Journal of Applied Analysis and Computation
Volume 12, Number 6, December 2022, 2208–2219
Website:http://www.jaac-online.com
DOI:10.11948/20210232

NONTRIVIAL RADIAL SOLUTIONS FOR A SYSTEM OF SECOND ORDER ELLIPTIC EQUATIONS*

Haiyan Zhang[1], Jiafa Xu[2,†] and Donal O'Regan[3]

Abstract In this paper we use the topological degree and the Krein-Rutman theorem to investigate the existence of nontrivial radial solutions for a system of second order elliptic equations. Our results are obtained under some conditions involving the eigenvalues of a relevant linear operator.

Keywords Elliptic equations, nontrivial radial solutions, topological degree, Krein-Rutman theorem.

MSC(2010) 34B18, 34B15, 35J25, 35J60, 47H11, 47N20.

1. Introduction

In this paper we investigate the existence of nontrivial radial solutions for the following system of second order elliptic equations

$$\begin{cases}\Delta u+g(|x|)f_1(v)=0, R_1<|x|<R_2,\\ \Delta v+g(|x|)f_2(u)=0, R_1<|x|<R_2,\\ u=0, |x|=R_1; u=0, |x|=R_2,\\ v=0, |x|=R_1; v=0, |x|=R_2,\end{cases}\tag{1.1}$$

where the functions $g, f_i (i=1,2)$ satisfy the following conditions

(H1). g is the nonnegative continuous function on $[R_1, R_2]$,

(H2). $f_i\in C(\mathbb{R},\mathbb{R})$, $i=1,2$.

For the elliptic boundary value problem

$$\begin{cases}-\Delta u=f(|x|,u), \ x\in\Omega,\\ u|_{\partial\Omega}=0,\end{cases}\tag{1.2}$$

[†]The corresponding author. Email address: liz.zhang@yeah.net(H. Zhang), xujiafa292@sina.com(J.Xu), donal.oregan@nuigalway.ie(D. O'Regan)

[1]School of Mathematics and Statistics, Suzhou University, Suzhou 234000, Anhui, China

[2]School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China

[3]School of Mathematical and Statistical Sciences, National University of Ireland, Galway, Ireland

*The authors were supported by Foundation of Anhui Provincial Education Department (Nos. KJ2020A0735, KJ2021ZD0136), Natural Science Foundation of Chongqing (cstc2020jcyj-msxmX0123), Technology Research Foundation of Chongqing Educational Committee(KJQN202000528, KJQN201900539), the open project of key laboratory (CSSXKFKTM202003), School of Mathematical Sciences, Chongqing Normal University.

where Ω is the unit ball in $\mathbb{R}^N$, there is a large number of papers studying the existence of radial solutions using different techniques; for example see [1–7, 9–12, 16–25] and the references therein. In [17] the author used the method of lower and upper solutions to obtain the positive radial solutions of the elliptic equation with a nonlinear gradient term

$$\begin{cases} -\Delta u = f(|x|, u, |\nabla u|),\ x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is as in (1.2). In [10], the authors used the fixed point index to study the existence of positive solutions for the following elliptic system on an annulus

$$\begin{cases} \Delta u + \lambda k_1(|x|) f(u, v) = 0,\ \text{in } \Omega, \\ \Delta v + \lambda k_2(|x|) g(u, v) = 0,\ \text{in } \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial n} = 0,\ \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial n} = 0, \quad \text{on } |x| = R_1, \\ \gamma_1 u + \delta_1 \dfrac{\partial u}{\partial n} = 0,\ \gamma_2 v + \delta_2 \dfrac{\partial v}{\partial n} = 0, \quad \text{on } |x| = R_2, \end{cases}$$

where λ is a positive parameter, $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$ with $\rho_i \equiv \gamma_i\beta_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$, $k_i : [R_1, R_2] \to [0, \infty)$ are continuous and do not vanish identically on any subinterval of $[R_1, R_2]$ for $i = 1, 2$, and the nonlinearities f, g satisfy the conditions

$$f_\infty \equiv \lim_{(u,v)\to\infty} \frac{f(u, v)}{u + v} = \infty, \quad g_\infty \equiv \lim_{(u,v)\to\infty} \frac{g(u, v)}{u + v} = \infty. \tag{1.3}$$

Inspired by the works mentioned, in this paper we use the topological degree and the Krein-Rutman theorem to investigate the existence of nontrivial radial solutions for (1.1). Our nonlinearities $f_i (i = 1, 2)$ grow superlinearly at infinity and they involve the eigenvalues of a relevant linear operator, which improves condition (1.3).

