Φ) is, in some way, related to the first two components u and φ. Note that if u is bounded in Ω, then it is straightforward that both notions of solutions (weak solution and capacity solution) are equivalent. Indeed, taking $S \in C_0^1(\mathbb{R})$ such that $S = 1$ in the interval $\left[-\|u\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}\right]$, then (C_3) becomes $\varphi - \varphi_0 \in H_0^1(\Omega)$ and $\Phi = \rho(u)\nabla\varphi$.

On the other hand, if u is not bounded, we take $m > 0$ and a function $S_m \in C_0^1(\mathbb{R})$ such that $S_m = 1$ on $\{|s| \leq m\}$. Using S_m in (2.3) and multiplying by $\chi_{\{|u| \leq m\}}$ we get

$$\chi_{\{|u| \leq m\}} \Phi = \chi_{\{|u| \leq m\}} \rho(u) \nabla \left(S_m(u)\varphi\right), \qquad \text{for all} \qquad m > 0$$

which yields $\Phi = \rho(u)\nabla\varphi$ almost everywhere in Ω.

In particular, the fundamental difference between a capacity solution and a weak solution is that, in the first case, $\nabla\varphi$ is considered in the almost everywhere sense whereas in the second case, $\nabla\varphi$ is regarded in the sense of distributions.

3. Main results

This section is devoted to formulate and prove the main results of this article. We consider the system (1.1) under the following assumptions on data:

(A.1) $\rho \in C(\mathbb{R})$ and there exists $\bar{\rho} \in \mathbb{R}$ such that $0 < \rho(s) \leq \bar{\rho}$, for all $s \in \mathbb{R}$.

(A.2) $\varphi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$.

Theorem 3.1. *Under the assumptions (A.1) and (A.2), the system* (1.1) *admits a capacity solution in the sense of Definition 2.1.*

In order to prove this result, we first show the existence of a weak solution to a similar problem but under a less restrictive assumption rendering the second equation uniformly elliptic, namely,

$$\text{(A.1)}' \quad \begin{cases} \rho \in C(\mathbb{R}) \text{ and there exist } \rho_1 \text{ and } \rho_2 \in \mathbb{R}, \text{ such that} \\ 0 < \rho_1 \leq \rho(s) \leq \rho_2, \quad \text{for all } s \in \mathbb{R}. \end{cases}$$

Theorem 3.2. *Assume (A.1)′ and (A.2). Then, the problem* (1.1) *admits a weak solution (u, φ), that is*

$$\begin{cases} u \in W_0^{1,\vec{p}}(\Omega), \quad \varphi - \varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i - 2} \partial_i u \partial_i v = \int_\Omega \rho(u) |\nabla\varphi|^2 v, \ \textit{for all } v \in W_0^{1,\vec{p}}(\Omega), \\ \displaystyle\int_\Omega \rho(u) \nabla\varphi \nabla\psi = 0, \quad \textit{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

Proof. In order to prove the existence of a weak solution of our problem, Schauder's fixed point theorem will be applied together with a result on the existence and uniqueness of a weak solution to a certain elliptic problem.

Let $\omega \in L^{p_1}(\Omega)$, we consider the elliptic problem

$$\begin{cases} \operatorname{div}(\rho(\omega)\nabla\varphi) = 0 & \text{in } \Omega, \\ \varphi = \varphi_0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Thanks to Lax-Milgram's theorem, (3.2) has a unique solution $\varphi \in H^1(\Omega)$, and thanks to (A.2) it is $\varphi \in L^\infty(\Omega) \cap H^1(\Omega)$. Indeed, by the maximum principle we have

$$\|\varphi\|_{L^\infty(\Omega)} \leq \|\varphi_0\|_{L^\infty(\Omega)}. \tag{3.3}$$

Moreover, by using $\varphi - \varphi_0 \in H_0^1(\Omega)$ as a test function in (3.2) we get,

$$\int_\Omega \rho(\omega)\nabla\varphi\nabla(\varphi - \varphi_0) = 0$$

hence,

$$\rho_1 \int_\Omega |\nabla\varphi|^2 \leq \int_\Omega \rho(\omega) |\nabla\varphi||\nabla\varphi_0| \leq \rho_2 \int_\Omega |\nabla\varphi| |\nabla\varphi_0|.$$

By the Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega} |\nabla\varphi|^2 \leq C\left(\rho_1, \rho_2, \varphi_0\right) = C. \tag{3.4}$$

This means that $\rho(\omega)\left|\nabla\varphi\right|^2 \in L^1(\Omega)$. We can use the equation for φ to show that this last term also belongs to $H^{-1}(\Omega)$. Indeed, let $\phi \in \mathcal{D}(\Omega)$ and take $\xi = \phi\varphi$ as a test function in (3.2). We have

$$\int_{\Omega} \rho(\omega)\nabla\varphi\nabla(\phi\varphi) = 0$$

that is,

$$\int_{\Omega} \rho(\omega)|\nabla\varphi|^2\phi = -\int_{\Omega} \rho(\omega)\varphi\nabla\varphi\nabla\phi = \langle \operatorname{div}(\rho(\omega)\varphi\nabla\varphi), \phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)},$$

and thus,

$$\rho(\omega)|\nabla\varphi|^2 = \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \quad \text{ in } \mathcal{D}'(\Omega). \tag{3.5}$$

Since $\rho(\omega)\varphi\nabla\varphi \in L^2(\Omega)^N$, we deduce the regularity

$$\rho(\omega)\left|\nabla\varphi\right|^2 \in H^{-1}(\Omega).$$

The identity (3.5) is the key that allows us to solve the classical thermistor problem (1.1) and to introduce the notion of a capacity solution as well.

Now, we set out the following nonlinear elliptic problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) = \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

The variational formulation of this problem is given as follows:

$$\begin{cases} \text{To find } u \in W_0^{1,\vec{p}}(\Omega) \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i-2}\partial_i u \partial_i v = -\sum_{i=1}^{N} \int_{\Omega} \rho(\omega)\varphi\partial_i\varphi\partial_i v, \text{ for all } v \in W_0^{1,\vec{p}}(\Omega). \end{cases} \tag{3.7}$$

Notice that $\operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \in H^{-1}(\Omega) \hookrightarrow W^{-1,\vec{p}'}(\Omega)$. The problem (3.6) is of the form $Au = f$, with $f \in W^{-1,\vec{p}'}(\Omega)$ and the operator A satisfies the Leray-Lions conditions on $W_0^{1,\vec{p}}(\Omega)$; then, by the Minty-Browder theorem, the problem (3.6) has at least one weak solution $u \in W_0^{1,\vec{p}}(\Omega)$.

We may define the operator $G\colon \omega \in L^{p_1}(\Omega) \mapsto G(\omega) = u \in W_0^{1,\vec{p}}(\Omega)$ with u being the unique solution to (3.7). Since $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_1}(\Omega)$, G maps $L^{p_1}(\Omega)$ into itself. Our strategy is to show that G satisfies the hypotheses of Schauder's fixed point theorem, which will then yield the desired weak solution to problem (3.1).

We first show the following estimate:

$$\|u\|_{L^{p_1}(\Omega)} \leq C(\varphi_0, \rho_2) = R. \tag{3.8}$$

Indeed, taking $v = u$ as a test function in (3.7), from (3.3) and (A.1)$'$ we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i} \le \rho_2 \|\varphi_0\|_{L^\infty(\Omega)} \sum_{i=1}^{N} \left(\int_\Omega |\partial_i \varphi|^{p_i'} \right)^{1/p_i'} \left(\int_\Omega |\partial_i u|^{p_i} \right)^{1/p_i}.$$

Moreover, using Young's inequality we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i} \le \sum_{i=1}^{N} \rho_2^{p_i'} \|\varphi_0\|_{L^\infty(\Omega)}^{p_i'} C(\Omega, p_i) \int_\Omega |\partial_i \varphi|^2 + \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i},$$

it follows by (3.4) that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i} \le C(\varphi_0, \rho_2).$$

Due to the Poincaré inequality, for some constant $C = C(\Omega) > 0$, it is

$$\|u\|_{L^{p_1}(\Omega)} \le C \|\partial_1 u\|_{L^{p_1}(\Omega)},$$

which yields the estimate (3.8).

