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P-DISTRIBUTION ALMOST PERIODIC SOLUTIONS OF SEMI-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH G-BROWNIAN MOTION*

Qigui Yang[1,†], Huoxia Liu[1] and Xiaofang Lin[1]

Abstract As a class of recurrence, almost periodicity has been studied in stochastic differential equations (SDEs) under the framework of linear expectation. However, in the framework of nonlinear expectation, there are few literatures on Poisson stable solutions for SDEs and (pseudo) almost periodic solutions for SDEs with exponential dichotomy. This paper is devoted to the existence and asymptotical stability of p-distribution Poisson stable solutions for nonhomogeneous linear and semi-linear SDEs driven by G-Brownian motion satisfying exponential stability. Moreover, some existence results of (pseudo) almost periodic solution in p-distribution are established for semi-linear SDEs driven by G-Brownian motion satisfying exponential dichotomy. Meanwhile, some examples are given to validate the obtained theoretical results.

Keywords Almost periodic solution, semi-linear SDE, comparability approach, exponential dichotomy, G-Brownian motion.

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1. Introduction

Recurrence describes that a motion returns infinitely often to any small neighborhood of the initial position. It characterizes the asymptotic behaviors and the complexity of systems. By Poincaré recurrence theorem and Birkhoof recurrence theorem, recurrence exists widely in lots of dynamical systems, such as in the fields of physics, chemistry, biology, engineering and economics. In addition, in probability theory, the existence of recurrence actually means the existence of invariant probability measures for Markov processes. Poisson stable motions are sometimes called recurrent motions in dynamical systems. It includes particularly pseudo-recurrence, pseudo-periodicity, almost recurrence, Levitan almost periodicity, Birkhoff recurrence, Bohr almost automorphy, Bohr almost periodicity, quasi-periodicity, periodicity, stationarity. As a special case of Poisson stability, almost periodicity was founded by Bohr [2–4] in $1924-1926$ and was developed by Bochner [6] with a simpler characterization. In the early stage, almost periodicity was mainly researched

†The corresponding author. Email: qgyang@scut.edu.cn(Q. Yang)
1Department of Mathematics, South China University of Technology, 510640, Guangzhou, China
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in Fourier series theory. While it was found gradually that many differential equations admit almost periodic solutions, see Fink [21], Amerio and Prouse [1] for a survey. Based on these excellent work before, the concept of almost periodicity was generalized to pseudo almost periodicity and μ-pseudo almost periodicity, see [5,42] and references therein for more details.

In stochastic sense and in the framework of linear expectation, there are many results of almost periodic solutions of SDEs, such as references [7,9,38,43]. Note that these systems become unpractical since the volatility is a fixed constant. Considering the uncertainty of probabilities and distributions (such as uncertainty problem in statistics, measures of risk and the super-hedging in finance), Peng [29] introduced the notion of G-normal distribution, G-expectation and G-Brownian motion as the canonical process under such sublinear expectation, then as well as the related calculus of Itô's type. The study of such field was further improved by Peng [29–34], such as the law of large numbers and central limit theorem under nonlinear expectations. From then on, many scholars have been interested in the study of SDEs driven by G-Brownian motion from many different perspectives, see references [22–26,39,40,44]. Particularly, Zhang et al. [44] and Gu et al. [22] respectively showed the existence of almost periodic solutions and pseudo almost automorphic solutions for SDEs with G-Brownian motion. Yang et al. investigated Stepanov-like doubly weighted pseudo almost automorphic processes for Sobolev-type equations driven by G-Brownian motion in paper [40].

The first step of studying recurrent solutions of SDEs is to discuss the existence of solutions. To the best of our knowledge, there are three methods on this line. The first one is the fixed point approach, see [7, 9, 38, 40, 43] and the references therein. The second one is the Favard separation approach. Favard initially applied this method to study almost periodic solution for linear equation. Liu and Wang [27] further generalized Favard separation method and Amerio separation method to linear SDEs and nonlinear SDEs, respectively. Moreover, the Favard separation method was applicable to research almost automorphic solution of linear differential equation in paper [10]. The third one is the subvariant functional method. Cieutat and Ezzinbi [11] used it to show that every K-minimizing mild solution is compact almost automorphic for nonlinear differential equation.