2. Basic Notions

Our aim is to find radial solutions for the system (1.1). Let $|x|=r, x=(x_1, x_2, ..., x_N)$ and then (1.1) can be transformed into the system of second order ordinary differential equations

$$\begin{cases} u''(r) + \left(\frac{N-1}{r}\right) u'(r) + g(r) f_1(v(r)) = 0, R_1 < r < R_2, \\ v''(r) + \left(\frac{N-1}{r}\right) v'(r) + g(r) f_2(u(r)) = 0, R_1 < r < R_2, \\ u(R_1) = u(R_2) = 0, \\ v(R_1) = v(R_2) = 0. \end{cases} \tag{2.1}$$

If we choose $p(r) = r^{N-1}, a(r) = g(r) \cdot r^{N-1}$, then the system (2.1) can be rewritten as

$$\begin{cases} (p(r)u'(r))' + a(r) f_1(v(r)) = 0, R_1 < r < R_2, \\ (p(r)v'(r))' + a(r) f_2(u(r)) = 0, R_1 < r < R_2, \\ u(R_1) = u(R_2) = 0, \\ v(R_1) = v(R_2) = 0. \end{cases} \tag{2.2}$$

From [15] we obtain that the system (2.2) can be rewritten as

$$\begin{cases} u(r) = \displaystyle\int_{R_1}^{R_2} G(r,s)a(s)f_1(v(s))\mathrm{d}s, \\ v(r) = \displaystyle\int_{R_1}^{R_2} G(r,s)a(s)f_2(u(s))\mathrm{d}s, \end{cases} \tag{2.3}$$

where

$$G(r,s) = \begin{cases} \dfrac{\left(\int_{R_1}^{s} \frac{1}{p(\tau)}\mathrm{d}\tau\right)\left(\int_{r}^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau\right)}{\int_{R_1}^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau}, s \leqslant r, \\ \dfrac{\left(\int_{R_1}^{r} \frac{1}{p(\tau)}\mathrm{d}\tau\right)\left(\int_{s}^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau\right)}{\int_{R_1}^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau}, r \leqslant s. \end{cases} \tag{2.4}$$

Lemma 2.1 (see [15, Lemma 2.1])**.** *The Green function G has the following properties:*

(i) $q(r)\Phi(s) \le G(r,s) \le \Phi(s)$ *for* $r, s \in [R_1, R_2]$,

(ii) $\omega(r) = \int_{R_1}^{R_2} G(r,s)a(s)\mathrm{d}s \le \|a\|_\infty \xi_{\max} q(r)$ *for* $r \in [R_1, R_2]$,

where

$$q(r) = \min\left\{ \frac{\int_r^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau}{\int_{R_1}^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau}, \frac{\int_{R_1}^{r} \frac{1}{p(\tau)}\mathrm{d}\tau}{\int_{R_1}^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau} \right\},$$

$$\Phi(s) = \frac{\int_s^{R_2} \frac{1}{p(\tau)}d\tau \cdot \int_{R_1}^{s} \frac{1}{p(\tau)}d\tau}{\int_{R_1}^{R_2} \frac{1}{p(\tau)}\mathrm{d}\tau},\ R_1 < r < R_2,$$

$$\xi_1 = \int_{R_1}^{R_2} \frac{\tau - R_1}{p(\tau)}\mathrm{d}\tau,\ \xi_2 = \int_{R_1}^{R_2} \frac{R_2 - \tau}{p(\tau)}\mathrm{d}\tau,\ \xi_{\max} = \max\{\xi_1, \xi_2\}.$$

Let $X = C[R_1, R_2]$ and $\|u\| = \sup_{r\in[R_1,R_2]} |u(r)|$ for $u \in X$. Note $(X, \|\cdot\|)$ is a Banach space. Define the following sets as follows

$$P = \{u \in X : u(r) \ge 0, r \in [R_1, R_2]\},\ P_0 = \{u \in X : u(r) \ge q(r)\|u\|, r \in [R_1, R_2]\}.$$

Then P, P_0 are cones on X. Moreover, $X^2 = X \times X$ is a Banach space with the norm $\|(u,v)\| = \|u\| + \|v\|, (u,v) \in X^2$, and $P^2 = P \times P$ is a cone on X^2.

Lemma 2.2. *Let* $(Lu)(r) = \int_{R_1}^{R_2} G(r,s)a(s)u(s)\mathrm{d}s$. *Then* $L(P) \subset P_0$.

Proof. If $u \in P$, then from Lemma 2.1(i) we have

$$(Lu)(r) = \int_{R_1}^{R_2} G(r,s)a(s)u(s)\mathrm{d}s \le \int_{R_1}^{R_2} \Phi(s)a(s)u(s)\mathrm{d}s, \forall r \in [R_1, R_2].$$

Therefore, we obtain

$$\|Lu\| \le \int_{R_1}^{R_2} \Phi(s)a(s)u(s)\mathrm{d}s.$$

From Lemma 2.1(i) again, we find

$$(Lu)(r) = \int_{R_1}^{R_2} G(r,s)a(s)u(s)\mathrm{d}s \geq q(r)\int_{R_1}^{R_2} \Phi(s)a(s)u(s)\mathrm{d}s.$$

Thus

$$(Lu)(r) \geq q(r)\|Lu\|, \forall r \in [R_1, R_2].$$

□

Lemma 2.3 (Krein-Rutman, see [14], [8, Theorem 19.3], [26, Theorem 7.C])**.** *Let P be a reproducing cone in a real Banach space E and let $L : E \to E$ be a compact linear operator with $L(P) \subset P$. Let $r(L)$ be the spectral radius of L. If $r(L) > 0$, then there exists $\varphi \in P\backslash\{0\}$ such that $L\varphi = r(L)\varphi$.*

Lemma 2.4 (see [13, Theorem A.3.3])**.** *Let Ω be a bounded open set in a Banach space X, and $T : \Omega \to X$ be a continuous compact operator. If there exists $x_0 \in X\backslash\{0\}$ such that*

$$x - Tx \neq \mu x_0,\ \forall x \in \partial\Omega,\ \mu \geq 0,$$

then the topological degree $\deg(I - T, \Omega, 0) = 0$.