From (3.8), for $R > 0$ large enough, the operator G transforms the ball $B_R = \left\{ v \in L^{p_1}(\Omega) \ / \ \|v\|_{L^{p_1}(\Omega)} \le R \right\}$ into itself. Since the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_1}(\Omega)$ is compact, the operator G is compact. To complete the proof, it remains to show that G is continuous. Indeed, let $(\omega_n) \subset B_R$ such that $\omega_n \to \omega$ strongly in $L^{p_1}(\Omega)$ and consider the corresponding functions to ω_n and ω, that is, $u_n = G(\omega_n)$, φ_n, $u = G(\omega)$ and φ. Put $F_n^i = \rho(\omega_n) \varphi_n \partial_i \varphi_n$ and $F^i = \rho(\omega) \varphi \partial_i \varphi$, $1 \le i \le N$. We have to show that

$$u_n \to u = G(\omega) \text{ strongly in } L^{p_1}(\Omega).$$

It is easy to check that, for some subsequence still denoted in the same way, we have

$$F_n^i \to F^i \text{ strongly in } L^2(\Omega) \text{ for each } i,\ 1 \le i \le N.$$

By subtracting the respective equations of (3.7) for u_n and u, and taking $v = u_n - u$ as a test function, we obtain

$$\sum_{i=1}^{N} \int_\Omega \left[|\partial_i u_n|^{p_i - 2} \partial_i u_n - |\partial_i u|^{p_i - 2} \partial_i u \right] \partial_i (u_n - u) = - \sum_{i=1}^{N} \int_\Omega (F_n^i - F^i) \partial_i (u_n - u). \tag{3.9}$$

Since $p_i \ge 2$, for all $i = 1, \ldots, N$, then for some $\alpha > 0$, we have (see [6])

$$\left[|\partial_i u_n|^{p_i - 2} \partial_i u_n - |\partial_i u|^{p_i - 2} \partial_i u \right] \partial_i (u_n - u) \ge \alpha |\partial_i (u_n - u)|^{p_i} \tag{3.10}$$

which implies that, using (3.9),

$$\alpha \sum_{i=1}^{N} \int_\Omega |\partial_i (u_n - u)|^{p_i} \le \sum_{i=1}^{N} \|F_n^i - F^i\|_{L^{p_i'}(\Omega)} \|\partial_i (u_n - u)\|_{L^{p_i}(\Omega)}.$$

Using Young's inequality, we get

$$\alpha \sum_{i=1}^{N} \int_\Omega |\partial_i (u_n - u)|^{p_i} \le \sum_{i=1}^{N} \frac{1}{p_i'} \left(\frac{1}{\alpha} \right)^{p_i'/p_i} \|F_n^i - F^i\|_{L^{p_i'}(\Omega)}^{p_i'} + \sum_{i=1}^{N} \frac{\alpha}{p_i} \|\partial_i (u_n - u)\|_{L^{p_i}(\Omega)}^{p_i}.$$

Therefore

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i(u_n-u)|^{p_i}\leq\sum_{i=1}^{N}\frac{1}{p_i'}\left(\frac{1}{\alpha}\right)^{p_i'/p_i}C(\Omega,p_i)\|F_n^i-F^i\|_{L^2(\Omega)}^2\ \ \to 0.$$

Finally,

$$u_n\to u \text{ strongly in } W_0^{1,\vec{p}}(\Omega)\hookrightarrow L^{p_1}(\Omega).$$

consequently, the operator G has at least one fixed point: $u=G(u)$, which gives a weak solution to problem (3.6). This completes the proof of Theorem 3.2. □

Proof of Theorem 3.1. The proof of this result is divided into four steps. We first introduce a sequence of approximate problems and derive a priori estimates for the approximate solutions. Then, we show two ordinary results, namely the strong convergence, modulo a subsequence, of both (∇u_n) and (φ_n) in $L^1(\Omega)$, where (u_n,φ_n) stand for a weak solution to the approximate problem of (1.1). The passing to the limit will lead to the desired result.

Step 1 : Approximate problems and a priori estimates.

For every $n\in\mathbb{N}$, we introduce the following regularization of the data,

$$\rho_n(s)=\rho(s)+\frac{1}{n}. \tag{3.11}$$

and consider the approximate system given as

$$-\sum_{i=1}^{N}\partial_i\left(|\partial_i u_n|^{p_i-2}\partial_i u_n\right)=\rho_n(u_n)|\nabla\varphi_n|^2 \qquad \text{in } \Omega, \tag{3.12}$$

$$\operatorname{div}(\rho_n(u_n)\nabla\varphi_n)=0 \qquad \text{in } \Omega, \tag{3.13}$$

$$\varphi_n=\varphi_0 \qquad \text{on } \partial\Omega, \tag{3.14}$$

$$u_n=0 \qquad \text{on } \partial\Omega. \tag{3.15}$$

From (A.1), we have that $n^{-1}\leq\rho_n(s)\leq\bar{\rho}+1=\rho_3,$ for all $s\in\mathbb{R}$. Consequently, for every $n\in\mathbb{N}$, ρ_n satisfies $(\mathrm{A.1})'$. Using Theorem 3.2 we deduce the existence of a weak solution (u_n,φ_n) to the system (3.12)-(3.15).

The maximum principle yields

$$\|\varphi_n\|_{L^\infty(\Omega)}\leq\|\varphi_0\|_{L^\infty(\Omega)}\,, \tag{3.16}$$

therefore, there exists a function $\varphi\in L^\infty(\Omega)$ and a subsequence, still denoted in the same way, such that

$$\varphi_n\to\varphi \text{ weakly-}*\ \text{ in } L^\infty(\Omega). \tag{3.17}$$

Now let multiply (3.13) by $\xi=\varphi_n-\varphi_0\in H_0^1(\Omega)$ and integrate over Ω, we get

$$\int_\Omega\rho_n\left(u_n\right)\nabla\varphi_n\nabla\left(\varphi_n-\varphi_0\right)=0,$$

hence,

$$\int_\Omega\rho_n\left(u_n\right)\left|\nabla\varphi_n\right|^2\leq C_1,\ \text{for all } n\geq 1 \tag{3.18}$$

where $C_1 = C_1\left(\bar{\rho}, \|\varphi_0\|_{H^1(\Omega)}\right)$. Consequently, the sequence $(\rho_n(u_n)\nabla\varphi_n)$ is bounded in $L^2(\Omega)^N$. Therefore, there exists a function $\Phi \in L^2(\Omega)^N$ and a subsequence, still denoted in the same way, such that

$$\rho_n(u_n)\nabla\varphi_n \to \Phi \text{ weakly in } L^2(\Omega)^N. \tag{3.19}$$

This weak limit function $\Phi \in L^2(\Omega)^N$ is in fact the third component of the triplet appearing in the Definition 2.1 of a capacity solution. We have also

$$\left\langle \rho_n(u_n)|\nabla\varphi_n|^2, \xi \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = -\int_\Omega \rho_n(u_n)\varphi_n\nabla\varphi_n\nabla\xi, \quad \forall\xi \in H_0^1(\Omega)$$

with $(\operatorname{div}(\rho_n(u_n)\varphi_n\nabla\varphi_n)) \subset H^{-1}(\Omega) \hookrightarrow W^{-1,\vec{p}'}(\Omega)$ a bounded sequence due to $(A.1)$, (3.16) and (3.18).

