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PERIODIC DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS WITH PERTURBED AND SUB-LINEAR TERMS

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Abstract In this paper, we study a class of perturbed discrete nonlinear Schrödinger equations with sub-linear nonlinearities at infinity and obtain the existence of solitons for this class of equations by using a generalized saddle point theorem. To the best of our knowledge, there is no published result focusing on this class of perturbed discrete nonlinear equations by this method.

Keywords Discrete nonlinear Schrödinger equations, sub-linear, solitons, generalized saddle point theorem.

MSC(2010) 35Q51, 35Q55, 39A12, 39A70.

1. Introduction and main result

We consider the following discrete nonlinear equation

$$Lu_n - \omega u_n = \chi_n f_n(u_n) + h_n, \quad n \in Z, \tag{1.1}$$

where $\omega \in R$, L is a Jacobi operator [32] given by $Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n$, where $\{a_n\}$ and $\{b_n\}$ are real valued N-periodic sequences (N is a positive integer), i.e., $a_{n+N} = a_n$ and $b_{n+N} = b_n$ for $n \in Z$, $\{\chi_n\}$ and $\{h_n\}$ are real valued sequences.

As usual, solitons of (1.1) are spatially localized time-periodic solutions and decay to 0 at infinity, that is,

$$\lim_{|n|\to\infty} u_n = 0. \tag{1.2}$$

This problem appears when we look for the discrete solitons of the discrete nonlinear Schrödinger equation

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - \chi_n g_n(\psi_n), \quad n \in Z, \tag{1.3}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, $\{\varepsilon_n\}$ is assumed to be N-periodic and $g_n(e^{i\omega}s) = e^{i\omega}g_n(s)$ for any $\omega \in R$ and $(n, s) \in Z \times R$. Making use of the standing wave ansatz

$$\psi_n = u_n e^{-i\omega t},$$

where $\{u_n\}$ is a real valued sequence and $\omega \in R$ is the temporal frequency, we arrive at the equation

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = \chi_n g_n(u_n), \quad n \in Z. \tag{1.4}$$

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Clearly, the equation (1.4) is a special case of (1.1) with

$$a_n \equiv -1, \quad b_n = 2 + \varepsilon_n, \quad f_n \equiv g_n, \quad h_n \equiv 0, \quad n \in Z.$$

In fact, the non-perturbed equations (1.3) and (1.4) have been studied by many authors [4–6, 8, 17–19, 23–26, 28, 29, 34, 35].

As usual, the standard real sequence spaces l^q, $q \in [1, \infty]$, endowed with the norm

$$\|u\|_{l^q} := \left(\sum_{n=-\infty}^{+\infty} |u_n|^q \right)^{1/q}, \quad q \in [1, \infty), \quad \|u\|_{l^\infty} := \max_{n \in Z} |u_n|,$$

where $u = \{u_n\}_{n \in Z}$. The following embedding between such spaces is well-known:

$$l^q \subset l^p, \quad \|u\|_{l^p} \le \|u\|_{l^q}, \quad 1 \le q \le p \le \infty. \tag{1.5}$$

Note that every element of the space l^2 automatically satisfies (1.2), thus we shall study the existence of solutions for (1.1) in the space l^2.

Since the operator L is a bounded and self-adjoint operator in the space l^2 of two-sided infinite sequences, we consider (1.1) as a nonlinear equation in l^2 with (1.2) being satisfied automatically. The spectrum $\sigma(L)$ of L has a band structure, i.e., $\sigma(L)$ is a union of a finite number of closed intervals [32]. Thus the complement $R \setminus \sigma(L)$ consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite. The solitons of (1.1) with the temporal frequency ω belonging to a spectral gap are of considerable importance.

In this paper, we are mainly interested in the case where the problem (1.1) is strongly indefinite in the sense that $\dim(l^2)^- = +\infty$, but our result is new also in the definite case, where $(l^2)^-$ is the negative spectral subspace of $L - \omega$ in l^2. We assume that the temporal frequency ω belonging to a spectral gap: finite gap or semi-infinite gap, i.e.,

(L1) $\omega \notin \sigma(L)$, the spectrum of L.