When consider the Poisson stable solutions, it seemly becomes more challenging by the above methods since one can not obtain Poisson stable solutions in a unified framework by the above methods. On this line, Shcherbakov [35, 36] established comparability methods to study Poisson stable solutions of differential equations. Then Caraballo and Cheban [12] investigated almost periodic and almost automorphic solutions of linear differential equations. For scalar differential equations, the existence of Levitan almost periodic, Bohr almost periodic and almost automorphic solutions has been reported in paper [13]. The existence of Levitan almost periodic and almost automorphic solutions of V-monotone differential equations was also proved in paper [14]. Cheban [15] further researched Bohr almost periodic, Levitan almost periodic and almost automorphic solutions of linear SDEs. For semi-linear SDEs, Cheban and Liu [16] researched Poisson stability of solutions on the circumstance of Brownian motion and L^2-norm.

To our knowledge, almost all results about Poisson stable solutions are restricted to deterministic equations or SDEs with linear expectation; and in the framework of sublinear expectation, almost periodic solutions and almost automorphic solutions are just studied in the case that SDEs satisfy exponential stability. A natural question is raised: does the Poisson stable solutions survive to SDEs with sublinear expectation? This paper gives a positive response to this question. Under some sufficient conditions, Poisson stable (in particular, pseudo-recurrent, pseudo-periodic, almost recurrent, Levitan almost periodic, Birkhoff recurrent, Bohr almost automorphic, Bohr almost periodic, quasi-periodic with a limited spectrum, τ-periodic, stationary) solutions are obtained for linear and semi-linear SDEs with G-Brownian motion.

Precisely, the following nonhomogeneous linear SDEs are considered at first:

$$d\varpi(t) = (A\varpi(t) + G_1(t))dt + G_2(t)dB_t + G_3(t)d\langle B\rangle_t, \tag{1.1}$$

where B_t is a standard two-sided G-Brownian motion and $\langle B\rangle_t$ is its quadratic variation. This equation is explained as its integral type and the stochastic integrals with respect to B_t and $\langle B\rangle_t$ are given in subsection 2.1 below. By the virtue of estimates of these integrals (Proposition 2.1), one acquires some useful estimates relative to equations so that one obtains the asymptotic stability of Poisson stable solutions and the existence of (pseudo) almost periodic solution. For the nonhomogeneous linear SDE (1.1), one investigates the comparability characterizing by recurrence on $L_G^p(\Omega), p > 2$, and then obtains Poisson stable (particularly, pseudo-recurrent, pseudo-periodic, almost recurrent, Levitan almost periodic, Birkhoff recurrent, Bohr almost automorphic, Bohr almost periodic, quasi-periodic with a limited spectrum, τ-periodic, stationary) solutions in p-distribution.

Moreover, one considers the following semi-linear SDEs:

$$d\varpi(t) = (A\varpi(t) + G_1(t, \varpi(t)))dt + G_2(t, \varpi(t))dB_t + G_3(t, \varpi(t))d\langle B\rangle_t. \tag{1.2}$$

Except for the same Poisson stable results as equation (1.1) in the case that semigroup generated by A satisfies exponential stability, one studies (pseudo) almost periodic solution when the semigroup generated by A satisfies exponential dichotomy. Notice that there is no valid result different from Lemma 4.3 of paper [16] up to now so that one can not obtain Poisson stable solutions for SDEs with exponential dichotomy. In this case this paper acquires (pseudo) almost periodic solutions in p-distribution for (1.2) with exponential dichotomy.

The remaining part of this work is arranged as below. In Section 2, one recalls several necessary concepts and preliminaries. In Section 3, using the comparability method, Poisson stable solutions in p-distribution are gained for nonhomogeneous linear and semi-linear SDEs. Moreover, sufficient criteria for the asymptotic stability of the corresponding Poisson stable solutions of semi-linear SDEs are presented. In Section 4, for semi-linear SDEs with exponential dichotomy, some theorems about (pseudo) almost periodic solution in p-distributions are set up. In Section 5, some examples are presented to validate the theoretical claims.