Taking u_n as a test function in (3.12), we obtain

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} = -\sum_{i=1}^N \int_\Omega \rho_n(u_n)\varphi_n\partial_i\varphi_n\partial_i u_n.$$

In virtue of (3.11) and (3.16), and applying Hölder's inequality, we get

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} \leq \sum_{i=1}^N \|\varphi_0\|_{L^\infty(\Omega)}\rho_3^{1/2}\int_\Omega \rho_n(u_n)^{1/2}|\partial_i\varphi_n||\partial_i u_n|.$$

Using Young's inequality, we obtain

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} \leq \sum_{i=1}^N \frac{1}{p_i'}\|\varphi_0\|_{L^\infty(\Omega)}^{p_i'}\rho_3^{p_i'/2}\int_\Omega \rho_n(u_n)^{p_i'/2}|\partial_i\varphi_n|^{p_i'} + \sum_{i=1}^N \frac{1}{p_i}\int_\Omega |\partial_i u_n|^{p_i}.$$

That is

$$\sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i} \leq C. \tag{3.20}$$

Then (u_n) is bounded in $W_0^{1,\vec{p}}(\Omega)$. Therefore, there exists a function $u \in W_0^{1,\vec{p}}(\Omega)$ and a subsequence, still denoted in the same way, such that

$$u_n \to u \text{ weakly in } W_0^{1,\vec{p}}(\Omega). \tag{3.21}$$

Moreover, due to (3.20) and (3.21), we deduce that (u_n) is relatively compact in $L^{p_1}(\Omega)$, therefore we may assume that

$$u_n \to u \text{ strongly in } L^{p_1}(\Omega) \text{ and a.e. in } \Omega. \tag{3.22}$$

Finally, from (A.1), we deduce

$$\rho_n(u_n) \to \rho(u) \text{ weakly-}* \text{ in } L^\infty(\Omega) \text{ and a.e. in } \Omega. \tag{3.23}$$

On the other hand, due to (3.20), we have

$$\int_\Omega \left||\partial_i u_n|^{p_i-2}\partial_i u_n\right|^{p_i'} = \int_\Omega |\partial_i u_n|^{p_i} \leq C, \ \ i = 1, \ldots, N.$$

Thus $|\partial_i u_n|^{p_i-2}\partial_i u_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists function $\vartheta_i \in L^{p_i'}(\Omega)$ and a subsequence, still denoted in the same way, such that

$$|\partial_i u_n|^{p_i-2}\partial_i u_n \to \vartheta_i \text{ weakly in } L^{p_i'}(\Omega), \text{ for all } i=1,\dots,N. \tag{3.24}$$

Step 2: Almost everywhere convergence of the gradients (∇u_n).

In this step we prove that $\vartheta_i = |\partial_i u|^{p_i-2}\partial_i u$, for all $i=1,\dots,N$. To do so, we need, modulo a subsequence, that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega$$

i.e.

$$\partial_i u_n \to \partial_i u \text{ a.e. in } \Omega, \quad \text{for all } i=1,\dots,N. \tag{3.25}$$

In fact we will establish the following proposition.

Proposition 3.1. *Let* (u_n,φ_n) *be a solution to the problem* (3.12)-(3.15) *and* $u \in W_0^{1,\vec{p}}(\Omega)$ *given* (3.21). *Then, for a suitable subsequence, still denoted in the same way,*

$$\partial_i u_n \to \partial_i u \text{ a.e. in } \Omega, \text{ for all } i=1,\dots,N.$$

Proof. Let $0<\theta<1$ and $k>0$. We define the truncation function at height k, T_k, as

$$T_k(s) = \min(k, \max(s,-k)) = \begin{cases} s & \text{if } |s| \le k \\ ks/|s| & \text{if } |s| > k \end{cases}$$

Consider

$$I_{\Omega,n} = \int_\Omega \left\{ \sum_{i=1}^N \left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u \right) \partial_i (u_n - u) \right\}^\theta .$$

We will prove that $I_{\Omega,n} \to 0$ as $n\to\infty$. Indeed,

$$\begin{aligned} I_{\Omega,n} &= \int_{A_k} \left\{ \sum_{i=1}^N \left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u \right) \partial_i (u_n - u) \right\}^\theta \\ &\quad + \int_{B_k} \left\{ \sum_{i=1}^N \left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u \right) \partial_i (u_n - u) \right\}^\theta \\ &= I_{A_k,n} + I_{B_k,n}, \end{aligned}$$

where

$$A_k = \{x\in\Omega : |u(x)| \le k\}, \quad B_k = \{x \in \Omega : |u(x)| > k\}.$$

We can write $I_{A_k,n}$ as:

$$I_{A_k,n} = \int_{A_k} \left\{ \sum_{i=1}^N \left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u) \right) \partial_i (u_n - T_k(u)) \right\}^\theta ,$$

which is smaller than

$$\int_{\Omega} \left\{ \sum_{i=1}^N \left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u) \right) \partial_i (u_n - T_k(u)) \right\}^\theta = J_{\Omega,n}.$$

According to the a priori estimate proved in the previous step,

$$\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i u|^{p_i-2}\partial_i u\right)\partial_i\left(u_n-u\right)$$

is bounded in $L^1(\Omega)$. Using Hölder inequality, we get

$$I_{B_k,n}\le\left[\int_{B_k}\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i u|^{p_i-2}\partial_i u\right)\partial_i\left(u_n-u\right)\right]^{\theta}\operatorname{meas}\left(B_k\right)^{1-\theta}$$

hence

$$I_{A_k,n}+I_{B_k,n}\le J_{\Omega,n}+c_2\operatorname{meas}\left(B_k\right)^{1-\theta}=J_{\Omega,n}+\omega_1(k).$$

On the other hand, $J_{\Omega,n}$ can be written as $(\varepsilon>0)$

$$\begin{aligned}J_{\Omega,n}=&\int\limits_{\{|u_n-T_k(u)|\le\varepsilon\}}\Big\{\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)\right)\partial_i\left(u_n-T_k(u)\right)\Big\}^{\theta}\\&+\int\limits_{\{|u_n-T_k(u)|>\varepsilon\}}\Big\{\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)\right)\partial_i\left(u_n-T_k(u)\right)\Big\}^{\theta}.\end{aligned}\tag{3.26}$$

The first integral can be written as

$$\int_{\Omega}\left\{\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)\right)\partial_i T_\varepsilon\left(u_n-T_k(u)\right)\right\}^{\theta}.$$

Using Hölder's inequality and the a priori estimate of the previous step we obtain

$$\begin{aligned}J_{\Omega,n}\le&\left\{\sum_{i=1}^{N}\int_{\Omega}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u)\right)\partial_i T_\varepsilon\left(u_n-T_k(u)\right)\right\}^{\theta}\\&\times(\operatorname{meas}(\Omega))^{1-\theta}+c_3\operatorname{meas}\left\{x\in\Omega:|u_n-T_k(u)|>\varepsilon\right\}^{1-\theta}.\end{aligned}$$

On the other hand, using $T_\varepsilon\left(u_n-T_k(u)\right)$ in (3.12) as a test function it yields

$$\begin{aligned}J_{\Omega,n}\le c_4\Big[&\int_{\Omega}\rho_n\left(u_n\right)\left|\nabla\varphi_n\right|^2T_\varepsilon\left(u_n-T_k(u)\right)\\&-\sum_{i=1}^{N}\int_{\Omega}(|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u))\partial_i T_\varepsilon\left(u_n-T_k(u)\right)\Big]^{\theta}\\&+c_3\operatorname{meas}\left\{x\in\Omega:|u_n-T_k(u)|>\varepsilon\right\}^{1-\theta}.\end{aligned}$$

We remark from (3.18) that

$$\int_{\Omega}\rho_n\left(u_n\right)\left|\nabla\varphi_n\right|^2T_\varepsilon\left(u_n-T_k(u)\right)\le\varepsilon\int_{\Omega}\rho_n\left(u_n\right)\left|\nabla\varphi_n\right|^2\le\varepsilon C,$$

we have

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega (|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u))\partial_i T_\varepsilon\left(u_n - T_k(u)\right)\\
&=\sum_{i=1}^{N}\int_\Omega (|\partial_i T_k(u)|^{p_i-2}\partial_i T_k(u))\partial_i T_\varepsilon\left(u - T_k(u)\right)\\
&=\omega_2(k),
\end{aligned}$$

and also

$$\limsup_{n\to\infty} \text{meas}\left\{|u_n - T_k(u)| > \varepsilon\right\}^{1-\theta} \leq \text{meas}\left\{|u - T_k(u)| \geq \varepsilon\right\}^{1-\theta} = \omega_3(k),$$

Thus

$$\limsup_{n\to\infty} J_{\Omega,n} \leq c_4\left(\varepsilon C - \omega_2(k)\right)^\theta + c_3\omega_3(k).$$

Hence we have shown that

$$\limsup_{n\to\infty}\left[I_{A_k,n} + I_{B_k,n}\right] \leq \omega_1(k) + c_4\left(\varepsilon C - \omega_2(k)\right)^\theta + c_3\omega_3(k),$$

where $\omega_i(k)$ converge to zero as k tends to infinity. Letting $k \to \infty$ then $\varepsilon \to 0$, we obtain that $I_{\Omega,n} \to 0$, that is

$$\left\|\left\{\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u\right)\partial_i\left(u_n - u\right)\right\}^\theta\right\|_{L^1(\Omega)} \to 0.$$

Thus, for a suitable subsequence still denoted by (u_n), we have

$$\left\{\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u\right)\partial_i\left(u_n - u\right)\right\}^\theta \to 0 \quad \text{a.e.}$$

Since θ is positive, we have

$$\left\{\sum_{i=1}^{N}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n - |\partial_i u|^{p_i-2}\partial_i u\right)\partial_i\left(u_n - u\right)\right\} \to 0 \quad \text{a.e.}$$

Finally, from (3.10), it yield

$$\partial_i u_n \to \partial_i u \text{ a.e.} \quad \text{for } i = 1, \ldots, N$$

and so

$$\nabla u_n \to \nabla u \text{ a.e.}$$

□

Step 3 : L^1-convergence of (φ_n).