Lemma 2.5 (see [13, Lemma 2.5.1])**.** *Let Ω be a bounded open set in a Banach space X with $0 \in \Omega$, and $T : \Omega \to X$ be a continuous compact operator. If*

$$Tx \neq \mu x,\ \forall x \in \partial\Omega,\ \mu \geq 1,$$

then the topological degree $\deg(I - T, \Omega, 0) = 1$.

3. Main Results

From (2.3) we can define operators $T_i (i = 1, 2) : X \to X$, and $T : X^2 \to X^2$ as follows:

$$(T_1 v)(r) := \int_{R_1}^{R_2} G(r,s)a(s)f_1(v(s))\mathrm{d}s,$$
$$(T_2 u)(r) := \int_{R_1}^{R_2} G(r,s)a(s)f_2(u(s))\mathrm{d}s,$$

and

$$T(u,v)(r) = ((T_1 v), (T_2 u))(r), r \in [R_1, R_2], u, v \in X,$$

where G is as in (2.4). We note that $T_i (i = 1, 2)$ and T are completely continuous operators, and (u, v) solves (1.1) if and only if (u, v) is a fixed point of the operator T.

Theorem 3.1. *$r(L) > 0$, where $r(L)$ is the spectral radius of L in Lemma 2.2.*

Proof. From the definition of the norm, we have

$$\|L\| = \max_{r\in[R_1,R_2]} \int_{R_1}^{R_2} G(r,s)a(s)\mathrm{d}s \geq \max_{r\in[R_1,R_2]} q(r) \cdot \int_{R_1}^{R_2} \Phi(s)a(s)\mathrm{d}s.$$

Similarly, for all $n \in \mathbb{N}_+$ we obtain

$$\begin{aligned}\|L^n\| &= \max_{r\in[R_1,R_2]} \underbrace{\int_{R_1}^{R_2}\cdots\int_{R_1}^{R_2}}_{n} G(r,s_1)a(s_1)G(s_1,s_2)a(s_2)\cdots G(s_{n-1},s_n)a(s_n)\mathrm{d}s_1\cdots\mathrm{d}s_n \\ &\geq \max_{r\in[R_1,R_2]} q(r)\cdot \underbrace{\int_{R_1}^{R_2}\cdots\int_{R_1}^{R_2}}_{n} \Phi(s_1)a(s_1)q(s_1)\Phi(s_2)a(s_2) \\ &\quad \cdots q(s_{n-1})\Phi(s_n)a(s_n)\mathrm{d}s_1\cdots\mathrm{d}s_n \\ &= \max_{r\in[R_1,R_2]} q(r)\cdot\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s\cdot\left(\int_{R_1}^{R_2} q(s)\Phi(s)a(s)\mathrm{d}s\right)^{n-1},\end{aligned}$$

and from Gelfand's theorem we have

$$r(L) = \lim_{n\to\infty}\sqrt[n]{\|L^n\|} \geq \int_{R_1}^{R_2} q(s)\Phi(s)a(s)\mathrm{d}s > 0.$$

□

Now from Theorem 3.1 we have $r(L) > 0$, and thus from the Krein-Rutman theorem, there exists $\varphi \in P\backslash\{0\}$ such that

$$(L\varphi)(r) = r(L)\varphi(r), r \in [R_1, R_2], \tag{3.1}$$

i.e.

$$\int_{R_1}^{R_2} G(r,s)a(s)\varphi(s)\mathrm{d}s = r(L)\varphi(r), r\in[R_1,R_2], \tag{3.2}$$

and this means that φ is a positive solution for the boundary value problem

$$\begin{cases}(p(r)u'(r))' + \lambda_1 a(r)u(r) = 0, R_1 < r < R_2,\\ u(R_1) = u(R_2) = 0,\end{cases} \tag{3.3}$$

where $\lambda_1 = \frac{1}{r(L)}$. Moreover, from Lemma 2.2 and (3.2) we obtain

$$\varphi \in P_0. \tag{3.4}$$

Theorem 3.2. *Suppose that (H1)-(H2) and the following conditions hold:*
(H3). There exist $b_i, c_i > 0$ *and* $K_1(v), K_2(u) \in C[\mathbb{R},\mathbb{R}^+]$ *such that*

$$f_1(v) \geq -b_1 - c_1K_1(v),\ \ f_2(u) \geq -b_2 - c_2K_2(u), \forall u,v\in\mathbb{R}, i = 1,2,$$