To sate our main result, we still need the following assumptions:

(X1) $\chi_n \ge 0$ for all $n \in Z$, $\chi = \{\chi_n\}_{n \in Z} \in l^2$ and $h = \{h_n\}_{n \in Z} \in l^2$.

(F1) $f_n \in C(R, R)$, $f_n(s)s \ge 0$ for all $s \in R$ and $n \in Z$, $\left|\frac{f_n(s)}{s}\right| < \infty$ if $|s| < \infty$ for all $n \in Z$, and

$$\lim_{|s| \to \infty} \frac{f_n(s)}{s} = 0 \ \text{ uniformly in } n \in Z. \tag{1.6}$$

As usual, we say that the nonlinearity f_n is sub-linear growth at infinity if (1.6) holds. Now, our main result reads as follows:

Theorem 1.1. *If* $(L1)$, $(X1)$ *and* $(F1)$ *hold, then (1.1) has at least one soliton.*

Remark 1.1. Note that $(F1)$ implies that $f_n(0) = 0$ for all $n \in Z$. Therefore if $h_n = 0$ for all $n \in Z$, it is easy to see that the zero function $u = \{u_n\}_{n \in Z} = \{0\}_{n \in Z}$ is a solution of (1.1).

Remark 1.2. To the best of our knowledge, there is no published result concerning the discrete nonlinear Schrödinger equation (1.1). If $h_n \equiv 0$ ($n \in Z$), some authors [4–6, 23–26, 28, 29, 34, 35] obtained the existence (or multiplicity) of nontrivial solitons for (1.1) with the nonlinearity f_n being *asymptotically linear* (see [5,6,25,26,28,29,34,35]) or *super linear* (see [4,23,24]) growth at infinity. i.e., f_n satisfies

$$\lim_{|s|\to\infty} \frac{f_n(s)}{s} = a_n \ (0 < a_n < \infty) \quad \text{or} \quad \lim_{|s|\to\infty} \frac{f_n(s)}{s} = \infty, \quad n \in Z. \tag{1.7}$$

In our setting, the authors (in the aforementioned references) all used the following condition near zero on the nonlinearity:

$$f_n(s) = o(s) \ \text{ as } \ s \to 0, \quad n \in Z. \tag{1.8}$$

Besides, they all assumed that f_n and χ_n are periodic in $n \in Z$. However, in this paper, we impose the more weaker condition $(F1)$ near zero and study (1.1) with general h_n and f_n. Moreover, we assume the nonlinearity f_n is *sub-linear* growth at infinity (see (1.6)) and *do not* need assume χ_n, h_n and f_n are periodic. As is shown in the next example, our assumption $(F1)$ is reasonable and there is a function such that the condition (1.7) is not satisfied.

Example 1.1. Let

$$f_n(s) = \begin{cases} \delta_n s, & |s| < 1, \\ \delta_n |s|^{\mu-2} s, & |s| \geq 1. \end{cases}$$

where $(n, s) \in Z \times R$, $\mu \in (1, 2)$ is a constant, $0 < \inf_{n\in Z} \delta_n \leq \sup_{n\in Z} \delta_n < \infty$. It is not hard to check that it satisfies the assumption $(F1)$ but does not satisfy the conditions (1.7) and (1.8).

The discrete nonlinear Schrödinger (DNLS) equations are one of the most important inherently discrete models, having a crucial role in modelling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology such as nonlinear optics [2], biomolecular chains [16] and Bose-Einstein condensates [21]. For related problems, we refer the reader to [11–13,30].

The study of the dynamics of the DNLS equations have been an active theme of research in the past decade, see [1,3,9,10,15] and their references therein. Among the methods used are the principle of anticontinuity [1], variational methods, centre manifold reduction [15] and the Nehari manifold approach [25]. If ω is below or above the spectrum of the difference operator L, the existence and nonexsitence of solutions ware considered by many authors [25,34]. If ω lies in a finite gap, then the associated energy functional is strongly indefinite, thus it is much more difficult to obtain the existence results (see [7] for discussions on strongly indefinite problem). We point out that some authors have obtained gap solitons of periodic DNLS with superlinear nonlinearity, see [4, 23, 24] and their references therein. However, to the best of our knowledge, little has been done for periodic DNLS equations with saturable nonlinearities, see [5, 6, 25, 26, 28, 29, 34, 35] and their references therein. For related results, we refer the readers to see [14,22], and so on.