2. Preliminaries

2.1. G-Brownian motion

In this subsection, one introduces some notations and preliminaries of the theory of sublinear expectation and G-stochastic analysis from Peng [34]. Throughout the paper, our results are established in the space $L_G^p(\Omega)$ satisfying $p > 2$ except that p is specifically specified, and the $L_G^p(\Omega)$ space will be introduced below. First, for a given set Ω, $\mathcal{H}$ is a linear space of real valued functions on Ω. A function $E : \mathcal{H} \to \mathbb{R}$ is said to be a sublinear expectation provided that the following properties hold, for all $x, y \in \mathcal{H}$,

(a_1) $x \geq y$ implies $E[x] \geq E[y]$;

(a_2) $E[c] = c$, $\forall c \in \mathbb{R}$;

(a_3) $E[x + y] \leq E[x] + E[y]$;

(a_4) $E[\lambda x] = \lambda E[x]$, $\forall \lambda \geq 0$.

The triplet $(\Omega, \mathcal{H}, E)$ is called sublinear expectation space.

Definition 2.1. Let $(\Omega, \mathcal{H}, E)$ be a sublinear expectation space. For any $x \in \mathcal{H}$ with $\overline{\sigma}^2 := E[x^2]$, $\underline{\sigma}^2 := E[-x^2]$, if for any $c, d \geq 0$, $cx + dy = \sqrt{c^2 + d^2}x$, then x is called $\mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$-distributed or G-distributed, where y is an independent copy of x.

Let X be an $\mathbb{R}^d$-valued G-distributed stochastic variable and $C_{b,lip}(\mathbb{R}^d, \mathbb{R})$ be the space of bounded Lipschitz functions defined on $\mathbb{R}^d$. For any $\varphi \in C_{b,lip}(\mathbb{R}^d, \mathbb{R})$, define $u(t, x)$ given by

$$u(t, x) = E[\varphi(x + \sqrt{t}X)].$$

Then $u(t, x)$ is the solution of the following parabolic partial differential equation

$$\begin{cases} \partial_t u - G(\partial^2_{xx} u) = 0, \\ u(0, x) = \varphi(x), \end{cases} \tag{2.1}$$

where $G(r) = \frac{1}{2}(\overline{\sigma}^2 r^+ - \underline{\sigma}^2 r^-)$ for $r \in \mathbb{R}$, $r^+ = \max\{0, r\}$ and $r^- = (-r)^+$.

Definition 2.2. The process $B = \{B_t, t \geq 0\}$ in a sublinear expectation space $(\Omega, \mathcal{H}, E)$ is defined as a G-Brownian motion provided that the following statements hold:

(a_1) $B_0 = 0$;

(a_2) for every $t, r \geq 0$, the difference $B_{t+r} - B_t$ is $\mathcal{N}(0, [\underline{\sigma}^2 r, \overline{\sigma}^2 r])$-distributed and is independent of $(B_{t_1}, \cdots, B_{t_n})$, for all $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

Now, one considers that B_t is a $\mathbb{R}^d$-valued G-Brownian motion on $(\Omega, \mathcal{H}, E)$. Let

$$L_{ip}(\Omega_T) := \left\{\varphi(B_{t_1}, \cdots, B_{t_n}), n \in \mathbb{N}, t_j \in [0, T], j = 1, 2, \cdots, n, \varphi \in C_{b,lip}(\mathbb{R}^{d \times n})\right\}.$$

It is clear that $L_{ip}(\Omega_t) \subset L_{ip}(\Omega_T)$ for $t \leq T$. Set

$$L_{ip}(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

Denote $L^p_G(\Omega_T)$ and $L^p_G(\Omega)$ by the completion of $L_{ip}(\Omega_T)$ and $L_{ip}(\Omega)$ respectively under the norm $\|\cdot\|_p := (E|\cdot|^p)^{\frac{1}{p}}$. For $t > 0$, a partition of $[0, t]$ is a finite-ordered subset $\{\pi_t\}$ satisfying

$$\pi_t : 0 = t_0 < t_1 < \cdots < t_{N-1} = t$$

with $\max\{t_i - t_{i-1}, i = 1, \cdots, N\} \to 0$ as $N \to \infty$. Then one further sets