(H4). $\displaystyle\lim_{|v|\to+\infty}\frac{K_1(v)}{|v|} = 0,\ \lim_{|u|\to+\infty}\frac{K_2(u)}{|u|} = 0,$

(H5). $\displaystyle\liminf_{|v|\to+\infty}\frac{f_1(v)}{|v|} > \lambda_1,\ \liminf_{|u|\to+\infty}\frac{f_2(u)}{|u|} > \lambda_1,$

(H6). $\displaystyle\limsup_{|v|\to 0}\frac{|f_1(v)|}{|v|} < \lambda_1,\ \limsup_{|u|\to 0}\frac{|f_2(u)|}{|u|} < \lambda_1.$

Then the system (1.1) *has at least one nontrivial radial solutions.*

Proof. From (H6) there exist $\varepsilon_1 \in (0, \lambda_1)$ and $r_1 > 0$ such that

$$|f_1(v)| \leq (\lambda_1 - \varepsilon_1)|v|, \ |f_2(u)| \leq (\lambda_1 - \varepsilon_1)|u|, \ \forall u, v \in \mathbb{R} \text{ with } |u|, |v| \leq r_1.$$

This gives us

$$|\left(T_1 v\right)(r)| \leq \int_{R_1}^{R_2} G(r,s)a(s)|f_1(v(s))|\mathrm{d}s \leq (\lambda_1 - \varepsilon_1) \int_{R_1}^{R_2} G(r,s)a(s)|v(s)|\mathrm{d}s,$$

and

$$|\left(T_2 u\right)(r)| \leq \int_{R_1}^{R_2} G(r,s)a(s)|f_2(u(s))|\mathrm{d}s \leq (\lambda_1 - \varepsilon_1) \int_{R_1}^{R_2} G(r,s)a(s)|u(s)|\mathrm{d}s.$$

Now we prove that

$$(u, v) \neq \mu T(u, v) \text{ for all } u, v \in \partial B_{r_1} \text{ and } \mu \in [0, 1]. \tag{3.5}$$

Suppose that there exist $u, v \in \partial B_{r_1}$ and $\mu \in [0, 1]$ such that

$$(u, v) = \mu T(u, v).$$

Then,

$$u = \mu T_1 v, \text{ and } v = \mu T_2 u.$$

This implies that

$$|u(r)| = \mu \left|\left(T_1 v\right)(r)\right| \leq (\lambda_1 - \varepsilon_1) \int_{R_1}^{R_2} G(r,s)a(s)|v(s)|\mathrm{d}s, r \in [R_1, R_2],$$

and

$$|v(r)| = \mu \left|\left(T_2 u\right)(r)\right| \leq (\lambda_1 - \varepsilon_1) \int_{R_1}^{R_2} G(r,s)a(s)|u(s)|\mathrm{d}s, r \in [R_1, R_2].$$

Consequently, we have

$$|u(r)| + |v(r)| \leq (\lambda_1 - \varepsilon_1) \int_{R_1}^{R_2} G(r,s)a(s)(|u(s)| + |v(s)|)\mathrm{d}s.$$

Let $z(r) = |u(r)| + |v(r)|$. Then $z \in P$ and

$$z(r) \leq (\lambda_1 - \varepsilon_1) \int_{R_1}^{R_2} G(r,s)a(s)z(s)\mathrm{d}s = (\lambda_1 - \varepsilon_1)(Lz)(r), r \in [R_1, R_2].$$

The nth iteration of this inequality shows that

$$z(r) \leq (\lambda_1 - \varepsilon_1)^n \left(L^n z\right)(r)(n = 1, 2, \ldots),$$

and then

$$\|z\| \leq (\lambda_1 - \varepsilon_1)^n \|L^n\| \cdot \|z\|, \text{ i.e., } 1 \leq (\lambda_1 - \varepsilon_1)^n \|L^n\|.$$

This yields

$$1 \leq (\lambda_1 - \varepsilon_1) \lim_{n \to \infty} \sqrt[n]{\|L^n\|} = (\lambda_1 - \varepsilon_1) r(L) = \frac{\lambda_1 - \varepsilon_1}{\lambda_1} < 1.$$

This is a contradiction. Hence, (3.5) holds, and Lemma 2.5 guarantees that

$$\deg\left(I-T,B_{r_1},0\right)=1. \tag{3.6}$$

On the other hand, from (H5) there exist $\varepsilon_2>0$ and $r_2>0$ such that

$$f_1(v)\geq(\lambda_1+\varepsilon_2)\,|v|,\ \ f_2(u)\geq(\lambda_1+\varepsilon_2)\,|u|,\ \text{ for } |u|,|v|>r_2.$$