The rest of our paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1), and we also give the detailed proof of our main result.

2. Variational frameworks and proof of the main result

In this section, we shall denote by $\|\cdot\|_{l^q}$ the usual l^q norm and C for different positive constants.

Let E^- be a separable closed subspace of a Hilbert space E with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and $E^+ = (E^-)^\perp$. For some $R > 0$, set

$$M := \{u \in E^- : \ \|u\| \le R\}. \tag{2.1}$$

Then M is a submanifold of E^- with boundary ∂M. On E we will also use a topology τ generated by the norm

$$\|u\|_\tau := \max\left(\|P_+u\|, \ \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}|(P_-u, e_k)|\right),$$

where $P_\pm : E \to E^\pm$ is the orthogonal projection of E onto $E^\pm$ and $\{e_k\}$ is a total orthonormal sequence in E^-. Obviously,

$$u^j \xrightarrow{\tau} u \ \text{ implies that } \ P_+u^j \to P_+u \ \text{ and } \ P_-u^j \rightharpoonup P_-u. \tag{2.2}$$

Let $\Phi \in C^1(E, R)$. We say that Φ is τ-upper semicontinuous if $u^j \xrightarrow{\tau} u$ implies $\Phi(u) \ge \overline{\lim}_{j\to\infty}\Phi(u^j)$, and Φ' is weakly sequentially continuous if $u^j \rightharpoonup u$ implies $\Phi'(u^j) \rightharpoonup \Phi'(u)$.

Next, we shall use the following generalized saddle point theorem to prove our main result.

Lemma 2.1 ([20]). *Suppose that $\Phi \in C^1(E, \mathrm{R})$ is τ-upper semicontinuous and Φ' is weakly sequentially continuous. If*

$$b := \inf_{E^+} \Phi > \sup_{\partial M} \Phi, \quad d := \sup_M \Phi < \infty, \tag{2.3}$$

then for some $c \in [b, d]$, there is a sequence $\{u^j\} \subset E$ such that

$$\Phi(u^j) \to c, \quad \Phi'(u^j) \to 0.$$

Such a sequence is called a $(PS)_c$ sequence.

Let $E := l^2$ and $F_n(s) := \int_0^s f_n(t)dt$, $(n, s) \in Z \times R$. The corresponding functional of (1.1) is

$$\Phi(u) = \frac{1}{2}((L-\omega)u, u)_{l^2} - \sum_{n=-\infty}^{+\infty} (\chi_n F_n(u_n) + h_n u_n), \quad u \in E,$$

where $(\cdot,\cdot)_{l^2}$ is the inner product in E and the corresponding norm in E is denoted by $\|\cdot\|_{l^2}$. By (L_1), we have the decomposition $E = E^- \oplus E^+$, where E^+ and E^- are the positive and negative spectral subspaces of $L - \omega$ in E, respectively. If $\sigma(L-\omega) \subset (0, +\infty)$, then $\dim E^- = 0$, otherwise E^- is infinite-dimensional. Let

$$Q(u) := ((L-\omega)u, u)_{l^2}.$$

Obviously, the quadratic part of Φ, $Q(u)$ is positive on E^+ and negative on E^-. Moreover, we may define an new inner product $(\cdot,\cdot)$ on E with corresponding norm $\|\cdot\|$ such that

$$((L-\omega)u,u)_{l^2} = \pm\|u\|^2, \quad \forall u \in E^{\pm}.$$

Obviously, the decomposition $E = E^- \oplus E^+$ orthogonal with respect to both $(\cdot,\cdot)$ and $(\cdot,\cdot)_{l^2}$. Therefore, Φ can be rewritten

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \sum_{n=-\infty}^{+\infty}\left(\chi_n F_n(u_n) + h_n u_n\right).$$

By assumptions $(L1)$, $(X1)$ and $(F1)$, it is easy to verify that $\Phi \in C^1(E,R)$ and the derivative is given by

$$\langle \Phi'(u), v\rangle = (u^+,v^+) - (u^-,v^-) - \sum_{n=-\infty}^{+\infty}\left(\chi_n f_n(u_n)v_n + h_n v_n\right), \quad \forall u,v \in E. \tag{2.4}$$

Equation (2.4) implies that (1.1) is the corresponding Euler-Lagrange equation for Φ. Therefore, we have reduced the problem of finding a solution of (1.1) to that of seeking a critical point of the functional Φ on E.