$$M_G^{p,0}([0,T]) := \Bigg\{ \xi_t := \sum_{j=0}^{N-1} \xi_j I_{[t_j,t_{j+1})}(t); \xi_j \in L_G^p(\Omega_{t_j}), N \in \mathbb{N}, \\ 0 = t_0 < t_1 < \cdots < t_N = T \Bigg\},$$

and denotes $M_G^p([0,T])$ by the completion of $M_G^{p,0}([0,T])$ under the norm

$$\|\xi\|_{M_G^p([0,T])} = \left(\int_0^T |\xi_t|^p dt \right)^{\frac{1}{p}}.$$

The Itô integral is given by

$$\int_0^T \xi_t dB_t := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}), \quad \xi_t(\omega) = \sum_{j=0}^{N-1} \xi_{t_j}(\omega) I_{[t_j,t_{j+1})} \in M_G^{p,0}([0,T]).$$

Definition 2.3. The quadratic variation process $\langle B \rangle_t, t \geq 0$ of B_t is defined by

$$\langle B \rangle_t := \lim_{N \to \infty} \sum_{j=0}^{N-1} (B_{t_{j+1}} - B_{t_j})^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

The integral with respect to $\langle B \rangle_t$ is given by

$$\int_0^T \xi_t d\langle B \rangle_t := \sum_{j=0}^{N-1} \xi_j (\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}), \quad \xi_t \in M_G^0(0,T).$$

Remark 2.1. Because integrals $\int_0^T \xi_t dB_t$ and $\int_0^T \xi_t d\langle B \rangle_t$ are continuous linear mappings from $M_G^{p,0}(0,T)$ to $L_G^p(\Omega_T)$, they can be extended to maps from $M_G^p(0,T)$ to $L_G^p(\Omega_T)$ continuously.

Proposition 2.1. *For $0 \leq t \leq T < +\infty$, $p \geq 1$,*

(a_1) $\forall\ \xi_t \in M_G^2([0,T])$, $E\left[\left|\int_0^T \xi_t^2 d\langle B \rangle_t\right|\right] \leq \overline{\sigma}^2 E\left[\left|\int_0^T \xi_t^2 dt\right|\right]$;

(a_2) $\forall\ \xi_t \in M_G^p([0,T])$, $E\left[\left|\int_0^T \xi_t dB_t\right|^p\right] \leq C_p E\left[\left|\int_0^T \xi_t^2 d\langle B \rangle_t\right|^{\frac{p}{2}}\right]$.

2.2. Poisson stability

In this subsection, one recalls the types of Poisson stability. One refers the readers to [13, 14, 16] and the references therein for more details.

Note that $(L_G^p(\Omega), \|\cdot\|_p)$ is a Banach space. Let $C(\mathbb{R}, L_G^p(\Omega))$ be the space consisting of continuous functions $\psi : \mathbb{R} \to L_G^p(\Omega)$ endowed with the distance

$$d(\psi, \phi) := \sup_{L_1 > 0} \min\left\{ \max_{|t| \leq L_1} \left(E\,|\psi(t) - \phi(t)|^p\right)^{\frac{1}{p}}, L_1^{-1} \right\}.$$

Then $(C(\mathbb{R}, L_G^p(\Omega)), d)$ is a complete Banach space.

Definition 2.4. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is said to be stationary (τ-periodic) provided that for all $t \in \mathbb{R}$ $\psi(t) = \psi(0)$ ($\psi(t+\tau) = \psi(t)$ for some $\tau > 0$).

Definition 2.5. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is defined as quasi-periodic with limited spectrum of frequencies $\mu_1, \mu_2, \cdots, \mu_k$ provided that the following conditions are fulfilled:
(a_1) the numbers $\mu_1, \mu_2, \cdots, \mu_k$ are rationally independent;
(a_2) there is a continuous function for all $(t_1, t_2, \cdots, t_k) \in \mathbb{R}^k$ $F : \mathbb{R}^k \to L_G^p(\Omega)$ fulfilling

$$F(t_1 + 2\pi, t_2 + 2\pi, \cdots, t_k + 2\pi) = F(t_1, t_2, \cdots, t_k);$$

(a_3) for $t \in \mathbb{R}$, $\psi(t) = F(\mu_1 t, \mu_2 t, \cdots, \mu_k t)$.