In this step, we will show that $\varphi_n \to \varphi$ strongly in $L^1(\Omega)$ modulo a subsequence. This result generalizes Lemma 4 of González Montesinos and Ortegón Gallego in [11] which, in its turn, is an adaptation of a result due to Xu in [20] (see also [4]).

Lemma 3.1. *Let $\vec{p} = (p_1, \dots, p_N)$, such that $p_i \geq 2$ for all $i = 1, \dots, N$ and let (u_n) be a bounded sequence in $W_0^{1,\vec{p}}(\Omega)$. Then, there exists a subsequence $(u_{n(k)}) \subset (u_n)$, such that for every $\varepsilon > 0$, there correspond a positive number $M = M(\varepsilon)$ and a function $\psi \in W^{1,1}(\Omega)$ satisfying the following properties:*

$$0 \leq \psi \leq 1, \tag{3.27}$$

$$\|\psi - 1\|_{L^1(\Omega)} + \|\nabla \psi\|_{L^1(\Omega)} \leq \varepsilon, \tag{3.28}$$

$$|u|,\ \left|u_{n(k)}\right| \leq M \ \text{ on } \{\psi > 0\} \text{ for all } k \geq 1, \tag{3.29}$$

where $u_{n(k)} \to u$ strongly in $L^{p_1}(\Omega)$ and a.e. in Ω.

Proof. Since $p_1 \geq 2$, we have

$$L^{p_1}(\Omega) \hookrightarrow L^{p_1'}(\Omega). \tag{3.30}$$

Since (u_n) is relatively compact in $L^{p_1}(\Omega)$ and owing to (3.30), we can extract a subsequence $\left(u_{n(k)}\right) \subset (u_n)$ such that

$$\sum_{k=1}^{\infty} \left\|u_{n(k)} - u\right\|_{L^{p_1'}(\Omega)} \leq 1. \tag{3.31}$$

Fix $K > 0$, we define the function γ as

$$\gamma = (|u| - K)^+ + \sum_{k=1}^{\infty} \left(\left|u_{n(k)} - u\right| - K\right)^+.$$

Putting $v_k = u_{n(k)} - u, k \geq 1$, and $v_0 = u$, we have

$$\begin{aligned}
&\int_\Omega (|v_k| - K)^+ + \int_\Omega \left|\nabla (|v_k| - K)^+\right| \\
&= \int_{\{|v_k|>K\}} (|v_k| - K)^+ \frac{|v_k|}{|v_k|} + \int_{\{|v_k|>K\}} \left|\nabla (|v_k| - K)^+\right| \frac{|v_k|}{|v_k|} \\
&\leq \frac{1}{K} \left(\|v_k\|_{L^{p_1}(\Omega)} + \|\nabla v_k\|_{(L^{p_1}(\Omega))^N}\right) \|v_k\|_{L^{p_1'}(\Omega)}.
\end{aligned}$$

Summing these inequalities, bearing in mind that $\left(u_{n(k)}\right)$ and (v_k) are bounded in $W_0^{1,\vec{p}}(\Omega)$, and using (3.31), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left(\left\|(|v_k| - K)^+\right\|_{L^1(\Omega)} + \left\|\nabla (|v_k| - K)^+\right\|_{L^1(\Omega)}\right) \\
\leq &\frac{C_0}{K} \sum_{k=0}^{\infty} \|v_k\|_{L^{p_1'}(\Omega)} = \frac{C_0}{K} \left(\|u\|_{L^{p_1'}(\Omega)} + \sum_{k=1}^{\infty} \left\|u_{n(k)} - u\right\|_{L^{p_1'}(\Omega)}\right) \\
\leq &\frac{C_0}{K} \left(\|u\|_{L^{p_1'}(\Omega)} + 1\right) = \frac{C}{K}.
\end{aligned}$$

Hence,

$$\|\gamma\|_{W^{1,1}(\Omega)} \leq \frac{C}{K}.$$

Finally, for a given $\varepsilon > 0$, take $K = C/\varepsilon$, then

$$\|\gamma\|_{L^1(\Omega)} + \|\nabla\gamma\|_{L^1(\Omega)} \leq \varepsilon.$$

If we take the function $\psi = (1-\gamma)^+$, then it is easy to check that the conditions (3.27)-(3.29) are all satisfied for $K \geq C/\varepsilon$ and $M = K + 1$. $\square$

Lemma 3.2. *For any function $S \in C_0^1(\mathbb{R})$ there exists a subsequence, still denoted in the same way, such that*

$$S(u_n)\varphi_n \to S(u)\varphi \text{ weakly in } H^1(\Omega). \tag{3.32}$$

Furthermore, if $0 \leq S \leq 1$, then there exists a positive constant C, independent of S, such that

$$\limsup_{n\to\infty} \int_\Omega \rho_n(u_n)\left|\nabla\left[S(u_n)\varphi_n - S(u)\varphi\right]\right|^2 \leq C\left\|S'\right\|_\infty\left(1 + \left\|S'\right\|_\infty\right).$$

Lemma 3.3. *There exists a subsequence $\left(\varphi_{n(k)}\right) \subset (\varphi_n)$ such that*

$$\lim_{k\to\infty}\int_\Omega \left|\varphi_{n(k)} - \varphi\right| = 0.$$

Proof. The proof of these lemmas are identical to those of Lemma 3.5 and 3.6 in [4]. $\square$

Now we are ready to end the proof of Theorem 3.1.

Step 4 : Passing to the limit.

Owing to (3.17), (3.19) and (3.21) it is straightforward to check the condition (C_1) of Definition 2.1. The convergences in Proposition 3.1 and Lemma 3.3 lead us to the condition (C_2) of Definition 2.1. In order to obtain the condition (C_3), using Proposition 3.1 and Lemma 3.3 again with (3.32), it is enough to make $k \to \infty$ in the following expression:

$$S\left(u_{n(k)}\right)\rho_{n(k)}\left(u_{n(k)}\right)\nabla\varphi_{n(k)} = \rho_{n(k)}\left(u_{n(k)}\right)\left[\nabla\left(S\left(u_{n(k)}\right)\varphi_{n(k)}\right) - \varphi_{n(k)}\nabla S\left(u_{n(k)}\right)\right].$$

This completes the proof of Theorem 3.1. $\square$

Remark 3.1. We recall that if we choose $p_i = p$, for any $i = 1, \ldots, N$, we obtain the classic results.

Remark 3.2. All the results in Section 3 also hold if our anisotropic operator is changed to a more general one, i.e., a non linear differential operator A from $W_0^{1,\vec{p}}(\Omega)$ into its dual of the form

$$A(u) = -\sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u)$$

where each $a_i(x, s, \xi)$ is a Caratheodory function on $\Omega\times\mathbb{R}\times\mathbb{R}^N$ such that, for some constants α, $\beta > 0$, and for all ξ, $\xi' \in \mathbb{R}^N$, and a.e. $x \in \Omega$, the following inequalities hold:

$$\left[a_i(x,s,\xi) - a_i(x,s,\eta)\right](\xi_i - \eta_i) \geq \alpha|\xi_i - \eta_i|^{p_i}, \text{ for all } i = 1, \ldots, N,$$

and

$$|a_i(x,s,\xi)| \leq \beta\left[c_i(x) + |s|^{p_1/p_i'} + |\xi|^{p_i - 1}\right], \text{ for all } i = 1, \ldots, N,$$

where $c_i \in L^{p_i'}(\Omega)$, for all $i = 1, \ldots, N$ and $a_i(x, s, 0) = 0$ for all $i = 1, \ldots, N$.