In order to apply Lemma 2.1 to prove our result, we need the following two lemmas.

Lemma 2.2. *Under conditions of Theorem 1.1, the functional Φ is τ-upper semicontinuous and Φ' is weakly sequentially continuous.*

Proof. First, we show that the functional Φ is τ-upper semicontinuous.

Let $u^j \xrightarrow{\tau} u$ and $\Phi(u^j) \geq C_0$ for some constant C_0. By (2.2), we have

$$(u^j)^+ \to u^+, \quad (u^j)^- \rightharpoonup u^- \quad \text{and} \quad u^j \rightharpoonup u \quad \text{in } E, \quad u_n^j \to u_n \text{ for all } n \in Z. \tag{2.5}$$

Clearly, $(F1)$ implies $F_n(s) \geq 0$ for all $(n,s) \in Z \times R$. It follows from $(X1)$ that $\chi_n F_n(s) \geq 0$ for all $(n,s) \in Z \times R$, which together with (2.5) and the Fatou's lemma implies

$$\underline{\lim}_{j\to\infty} \sum_{n=-\infty}^{+\infty} \chi_n F_n(u_n^j) \geq \sum_{n=-\infty}^{+\infty} \chi_n F_n(u_n). \tag{2.6}$$

Note that $h = \{h_n\}_{n\in Z} \in l^2$ in $(X1)$, it follows from $u^j \rightharpoonup u$ in $E = l^2$ that

$$\lim_{j\to\infty} \sum_{n=-\infty}^{+\infty} h_n u_n^j = \lim_{j\to\infty}(h,u^j) = (h,u) = \sum_{n=-\infty}^{+\infty} h_n u_n. \tag{2.7}$$

By (2.6), (2.7), $\Phi(u^j) \geq C_0$, the definition of Φ and the weak lower semicontinuity of the norm, we get

$$\begin{aligned}
-C_0 &\geq \underline{\lim}_{j\to\infty}(-\Phi(u^j)) \\
&= \underline{\lim}_{j\to\infty}\frac{1}{2}\left(\|(u^j)^-\|^2 - \|(u^j)^+\|^2\right) + \sum_{n=-\infty}^{+\infty}\left(\chi_n F_n(u_n^j) + h_n u_n^j\right) \\
&\geq \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) + \sum_{n=-\infty}^{+\infty}\left(\chi_n F_n(u_n) + h_n u_n\right) \\
&= -\Phi(u).
\end{aligned}$$

It implies that $\Phi(u) \geq C_0$. Therefore, Φ is τ-upper semicontinuous.

Now, we show that Φ' is weakly sequentially continuous. Clearly, $(F1)$ implies that

$$|f_n(s)| \leq C(1+|s|), \quad (n,s) \in Z \times R. \tag{2.8}$$

Let $v \in E = l^2$, then for any $\varepsilon > 0$ there exists a positive constant $N_0 \in Z$ such that

$$\left(\sum_{\{n \in Z:\ |n| > N_0\}} v_n^2 \right)^{1/2} \leq \varepsilon. \tag{2.9}$$

Let $u^j \rightharpoonup u$ in E, then $\|u^j\| \leq C$ and $u_n^j \to u_n$ for all $n \in Z$. By $(X1)$, (2.8), (2.9), $\|u^j\|_{l^2} \leq C$ ($\|\cdot\|$ and $\|\cdot\|_{l^2}$ are equivalent), $\|\chi\|_{l^\infty} \leq \|\chi\|_{l^2}$ (see (1.5)) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\left| \sum_{\{n \in Z:\ |n| > N_0\}} \chi_n f_n(u_n^j) v_n \right| \\
\leq & C \left[\sum_{\{n \in Z:\ |n| > N_0\}} |\chi_n||v_n| + \|\chi\|_{l^\infty} \sum_{\{n \in Z:\ |n| > N_0\}} |u_n^j||v_n| \right] \\
\leq & C \left(\|\chi\|_{l^2} + \|\chi\|_{l^\infty} \|u^j\|_{l^2} \right) \left(\sum_{\{n \in Z:\ |n| > N_0\}} |v_n|^2 \right)^{1/2} \\
\leq & C\varepsilon.
\end{aligned}$$