Let $M_1=\max_{|v|\leq r_2}[|f_1(v)|+(\lambda_1+\varepsilon_2)\,|v|]$, $M_2=\max_{|u|\leq r_2}[|f_2(u)|+(\lambda_1+\varepsilon_2)\,|u|]$. Then

$$f_1(v)\geq(\lambda_1+\varepsilon_2)\,|v|-M_1,\ \ f_2(u)\geq(\lambda_1+\varepsilon_2)\,|u|-M_2,\forall u,v\in\mathbb{R}. \tag{3.7}$$

For any given $\epsilon,\overline{\epsilon}$ with $\varepsilon_2-c_1\epsilon>0$, $\varepsilon_2-c_2\overline{\epsilon}>0$, by (H4) there exists $r_3>r_2$ such that

$$K_1(v)\leq\epsilon|v|,\ \ K_2(u)\leq\overline{\epsilon}|u|,\ \ \forall|u|,|v|>r_3.$$

Let $K_1^*=\max_{|v|\leq r_3}K_1(v)$, and $K_2^*=\max_{|u|\leq r_3}K_2(u)$. Then we obtain

$$K_1(v)\leq\epsilon|v|+K_1^*,\ \ K_2(u)\leq\overline{\epsilon}|u|+K_2^*,\forall u,v\in\mathbb{R}. \tag{3.8}$$

Note $\epsilon,\overline{\epsilon}$ can be chosen arbitrarily small, so we can let $\Lambda_1>\max\{r_1,N_1,N_2,N_3,N_4\}$, and let

$$N_1=\frac{2(2b_2+2c_2K_2^*+M_2)\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s}{1-2\overline{\epsilon}c_2\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s},$$

$$N_2=\frac{2(2b_1+2c_1K_1^*+M_1)\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s}{1-2\epsilon c_1\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s},$$

$$N_3=\frac{N_5\left[(\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s+(\lambda_1+\varepsilon_2-c_1\epsilon)\,\|a\|_\infty\xi_{\max}\right]}{N_6(\varepsilon_2-c_1\epsilon)-(\lambda_1+\varepsilon_2-c_1\epsilon)\,(c_1\epsilon+c_2\overline{\epsilon})\|a\|_\infty\xi_{\max}},$$

$$N_4=\frac{N_5\left[(\varepsilon_2-c_2\overline{\epsilon})\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s+(\lambda_1+\varepsilon_2-c_2\overline{\epsilon})\,\|a\|_\infty\xi_{\max}\right]}{N_6(\varepsilon_2-c_2\overline{\epsilon})-(\lambda_1+\varepsilon_2-c_2\overline{\epsilon})\,(c_1\epsilon+c_2\overline{\epsilon})\|a\|_\infty\xi_{\max}},$$

where

$$N_5=2b_1+2b_2+M_1+M_2+2c_1K_1^*+2c_2K_2^*,$$
$$N_6=1-(c_1\epsilon+c_2\overline{\epsilon})\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s.$$

Now we claim that

$$(u,v)-T(u,v)\neq\mu\,(\varphi,\varphi)\,,\forall u,v\in\partial B_{\Lambda_1},\mu\geq0, \tag{3.9}$$

where φ is as in (3.1). Suppose that there exist $u,v\in\partial B_{\Lambda_1}$ and $\mu\geq0$ such that

$$(u,v)-T(u,v)=\mu\,(\varphi,\varphi)\,.$$

This means that

$$u=T_1v+\mu\varphi,\ \ v=T_2u+\mu\varphi. \tag{3.10}$$

Let

$$\overline{u}(r)=\int_{R_1}^{R_2}G(r,s)a(s)[2b_2+c_2K_2(u(s))+M_2+c_2K_2^*]\mathrm{d}s,$$

$$\overline{v}(r)=\int_{R_1}^{R_2}G(r,s)a(s)[2b_1+c_1K_1(v(s))+M_1+c_1K_1^*]\mathrm{d}s.$$

(i) Note $\|u\|=\|v\|=\Lambda_1$, and from (3.8) we have

$$\|\overline{u}\|\le\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s\cdot[2b_2+c_2(\overline{\epsilon}\|u\|+K_2^*)+M_2+c_2K_2^*]<\frac{1}{2}\Lambda_1,$$

and

$$\|\overline{v}\|\le\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s\cdot[2b_1+c_1(\epsilon\|v\|+K_1^*)+M_1+c_1K_1^*]<\frac{1}{2}\Lambda_1.$$

(ii) From Lemma 2.2, $\overline{u},\overline{v}\in P_0$.
(iii) $u+\overline{v}\in P_0, v+\overline{u}\in P_0$.
Indeed, from (3.10) we have

$$\begin{aligned}&u(r)+\overline{v}(r)\\=&(T_1v)(r)+\overline{v}(r)+\mu\varphi(r)\\=&\int_{R_1}^{R_2}G(r,s)a(s)[f_1(v(s))+2b_1+c_1K_1(v(s))+M_1+c_1K_1^*]\mathrm{d}s+\mu\varphi(r),\end{aligned}$$

and

$$\begin{aligned}&v(r)+\overline{u}(r)\\=&(T_2u)(r)+\overline{u}(r)+\mu\varphi(r)\\=&\int_{R_1}^{R_2}G(r,s)a(s)[f_2(u(s))+2b_2+c_2K_2(u(s))+M_2+c_2K_2^*]\mathrm{d}s+\mu\varphi(r).\end{aligned}$$

Note Lemma 2.2 and (3.4), so (iii) is true.