4. Numerical simulations

We have carried out some 2D numerical simulations for the approximation of the solution to system (1.1). In this case, this problem can be written as

$$\begin{cases} -\dfrac{\partial}{\partial x}\left[\left|\dfrac{\partial u}{\partial x}\right|^{p_1-2}\dfrac{\partial u}{\partial x}\right]-\dfrac{\partial}{\partial y}\left[\left|\dfrac{\partial u}{\partial y}\right|^{p_2-2}\dfrac{\partial u}{\partial y}\right]=\rho(u)|\nabla\varphi|^2 & \text{in } \Omega,\\ \operatorname{div}(\rho(u)\nabla\varphi)=0 & \text{in } \Omega,\\ \varphi=\varphi_0 & \text{on } \partial\Omega,\\ u=u_0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega=(0,L)\times(0,L)\subset\mathbb{R}^2$, $L=0.05$ m, $p_1,p_2\geq 2$, $\vec{p}=(p_1,p_2)$, the electric conductivity ρ is given by

$$\rho(s)=10e^{-|s-30|/20},\quad s\in\mathbb{R}. \tag{4.2}$$

The potential on the boundary has been taken as $\varphi_0(x,y)=V_0y/L$, where V_0 is the potential difference applied between the north and south sides of Ω. Finally, the boundary condition for the temperature is assumed to be constant, $u_0=30$ °C (that is, the u of system (1.1) is in fact $u-u_0$ of (4.1) and $\rho(s)$ is $\rho(s-u_0)$).

In order to compute the numerical approximation of (4.1), we propose the iterative algorithm below. First, take some values $\varepsilon_0,\varepsilon_1>0$ small enough, $\delta>0$ and introduce the functions $a_1(\nabla u)$ and $a_2(\nabla u)$ given by

$$a_1(\nabla u)=\varepsilon_0+\left|\frac{\partial u}{\partial x}\right|^{p_1-2},\quad a_2(\nabla u)=\varepsilon_0+\left|\frac{\partial u}{\partial y}\right|^{p_2-2}.$$

Then, proceed as follows:

Step 1: Initialization. Take an initial guess $u^0\in H^1(\Omega)$ with $u^0=u_0$ on $\partial\Omega$.

Step 2: Intermediate iteration. Assume u^n is already known. Compute $\varphi^{n+1}\in H^1(\Omega)$ solution to

$$\begin{cases} \operatorname{div}(\rho(u^n)\nabla\varphi^{n+1})=0 & \text{in } \Omega,\\ \varphi^{n+1}=\varphi_0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Then, compute $\tilde{u}^{n+1}\in H^1(\Omega)$ solution to

$$\begin{cases} \dfrac{\tilde{u}^{n+1}-u^n}{\delta}-\dfrac{\partial}{\partial x}\left[a_1(\nabla u^n)\dfrac{\partial\tilde{u}^{n+1}}{\partial x}\right] & \\ -\dfrac{\partial}{\partial y}\left[a_2(\nabla u^n)\dfrac{\partial\tilde{u}^{n+1}}{\partial y}\right]=\rho(u^n)|\nabla\varphi^{n+1}|^2 & \text{in } \Omega,\\ \tilde{u}^{n+1}=u_0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Step 3: New iteration u^{n+1}. Define the new iteration as

$$u^{n+1}=\frac{\tilde{u}^{n+1}+u^n}{2}.$$

Step 4: Termination test. If $\|\tilde{u}^{n+1} - u^n\| < \varepsilon_1$ then stop and we keep u^{n+1} as the approximate solution to (4.1), otherwise we increase n by one and repeat the procedure from Step 2.

Remark 4.1. The numerical algorithm described in steps 1–4 generates a sequence (u^n, φ^n), $n = 0, 1, 2, \ldots$ Each function u^n and φ^n is computed from the resolution of certain linear elliptic problems in Ω. The equations (4.3) and (4.4) consist of a fixed point technique. We could consider the same procedure without the first term in (4.4), that is, $(\tilde{u}^{n+1} - u^n)/\delta$. In this case, in general, the resulting algorithm does not show good convergence properties. The introduction of the term $(\tilde{u}^{n+1} - u^n)/\delta$ may lead to the stabilization of the method and to the convergence of the sequence (u^n, φ^n). However, in general, the convergence of this algorithm is not guaranteed and our numerical computations have shown that this property is strongly related to the choice of the parameter δ for values of p_1 or p_2 close to 2 or 5. In a certain interval (p_*, p^*), with $2 \le p_* < p^* \le 5$, this algorithm has shown to be convergent for $p_1, p_2 \in (p_*, p^*)$ and a wide range of values of δ even for $\delta > 1$ (see Tables 1–3 below).

Remark 4.2. The introduction of the small parameter $\varepsilon_0 > 0$, in the definition of the diffusion functions $a_1(\nabla u)$ and $a_2(\nabla u)$, is necessary in order to assure the well-posedness of the problem (4.4). In our numerical simulations, we have taken $\varepsilon_0 = 10^{-12}$.

Remark 4.3. We may consider a different approach by just taking $u^{n+1} = \tilde{u}^{n+1}$ so that the Step 3 is skipped. This approach generates a new sequence (u^n, φ^n) which, in most cases, is not convergent. In fact, the sequence (u^n) shows an oscillatory behavior. Thus, by defining u^{n+1} as the average between $\tilde{u}^{n+1}$ and u^n given in Step 3, we stabilize the sequence (u^n) and, in all cases for $p_1, p_2 \in (2, 5)$, its convergence as it is shown in Tables 1–3.

Remark 4.4. In order to select the initial guess u^0 we did the following. First, we compute the function φ^0 solution to

$$\begin{cases} \operatorname{div}(\rho(u_0)\nabla\varphi^0) = 0 & \text{in } \Omega, \\ \varphi^0 = \varphi_0 & \text{on } \partial\Omega. \end{cases}$$

Then, we compute u^0 as the unique solution to the elliptic problem

$$\begin{cases} -\Delta u^0 = \rho(u_0)|\nabla\varphi^0|^2 & \text{in } \Omega, \\ u^0 = u_0 & \text{on } \partial\Omega. \end{cases}$$

Remark 4.5. The norm $\|\cdot\|$ in the termination test of Step 4 may be $\|\cdot\|_{H_0^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^1(\Omega)}$ or $\|\cdot\|_{W^{1,\vec{p}}(\Omega)}$.

Since the boundary-value problems (4.3) and (4.4) have been solved by means of the FEM, where the discrete solutions are always bounded, we have used instead the L^∞-norm in Ω with tolerance $\varepsilon_1 = 0.5 \times 10^{-11}$, that is, the algorithm is stopped at iteration $n+1$ if $\|\tilde{u}^{n+1} - u^n\|_\infty < 0.5 \times 10^{-11}$.

Obviously, since the convergence of this method is not in general guaranteed, this stop condition may or may not be ever achieved. Thus, we need to consider a maximum number of iterations $N_{\text{max\_iter}}$, high enough, so that whenever $n =$

$N_{\text{max\_iter}}+1$ the algorithm is forced to stop since it may not be convergent. In that case, one may try to change the value of the parameter δ to a smaller one and start over again. In fact, we have successfully proceeded this way when one of the exponent p_1 or p_2 are close to 2 or 5, where for $\delta=10^{-2}$ we reached the maximum number of iterations $N_{\text{max\_iter}}$ whereas for $\delta=10^{-6}$ the algorithm seems to be convergent in some cases but with a very low rate of convergence (tables 1 and 2).

In our numerical simulations, for $\delta=10^{-2}$ we took $N_{\text{max\_iter}}=600$ whereas for $\delta=10^{-6}$ we took $N_{\text{max\_iter}}=1200$.

Remark 4.6. Another possible algorithm consists of keeping the monotone structure of the original operator in the equation for u. Thus, instead of (4.4) we may consider for instance

$$\begin{cases} \dfrac{\tilde{u}^{n+1}-u^n}{\delta}-\dfrac{\partial}{\partial x}\left[\left|\dfrac{\partial\tilde{u}^{n+1}}{\partial x}\right|^{p_1-2}\dfrac{\partial\tilde{u}^{n+1}}{\partial x}\right] \\ -\dfrac{\partial}{\partial y}\left[\left|\dfrac{\partial\tilde{u}^{n+1}}{\partial y}\right|^{p_2-2}\dfrac{\partial\tilde{u}^{n+1}}{\partial y}\right]=\rho(u^n)|\nabla\varphi^{n+1}|^2 \text{ in } \Omega, \\ \tilde{u}^{n+1}=u_0 \text{ on } \partial\Omega. \end{cases} \tag{4.5}$$

This is a nonlinear problem and no direct method can be used to solve it. Of course, we may characterize its solution $\tilde{u}^{n+1}$ as the minimum of a certain energy functional and then, in its turn, solve this optimization problem by generating a minimizing sequence [5,6].