Similarly, we get

$$\left| \sum_{\{n \in Z:\ |n| > N_0\}} \chi_n f_n(u_n) v_n \right| \leq C\varepsilon.$$

Therefore, we have

$$\left| \sum_{\{n \in Z:\ |n| > N_0\}} \chi_n f_n(u_n^j) v_n - \sum_{\{n \in Z:\ |n| > N_0\}} \chi_n f_n(u_n) v_n \right| \leq C\varepsilon.$$

It implies that

$$\lim_{j \to \infty} \sum_{\{n \in Z:\ |n| > N_0\}} \chi_n f_n(u_n^j) v_n = \sum_{\{n \in Z:\ |n| > N_0\}} \chi_n f_n(u_n) v_n,$$

which together with $u_n^j \to u_n$ for all $n \in Z$ implies that

$$\begin{aligned}
&\lim_{j \to \infty} \sum_{n=-\infty}^{+\infty} \chi_n f_n(u_n^j) v_n \\
&= \lim_{j \to \infty} \left(\sum_{\{n \in Z:\ |n| \leq N_0\}} + \sum_{\{n \in Z:\ |n| > N_0\}} \right) \chi_n f_n(u_n^j) v_n \\
&= \sum_{n=-\infty}^{+\infty} \chi_n f_n(u_n) v_n.
\end{aligned} \tag{2.10}$$

Consequently, by (2.10) and $u^j \rightharpoonup u$ in E, we have

$$
\begin{aligned}
\lim_{j\to\infty}\langle \Phi'(u^j), v\rangle &= \lim_{j\to\infty}\left[((u^j)^+, v^+) - ((u^j)^-, v^-) - \sum_{n=-\infty}^{+\infty}(\chi_n f_n(u_n^j)v_n + h_n v_n)\right] \\
&= (u^+, v^+) - (u^-, v^-) - \sum_{n=-\infty}^{+\infty}(\chi_n f_n(u_n)v_n + h_n v_n) \\
&= \langle \Phi'(u), v\rangle, \quad \forall v \in E.
\end{aligned}
$$

That is, Φ' is weakly sequentially continuous. The proof is finished. □

Lemma 2.3. *Under conditions of Theorem 1.1, the geometric assumption (2.3) in Lemma 2.1 is true. i.e.,*

$$
b := \inf_{E^+}\Phi > \sup_{\partial M}\Phi, \quad d := \sup_{M}\Phi < \infty.
$$

Proof. Obviously, if $\chi_n \equiv 0$ ($n \in Z$), then assumption ($L1$) implies that (1.1) becomes to a linear equation and it is easy to see that it has a solution. Therefore, we may assume that $\|\chi\|_{l^\infty} \neq 0$. The equivalence of norms $\|\cdot\|$ and $\|\cdot\|_{l^2}$ implies that there is a constant $C_0 > 0$ such that

$$
C_0\|u\|_{l^2}^2 \leq \|u\|^2, \quad \forall u \in E = l^2. \tag{2.11}
$$

Clearly, ($F1$) implies that

$$
|f_n(s)| \leq \frac{C_0}{3\|\chi\|_{l^\infty}}|s| + C, \quad |F_n(s)| \leq \frac{C_0}{3\|\chi\|_{l^\infty}}|s|^2 + C|s|, \quad (n, s) \in Z \times R. \tag{2.12}
$$

For $u \in E^+$, by (2.11), (2.12) and the definition of Φ, we have

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty}(\chi_n F_n(u_n) + h_n u_n) \\
&\geq \frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty}|\chi_n|\left(\frac{C_0}{3\|\chi\|_{l^\infty}}|u_n|^2 + C|u_n|\right) - \sum_{n=-\infty}^{+\infty}|h_n||u_n| \\
&\geq \frac{1}{2}\|u\|^2 - \frac{C_0}{3}\|u\|_{l^2}^2 - (C\|\chi\|_{l^2} + \|h\|_{l^2})\|u\|_{l^2} \\
&\geq \frac{1}{6}\|u\|^2 - \frac{1}{C_0^{1/2}}(C\|\chi\|_{l^2} + \|h\|_{l^2})\|u\|.
\end{aligned}
$$

It follows form $\|\chi\|_{l^2} + \|h\|_{l^2} < \infty$ (see ($X1$)) that $b := \inf_{E^+}\Phi > -\infty$.