Note $\|u\|=\|v\|=\Lambda_1$, $u+\overline{u}+\overline{v}\in P_0, v+\overline{u}+\overline{v}\in P_0$. Hence, we obtain

$$u(r)+\overline{u}(r)+\overline{v}(r)\ge q(r)\|u+\overline{u}+\overline{v}\|\ge q(r)(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|),$$

and

$$v(r)+\overline{u}(r)+\overline{v}(r)\ge q(r)\|v+\overline{u}+\overline{v}\|\ge q(r)(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|),$$

for $r\in[R_1,R_2]$. Therefore

$$\begin{aligned}&(\varepsilon_2-c_1\epsilon)(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|)-(\lambda_1+\varepsilon_2-c_1\epsilon)\,\|a\|_\infty\xi_{\max}[N_5+c_1\epsilon\|v\|+c_2\overline{\epsilon}\|u\|]\\\ge&(\varepsilon_2-c_1\epsilon)\left(\Lambda_1-\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s\cdot[N_5+c_1\epsilon\Lambda_1+c_2\overline{\epsilon}\Lambda_1]\right)\\&-(\lambda_1+\varepsilon_2-c_1\epsilon)\,\|a\|_\infty\xi_{\max}[N_5+c_1\epsilon\Lambda_1+c_2\overline{\epsilon}\Lambda_1]\\\ge&0,\end{aligned}$$

and

$$\begin{aligned}&(\varepsilon_2-c_2\overline{\epsilon})\,(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|)-(\lambda_1+\varepsilon_2-c_2\overline{\epsilon})\,\|a\|_\infty\xi_{\max}[N_5+c_1\epsilon\|v\|+c_2\overline{\epsilon}\|u\|]\\\ge&(\varepsilon_2-c_2\overline{\epsilon})\left(\Lambda_1-\int_{R_1}^{R_2}\Phi(s)a(s)\mathrm{d}s\cdot[N_5+c_1\epsilon\Lambda_1+c_2\overline{\epsilon}\Lambda_1]\right)\\&-(\lambda_1+\varepsilon_2-c_2\overline{\epsilon})\,\|a\|_\infty\xi_{\max}[N_5+c_1\epsilon\Lambda_1+c_2\overline{\epsilon}\Lambda_1]\\\ge&0.\end{aligned}$$

Therefore, from Lemma 2.1(ii) we find

$$
\begin{aligned}
&(\varepsilon_2 - c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)[v(s)+\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
&\quad-(\lambda_1+\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)[\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
\geq&(\varepsilon_2 - c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)q(s)(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|)\mathrm{d}s\\
&\quad-(\lambda_1+\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)\int_{R_1}^{R_2} G(s,\tau)a(\tau)[N_5+c_1K_1(v(\tau))+c_2K_2(u(\tau))]\mathrm{d}\tau\mathrm{d}s\\
\geq&(\varepsilon_2 - c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)q(s)(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|)\mathrm{d}s\\
&\quad-(\lambda_1+\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)q(s)\|a\|_\infty\xi_{\max}[N_5+c_1\epsilon\|v\|+c_2\overline{\epsilon}\|u\|]\mathrm{d}s\\
\geq&0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\varepsilon_2 - c_2\overline{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)[u(s)+\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
&\quad-(\lambda_1+\varepsilon_2-c_2\overline{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)[\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
\geq&(\varepsilon_2 - c_2\overline{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)q(s)(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|)\mathrm{d}s\\
&\quad-(\lambda_1+\varepsilon_2-c_2\overline{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)\int_{R_1}^{R_2} G(s,\tau)a(\tau)[N_5+c_1K_1(v(\tau))+c_2K_2(u(\tau))]\mathrm{d}\tau\mathrm{d}s\\
\geq&(\varepsilon_2 - c_2\overline{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)q(s)(\Lambda_1-\|\overline{u}\|-\|\overline{v}\|)\mathrm{d}s\\
&\quad-(\lambda_1+\varepsilon_2-c_2\overline{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)q(s)\|a\|_\infty\xi_{\max}[N_5+c_1\epsilon\|v\|+c_2\overline{\epsilon}\|u\|]\mathrm{d}s\\
\geq&0.
\end{aligned}
$$