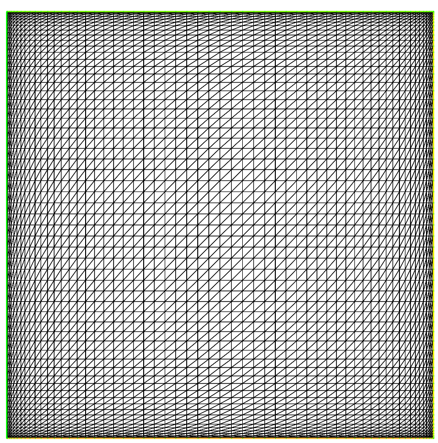


Figure 1. Initial triangulation of Ω.

Remark 4.7. Other iterative algorithms are possible in order to approximate a solution of the nonlinear problem (1.1). For instance, me may consider Newton's method or a least squares technique combined with the conjugate gradient method. In both cases, the corresponding algorithm requires to compute the gradient of certain operators with respect to u. In particular, we would need an additional regularity assumption for the electric conductivity, namely $\rho\in W^{1,\infty}(\mathbb{R})$. However, the definition of these two methods cannot be developed for the continuous problem (1.1) but for a FEM discretized version of the variational formulation (3.1).

The linear problems (4.3) and (4.4) have been solved by the FEM and implemented in Freefem++ software package [15]. We first build a mesh of the domain Ω with high density small triangles near the boundary, as shown in Figure 1. When the iteration $n = 15$ is reached, we perform a mesh adapting technique in order to better capture the possible high slope of the solution u. This adapting technique is applied again at iterations $n = 201$ and $n = 402$ whenever these values are reached. To do so, Freefem++ has a built-in function called adaptmesh based on the mesh adapting software BAMG [14]. In this way, the following line within a Freefem++ script

```
if (n==15 || n%201==0){Th=adaptmesh(Th,un);}
```

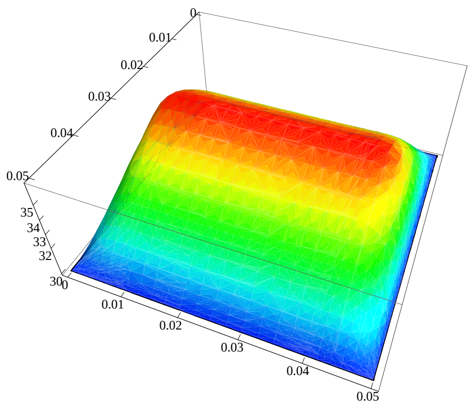
will execute the mesh adapting algorithm just at the desired iterations $n = 15$, $n = 201$ and $n = 402$. We use two degree Lagrange polynomial approximation.

Figure 2 exhibits the distribution of the temperature u for the exponents $p_1 = 3.6$ and $p_2 = 2.8$, 3.2, 3.6, 4.0, 4.4 and 4.8, respectively. The behavior of the solution puts in evidence the anisotropy of the semiconductor material occupying the region Ω. In figures 2(a) and 2(b) we have $p_1 > p_2$ and the solution takes values very close to its maximum along an interval parallel to the y-axis. This interval reduces to just one point when $p_1 = p_2$ (Figure 2(c)) which is quite similar to the isotropic case. Then, for $p_1 < p_2$ (figures 2(d)–2(f)) the situation is reversed so that the solution now takes values close to its maximum along an interval parallel to the x-axis.

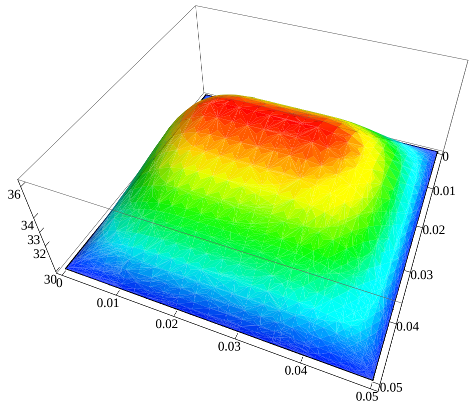
Figure 3 shows the final meshes after the adapting technique in several cases. We notice that the updated mesh is conveniently adapted and the high slope of the temperature u are adequately captured.

Tables 1–3 detail the number of iterations executed until the convergence condition of Step 4 is verified, or otherwise the corresponding entry equals the preset maximum number of iterations. It is remarkable how good the algorithm works for many values of (p_1, p_2), the closer to some value near (3,3), the better rate of convergence is shown. On the other hand, when we consider values of p_1 or p_2 near 5, the rate of convergence becomes worse or even no convergence at all. Amazingly, we have found a similar behavior of the algorithm when p_1 and p_2 are close to 2, at least for the case $V_0 = 220$.

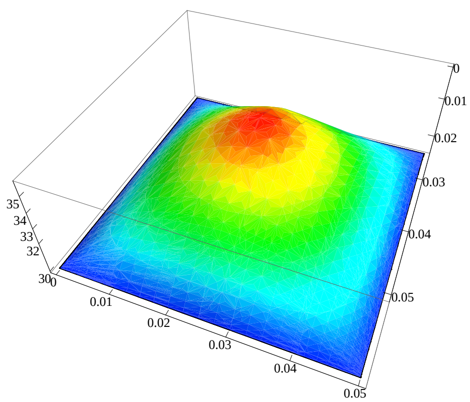
The information provided by these numerical simulations may be essential for the design of a thermistor with an anisotropic temperature gradient dependent diffusion. For instance, if we need a thermistor working inside an electrical circuit with normal voltage around 10 V, we want this device to switch off the current in the event of an unattended increase of the voltage, say 220 V. In this situation, it would be interesting to have a semiconductor material described by a model like (1.1) with p_1 and p_2 very close to the value 2.2, but not necessarily equals. Now, according to the figures 4(a) and 4(b), the normal working temperature would be less than 50°C. Thus, the thermistor could be connected to a thermometer that is programmed to switch off the current as soon as the measured temperature reaches the value of say 70°C. This surely would happen if suddenly the potential difference grows up to 220 V, since the thermistor would reach a temperature of about 152°C (Figure 4(b)). In this way, some sensitive parts (usually, very much expensive than a thermistor and a thermometer) inside the circuit would be protected from unattended potential difference increase.



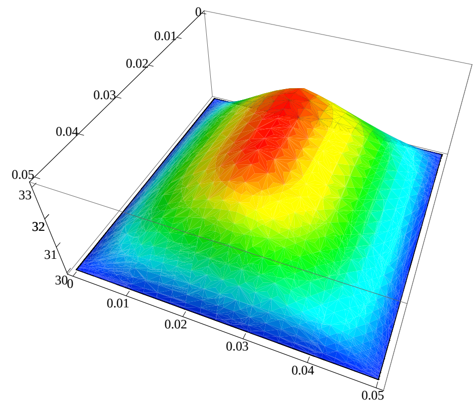
(a) $p2 = 2.8$, $\|u\|_\infty = 36.23$.



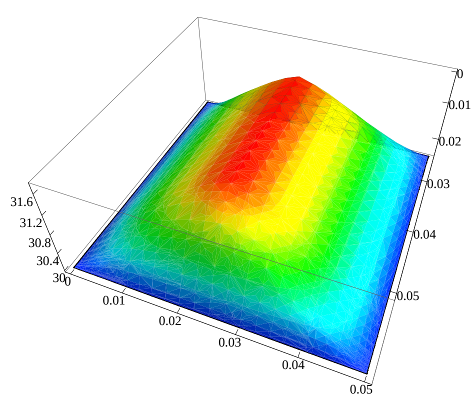
(b) $p2 = 3.2$, $\|u\|_\infty = 36.3018$.



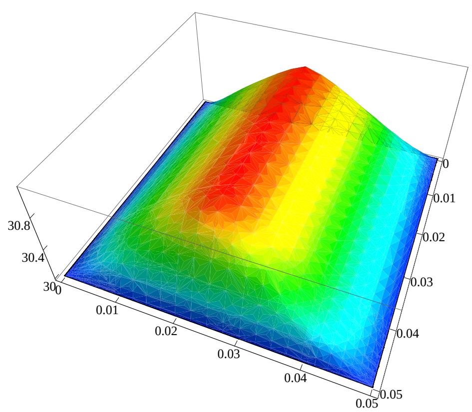
(c) $p2 = 3.6$, $\|u\|_\infty = 35.7709$.