For $u \in E^-$, by (2.11) and $\chi_n F_n(s) \geq 0$ for all $(n, s) \in Z \times R$ (see ($F1$) and ($X1$)), we have

$$
\begin{aligned}
\Phi(u) &= -\frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty}(\chi_n F_n(u_n) + h_n u_n) \\
&\leq -\frac{1}{2}\|u\|^2 + \|h\|_{l^2}\|u\|_{l^2} \\
&\leq -\frac{1}{2}\|u\|^2 + \frac{1}{C_0^{1/2}}\|h\|_{l^2}\|u\|.
\end{aligned}
$$

It follows form $\|h\|_{l^2} < \infty$ (see $(X1)$) that for R large enough we have

$$b := \inf_{E^+} \Phi > \sup_{\partial M} \Phi, \quad d := \sup_{M} \Phi < \infty,$$

where M is defined in (2.1). The proof is finished. □

Now, Lemmas 2.2 and 2.3 imply that Lemma 2.1 holds. We give the detailed proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.1, for some $c \in R$, there is a sequence $\{u^j\} \subset E$ such that

$$\Phi(u^j) \to c, \quad \Phi'(u^j) \to 0. \tag{2.13}$$

Let $\hat{u}^j = (u^j)^+ - (u^j)^-$, then $\|\hat{u}^j\| = \|u^j\|$. Therefore, by (2.11)-(2.13) and the spaces $E^{\pm}$ orthogonal with respect to $(\cdot,\cdot)_{l^2}$, we have

$$\begin{aligned}
C\|u^j\| &= C\|\hat{u}^j\| \\
&\geq \langle \Phi'(u^j), \hat{u}^j \rangle \\
&= \|(u^j)^+\|^2 + \|(u^j)^-\|^2 - \sum_{n=-\infty}^{+\infty} \left(\chi_n f_n(u_n^j)\hat{u}_n^j + h_n \hat{u}_n^j\right) \\
&\geq \|u^j\|^2 - \sum_{n=-\infty}^{+\infty} \left[|\chi_n| \left(\frac{C_0}{3\|\chi\|_{l^\infty}}|u_n^j| + C\right) + |h_n|\right] \left(|(u_n^j)^+| + |(u_n^j)^-|\right) \\
&\geq \|u^j\|^2 - \left(\frac{C_0}{3}\|u^j\|_{l^2} + C\|\chi\|_{l^2} + \|h\|_{l^2}\right) \left(\|(u^j)^+\|_{l^2} + \|(u^j)^-\|_{l^2}\right) \\
&\geq \|u^j\|^2 - 2\|u^j\|_{l^2} \left(\frac{C_0}{3}\|u^j\|_{l^2} + C\|\chi\|_{l^2} + \|h\|_{l^2}\right) \\
&\geq \frac{1}{3}\|u^j\|^2 - \frac{2}{C_0^{1/2}} \left(C\|\chi\|_{l^2} + \|h\|_{l^2}\right) \|u^j\|.
\end{aligned}$$

It follows from $\|\chi\|_{l^2} + \|h\|_{l^2} < \infty$ (see $(X1)$) that $\{u^j\}$ is bounded in E.

Consequently, up to a subsequence, we may assume that $u^j \rightharpoonup u$ in E. By the fact that Φ' is weakly sequentially continuous (see Lemma 2.2), we have

$$\lim_{j\to\infty} \langle \Phi'(u^j), v \rangle = \langle \Phi'(u), v \rangle = 0, \quad \forall v \in E.$$

Therefore, $\Phi'(u) = 0$ and u is a solution of (1.1). The proof is finished. □

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