As a result, from the above two inequalities we obtain

$$
\begin{aligned}
&\int_{R_1}^{R_2} G(r,s)a(s)[f_1(v(s))+2b_1+c_1K_1(v(s))+M_1+c_1K_1^*]\mathrm{d}s\\
\geq&\int_{R_1}^{R_2} G(r,s)a(s)[(\lambda_1+\varepsilon_2)\,|v(s)|-M_1-b_1-c_1K_1(v(s))+b_1+M_1+c_1K_1^*]\mathrm{d}s\\
\geq&\int_{R_1}^{R_2} G(r,s)a(s)[(\lambda_1+\varepsilon_2)\,|v(s)|-M_1-b_1-c_1(\epsilon|v(s)|+K_1^*)+b_1+M_1+c_1K_1^*]\mathrm{d}s\\
=&(\lambda_1+\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)|v(s)|\mathrm{d}s\\
\geq&(\lambda_1+\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)v(s)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= (\lambda_1+\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)[v(s)+\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
&\quad -(\lambda_1+\varepsilon_2-c_1\epsilon)\int_{R_1}^{R_2} G(r,s)a(s)[\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
&\geq \lambda_1\int_{R_1}^{R_2} G(r,s)a(s)[v(s)+\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
&\geq \lambda_1\int_{R_1}^{R_2} G(r,s)a(s)[v(s)+\overline{u}(s)]\mathrm{d}s\\
&= \lambda_1 L(v+\overline{u})(r),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{R_1}^{R_2} G(r,s)a(s)[f_2(u(s))+2b_2+c_2K_2(u(s))+M_2+c_2K_2^*]\mathrm{d}s\\
&\geq \int_{R_1}^{R_2} G(r,s)a(s)[(\lambda_1+\varepsilon_2)\,|u(s)|-M_2-b_2-c_2(\bar{\epsilon}|u(s)|+K_2^*)+b_2+M_2+c_2K_2^*]\mathrm{d}s\\
&= (\lambda_1+\varepsilon_2-c_2\bar{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)|u(s)|\mathrm{d}s\\
&\geq (\lambda_1+\varepsilon_2-c_2\bar{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)[u(s)+\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
&\quad -(\lambda_1+\varepsilon_2-c_2\bar{\epsilon})\int_{R_1}^{R_2} G(r,s)a(s)[\overline{u}(s)+\overline{v}(s)]\mathrm{d}s\\
&\geq \lambda_1 L(u+\overline{u}+\overline{v})(r)\\
&\geq \lambda_1 L(u+\overline{v})(r).
\end{aligned}
$$

Consequently, we have

$$T_1v+\overline{v}\geq \lambda_1 L(v+\overline{u}),\ T_2u+\overline{u}\geq \lambda_1 L(u+\overline{v}).$$

Thus from (3.10) we have

$$u+v+\overline{u}+\overline{v}=T_1v+T_2u+\overline{u}+\overline{v}+2\mu\varphi\geq \lambda_1 L(u+v+\overline{u}+\overline{v})+2\mu\varphi\geq 2\mu\varphi.$$

Define $\mu^*=\sup S_\mu:=\sup\{\mu>0: u+v+\overline{u}+\overline{v}\geq 2\mu\varphi\}$. Then $S_\mu\neq\varnothing$, $\mu^*\geq\mu$ and $u+v+\overline{u}+\overline{v}\geq 2\mu^*\varphi$. From $\varphi=\lambda_1 L\varphi$, we obtain

$$\lambda_1 L(u+v+\overline{u}+\overline{v})\geq \lambda_1 L\left(2\mu^*\varphi\right)=2\mu^*\lambda_1 L\varphi=2\mu^*\varphi.$$

Hence

$$u+v+\overline{u}+\overline{v}\geq \lambda_1 L(u+v+\overline{u}+\overline{v})+2\mu\varphi\geq 2\left(\mu+\mu^*\right)\varphi,$$

which contradicts the definition of μ^*. Therefore, (3.9) holds, and from Lemma 2.4 we obtain

$$\deg\left(I-T,B_{\Lambda_1},0\right)=0. \tag{3.11}$$

Now (3.6) and (3.11) together imply that

$$\deg\left(I-T,B_{\Lambda_1}\backslash\overline{B}_{r_1},0\right)=\deg\left(I-T,B_{\Lambda_1},0\right)-\deg\left(I-T,B_{r_1},0\right)=-1.$$

Therefore the operator T has at least one fixed point in $B_{\Lambda_1}\backslash\overline{B}_{r_1}$. Equivalently, (1.1) has at least one nontrivial solution. □

Example 3.1. Let

$$f_1(v) = \begin{cases} \sum\limits_{i=1}^{n} (-1)^i a_i - |v|^{\frac{1}{3}} \ln(|v|+1) + \ln 2, & v \in (-\infty, -1), \\ \sum\limits_{i=1}^{n} a_i v^i, & v \in [-1, +\infty), \end{cases}$$

$$f_2(u) = \begin{cases} \sum\limits_{i=1}^{n} (-1)^i a_i - |u|^{\frac{1}{3}} \ln\left(|u|^{\frac{1}{3}}+1\right) + \ln 2, & u \in (-\infty, -1), \\ \sum\limits_{i=1}^{n} a_i u^i, & u \in [-1, +\infty), \end{cases}$$

where $0 < a_1 < \lambda_1, a_i \geq 0$ and $a_i \not\equiv 0 (i = 2, 3, \cdots, n)$. Then $f_i (i = 1, 2)$ are unbounded from below. Choose $c_j = 1, b_j = \sum\limits_{i=1}^{n} a_i + \ln 2 (j = 1, 2)$, $K_1(v) = |v|^{\frac{1}{3}} \ln(|v|+1)$, $K_2(u) = |u|^{\frac{1}{3}} \ln\left(|u|^{\frac{1}{3}}+1\right)$, and we see that (H1)-(H6) hold. Therefore, (1.1) has at least one nontrivial solution.