(d) $p2 = 4.0$, $\|u\|_\infty = 33.1028$.

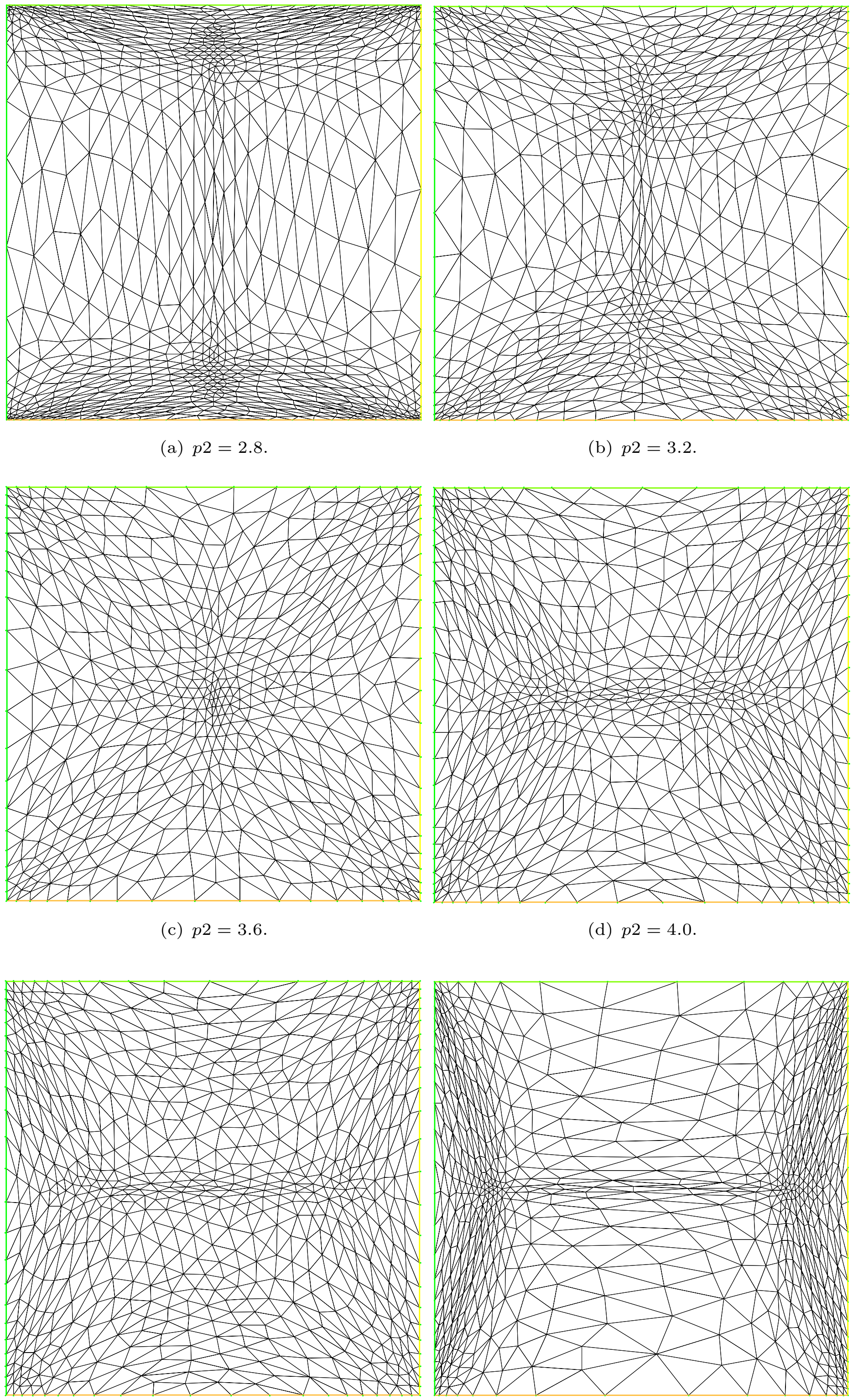


(e) $p2 = 4.4$, $\|u\|_\infty = 31.769$.



(f) $p2 = 4.8$, $\|u\|_\infty = 31.1297$.

Figure 2. Distribution of the temperature u for $p_1 = 3.6$ and the indicated p_2. We took $\delta = 10^{-2}$ and voltage $V_0 = 220$.



(a) $p2 = 2.8$.

(b) $p2 = 3.2$.

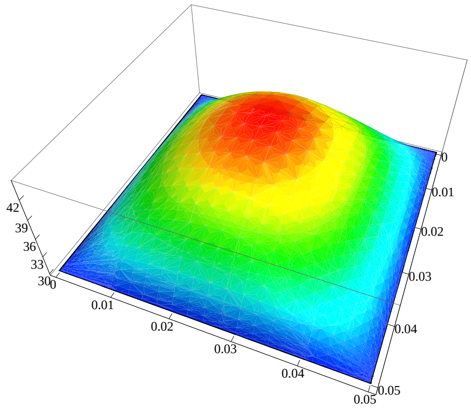
(c) $p2 = 3.6$.

(d) $p2 = 4.0$.

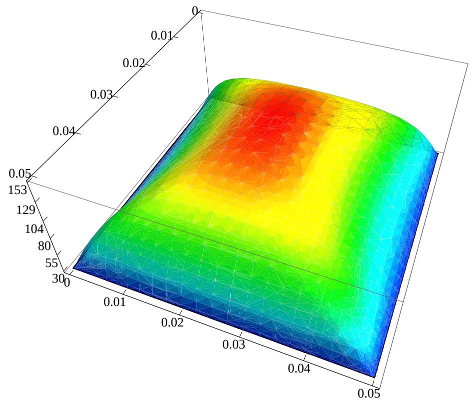
(e) $p2 = 4.4$.

(f) $p2 = 4.8$.

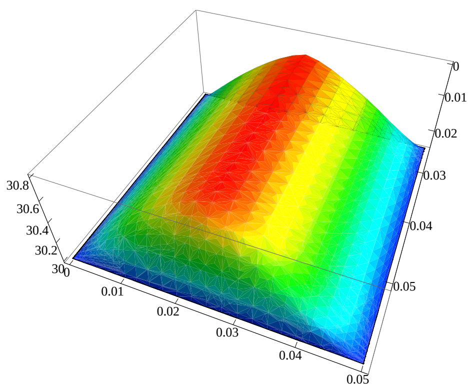
Figure 3. Final mesh after the adapting technique based on the temperature u of Figure 2 for $p_1 = 3.6$ and the indicated p_2, respectively, with $\delta = 10^{-2}$ and voltage $V_0 = 220$ V.



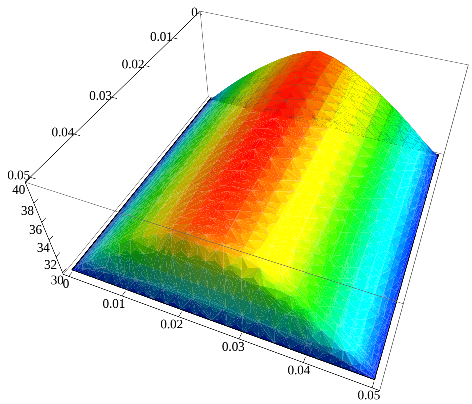
(a) $p_2 = 2.2$, $V_0 = 10$ V, $\|u\|_\infty = 44.4279$.



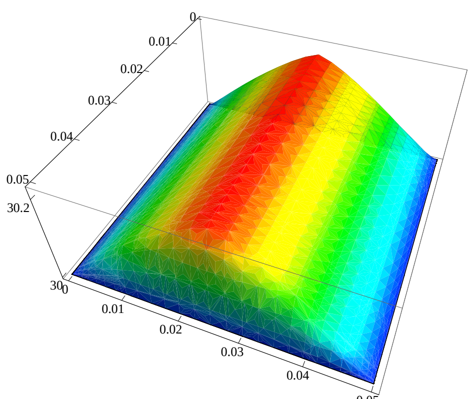
(b) $p_2 = 2.2$, $V_0 = 220$ V, $\|u\|_\infty = 152.676$.