References

[1] G. A. Afrouzi and T. A. Roushan, *Existence of positive radial solutions for some nonlinear elliptic systems,* Bull. Math. Anal. Appl., 2011, 3(4), 146–154.

[2] T. Alotaibi, D. Hai and R. Shivaji, *Existence and nonexistence of positive radial solutions for a class of p-Laplacian superlinear problems with nonlinear boundary conditions,* Commun. Pure Appl. Anal., 2020, 19(9), 4655–4666.

[3] V. Anuradha, D. Hai and R. Shivaji, *Existence results for superlinear semipositone BVP's,* Proc. Amer. Math. Soc., 1996, 124(3), 757–763.

[4] M. Ben Chrouda and K. Hassine, *Uniqueness of positive radial solutions for elliptic equations in an annulus,* Proc. Amer. Math. Soc., 2021, 149(2), 649–660.

[5] A. Castro and S. Song, *Infinitely many radial solutions for a super-cubic Kirchhoff type problem in a ball,* Discrete Contin. Dyn. Syst. Ser. S, 2020, 13(12), 3347–3355.

[6] K. Chu, D. Hai and R. Shivaji, *Uniqueness of positive radial solutions for infinite semipositone p-Laplacian problems in exterior domains,* J. Math. Anal. Appl., 2019, 472(1), 510–525.

[7] K. Chu, D. Hai and R. Shivaji, *Uniqueness of positive radial solutions for a class of infinite semipositone p-Laplacian problems in a ball,* Proc. Amer. Math. Soc., 2020, 148(5), 2059–2067.

[8] K. Deimling, *Nonlinear functional analysis,* Springer-Verlag, Berlin, 1985.

[9] X. Dong and Y. Wei, *Existence of radial solutions for nonlinear elliptic equations with gradient terms in annular domains,* Nonlinear Anal., 2019, 187, 93–109.

[10] D. R. Dunninger and H. Wang, *Multiplicity of positive radial solutions for an elliptic system on an annulus,* Nonlinear Anal., 2000, 42(5), 803–811.

[11] K. Du, R. Peng and N. Sun, *The role of protection zone on species spreading governed by a reaction-diffusion model with strong Allee effect,* J. Differential Equations, 2019, 266(11), 7327–7356.

[12] I. Flores, M. Franca and L. Iturriaga, *Positive radial solutions involving nonlinearities with zeros,* Discrete Contin. Dyn. Syst., 2019, 39(5), 2555–2579.

[13] D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones, volume 5 of Notes and Reports in Mathematics in Science and Engineering,* Academic Press, Inc., Boston, MA, 1988.

[14] M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space,* Amer. Math. Soc. Translation, 1950, 1950(26), 128.

[15] K. Lan, *Multiple positive solutions of semi-positone Sturm-Liouville boundary value problems,* Bull. London Math. Soc., 2006, 38(2), 283–293,.

[16] F. Li and Y. Fan, *Existence and multiplicity of positive radial solutions for elliptic systems,* Acta. Math. Sinica (Chin. Ser.), 1999, 42(4), 591–596.

[17] Y. Li, *Positive radial solutions for elliptic equations with nonlinear gradient terms on the unit ball,* Mediterr. J. Math., 2020, 17(6), 13.

[18] R. Ma, *Existence of positive radial solutions for elliptic systems,* J. Math. Anal. Appl., 1996, 201(2), 375–386.

[19] N. Sun and X. Han, *Asymptotic behavior of solutions of a reaction-diffusion model with a protection zone and a free boundary,* Appl. Math. Lett., 2020, 107, 7.

[20] N. Sun, *A time-periodic reaction-diffusion-advection equation with a free boundary and signchanging coefficients,* Nonlinear Anal. Real World Appl., 2020, 51, 28.

[21] N. Sun and J. Fang, *Propagation dynamics of Fisher-KPP equation with time delay and free boundaries,* Calc. Var. Partial Differential Equations, 2019, 58(4), 38.

[22] N. Sun, B. Lou and M. Zhou, *Fisher-KPP equation with free boundaries and time-periodic advections,* Calc. Var. Partial Differential Equations, 2017, 56(3), 36.

[23] N. Sun and C. Lei, *Long-time behavior of a reactiondiffusion model with strong allee effect and free boundary: Effect of protection zone,* J. Dyn. Differ. Equ., 2021.

[24] J. Tian and Y. Wei, *Radial solution of asymptotically linear elliptic equation with mixed boundary value in annular domain,* J. Appl. Anal. Comput., 2020, 10(6), 2787–2805.

[25] H. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus,* J. Differential Equations, 1994, 109(1), 1–7.

[26] E. Zeidler, *Nonlinear functional analysis and its applications,* I. Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.