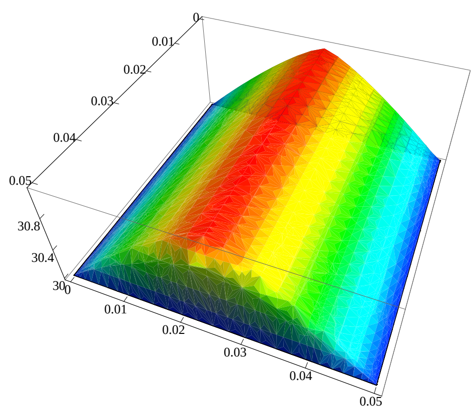
(c) $p_2 = 3.4$, $V_0 = 10$ V, $\|u\|_\infty = 30.8131$.



(d) $p_2 = 3.4$, $V_0 = 220$ V, $\|u\|_\infty = 39.8339$.



(e) $p_2 = 4.8$, $V_0 = 10$ V, $\|u\|_\infty = 30.2232$.



(f) $p_2 = 4.8$, $V_0 = 220$ V, $\|u\|_\infty = 31.1306$.

Figure 4. Distribution of the temperature u for $p_1 = 2.2$ and the indicated p_2. We took $\delta = 10^{-2}$ in all these cases but (b) where $\delta = 10^{-6}$.

Table 1. Number of iterations n executed by the algorithm described by steps 1–4 for $\delta = 10^{-2}$ and $V_0 = 220$ V. An entry below 600 means that the convergence condition $\|\tilde{u}^{n+1} - u^n\|_\infty < \varepsilon_1 = 0.5 \times 10^{-11}$ was verified for the first time for n equal to that entry. When this entry n equals $N_{\mathrm{max\_iter}} = 600$, the sequence (u_n, φ_n) does not converge, usually by exhibiting some oscillatory behavior, or else its convergence is very slow. In those cases, we repeat the algorithm for $\delta = 10^{-6}$.

$p_1 \backslash p_2$	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
2.0	600	600	600	600	72	44	43	43	42	44	53	65	85	128	379	600
2.2	600	600	600	600	71	42	37	36	39	44	53	65	85	128	380	600
2.4	600	600	600	380	66	41	35	36	39	44	53	65	85	128	377	600
2.6	389	103	99	80	53	40	35	36	39	44	53	65	85	128	381	600
2.8	78	61	50	45	41	36	35	36	39	44	53	65	85	128	380	600
3.0	53	41	39	37	34	32	32	36	39	44	53	65	85	128	378	600
3.2	47	39	33	33	33	31	31	34	40	44	53	65	85	128	577	600
3.4	40	39	35	35	34	34	33	33	38	44	53	65	85	127	396	600
3.6	43	39	38	38	37	36	36	36	38	43	53	65	84	127	402	600
3.8	47	45	45	44	44	43	43	42	42	42	52	66	82	127	577	600
4.0	55	50	49	49	49	49	48	48	48	49	52	64	80	128	383	600
4.2	67	65	65	64	64	64	63	62	61	61	61	64	78	129	393	600
4.4	87	84	84	84	83	83	82	81	80	79	78	77	78	126	392	600
4.6	123	119	119	118	117	302	117	116	116	116	116	116	117	125	376	600
4.8	589	377	378	376	377	377	377	374	374	372	374	370	371	367	382	600
5.0	600	600	600	600	600	600	600	600	600	600	600	600	600	600	600	600

Table 2. Same as Table 1 for $\delta = 10^{-6}$, $V_0 = 220$ V and $N_{\mathrm{max\_iter}} = 1200$. In this situation, the sequence (u_n, φ_n) converges very slowly for $p_1 = 2$ and $p_2 = 2$ or $p_2 = 2.2$, and also for $p_2 = 2$ and $p_1 = 2.2$, $p_2 = 2.4$ or $p_2 = 2.6$. The case $p_1 = 5$ or $p_2 = 5$ generates a sequence (u^n) with an oscillatory character.

$p_1 \backslash p_2$	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
2.0	1200	1200	120	51	44	44	43	43	42	42	45	57	77	117	368	1200
2.2	1200	791	122	50	38	38	37	36	35	37	45	57	77	115	372	1200
2.4	1200	598	90	48	33	32	32	32	33	38	45	57	77	113	369	1200
2.6	1200	180	104	40	31	29	28	30	33	37	45	57	76	114	370	1200
2.8	114	84	66	46	31	26	28	30	33	38	45	57	76	113	369	1200
3.0	59	43	42	40	32	26	27	31	33	37	45	57	76	112	368	1200
3.2	46	39	33	30	29	27	27	30	33	38	45	58	76	115	373	1200
3.4	36	39	31	30	30	30	29	29	33	38	45	57	77	113	375	1200
3.6	40	39	33	33	33	33	33	32	33	37	45	57	77	112	376	1200
3.8	46	39	38	37	38	38	38	37	37	38	44	57	76	111	377	1200
4.0	52	46	45	45	45	44	45	44	44	44	45	56	75	111	375	1200
4.2	64	57	58	57	56	57	56	55	55	54	54	56	73	113	378	1200
4.4	81	76	76	77	76	76	75	74	73	72	72	72	73	110	386	1200
4.6	117	111	109	110	109	109	109	108	109	109	108	107	105	111	379	1200
4.8	395	373	369	366	366	369	372	368	373	369	366	369	368	370	370	1200
5.0	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200

5. Conclusions

We have analyzed the existence of a certain kind of solutions to a nonlinear system of two coupled partial differential equations of nonuniformly elliptic type in the framework of anisotropic Sobolev spaces. This model arises in the study of a small semiconductor device inside an electrical circuit called thermistor. The unknowns of this system are the temperature and the electric potential. Since we are dealing

Table 3. Same as tables 1 and 2 for $\delta = 10^{-2}$, $V_0 = 10$ V and $N_{\text{max\_iter}} = 600$. In this situation, the sequence (u_n, φ_n) converges very slowly for $p_1 = 5$ or $p_2 = 5$. In the other cases, the maximum number of iterations was not reached.

p_1 \ p_2	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
2.0	44	42	42	41	40	39	39	38	38	38	40	50	68	101	199	600
2.2	45	36	36	35	34	34	33	34	34	34	40	50	68	102	200	600
2.4	45	35	32	31	30	30	30	30	31	34	41	51	68	102	201	600
2.6	46	34	31	28	27	27	27	27	31	35	40	51	68	102	351	600
2.8	46	34	30	27	24	24	24	27	30	35	41	51	68	102	351	600
3.0	46	34	30	27	24	13	24	27	30	35	41	51	68	102	353	600
3.2	45	33	30	26	24	24	24	27	31	34	41	51	68	102	350	600
3.4	45	33	29	27	27	27	27	28	31	35	41	52	68	102	201	600
3.6	45	33	31	30	31	31	31	30	31	34	41	51	68	102	199	600
3.8	45	36	35	35	35	35	35	34	35	35	41	51	69	102	197	600
4.0	45	41	41	40	40	40	41	41	40	40	41	50	68	102	196	600
4.2	54	51	51	50	50	50	50	50	50	50	51	51	67	101	195	600
4.4	69	68	68	68	68	68	68	68	67	67	67	67	68	99	196	600
4.6	101	101	101	101	101	101	101	101	101	100	100	99	99	101	197	600
4.8	200	200	200	201	201	200	199	198	197	196	195	196	196	197	197	600
5	600	600	600	600	600	600	600	600	600	600	600	600	600	600	600	600

with a temperature dependent electric conductivity, $\rho(u)$, under the assumption $\rho(s) > 0$ for all $s \in \mathbb{R}$, the search of weak solutions is not well-suited in our setting and we need to introduce the concept of capacity solution (Definition 2.1). The main result of this work is the existence of a capacity solution to the problem (1.1) in the setting of anisotropic Sobolev spaces (Theorem 3.1). This kind of solution is obtained by approximation so that the capacity solution is, in fact, the limit of solutions to certain regularized problems.

We have implemented an algorithm for the numerical resolution of system (1.1) for different values of the exponents p_1 and p_2. This algorithm has shown numerically to be convergent for any values of the exponents in the interval $(2, 5)$ and certain values of the parameter δ (tables 1–3) which may depend on the applied potential difference V_0 as well. These numerical simulations may yield a valuable information in the design of thermistors inside electric circuits.

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