Definition 2.6. For $\varepsilon > 0$, a number $\tau \in \mathbb{R}$ is defined as ε-almost period of the continuous function $h : \mathbb{R} \to L_G^p(\Omega)$ provided that

$$(E\,|h(t+\tau) - h(t)|^p)^{\frac{1}{p}} < \varepsilon$$

for all $t \in \mathbb{R}$. Denote by $\mathcal{T}(\varepsilon, h)$ is the set of ε-almost periods of h. If the set of ε-almost periods of h is relatively dense on $\mathbb{R}$, i.e. for each $\varepsilon > 0$ there is a number $l' = l'(\varepsilon) > 0$ satisfying $(b, b+l') \cap \mathcal{T}(\varepsilon, h) \neq \emptyset$ for any $b \in \mathbb{R}$, then it is called almost periodic. Thereafter, $\mathcal{T}(\varepsilon, h)$ is the set of ε-almost periods of h.

Definition 2.7. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is called Levitan almost periodic provided that there is a Bohr almost periodic function $\phi \in C(\mathbb{R}, L_G^p(\Omega))$ so that for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $d(\psi^\sigma, \psi) < \varepsilon$ for all $\sigma \in \mathcal{T}(\phi, \delta)$.

Definition 2.8. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is said to be Lagrange stable provided that $\{\psi^\sigma : \sigma \in \mathbb{R}\}$ is a relatively compact subset of $C(\mathbb{R}, L_G^p(\Omega))$. If it also is Levitan almost periodic, then it is called Bohr almost automorphic.

Definition 2.9. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is defined as almost recurrent (Bebutov sense) provided that for each $\varepsilon > 0$ the set $\{\sigma : d(\psi^\sigma, \psi) < \varepsilon\}$ is relatively dense. If it is Lagrange stable, then it is called Birkhoff recurrent.

Definition 2.10. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is defined as positive (negative) pseudo-periodic provided that for every $\varepsilon > 0$ and $l_0 > 0$ it has a ε-almost periodic $\sigma > l_0$ ($\sigma < -l_0$) corresponding to the function ψ. The function ψ is said to be pseudo-periodic provided that this is both positive pseudo-periodic and negative pseudo-periodic.

Definition 2.11. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is defined as pseudo-recurrent if for any $\varepsilon > 0$ and $l_1 \in \mathbb{R}$ it has $L_1 \geq l_1$ so that for any $\tau_1 \in \mathbb{R}$ one finds a number $\tau_2 \in [l_1, L_1]$ fulfilling

$$\sup_{|t| \leq \frac{1}{\varepsilon}} (E\,|\psi(t + \tau_1 + \tau_2) - \psi(t + \tau_1)|^p)^{\frac{1}{p}} \leq \varepsilon.$$

Definition 2.12. A function $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ is defined as positive (negative) Poisson stable provided that for each $\varepsilon > 0$ and $l_1 > 0$, it has $\sigma > l_1 (\sigma < -l_1)$ satisfying $d(\psi^\sigma, \psi) < \varepsilon$. The function ψ is said to be Poisson stable provided that it is Poisson stable in both positive direction and negative direction.

Remark 2.2. In general, except for Lagrange stable function, other functions introduced above are all Poisson stable.

Let $BC(\mathbb{R}, L_G^p(\Omega))$ be the Banach space of all bounded continuous functions $\phi : \mathbb{R} \to L_G^p(\Omega)$ endowed with the supremum norm $\|\phi\|_\infty := \sup_{t\in\mathbb{R}}(E|\phi(t)|^p)^{\frac{1}{p}}$.

Definition 2.13. For given $\psi \in C(\mathbb{R}, L_G^p(\Omega))$, denote by ψ^σ the σ-translation of ψ, i.e. $\psi^\sigma(t) = \psi(t+\sigma)$ for $t \in \mathbb{R}$. $H(\psi)$ is called the hull of ψ provided that the set of all the limits of ψ^{σ_n} exists in $C(\mathbb{R}, L_G^p(\Omega))$, i.e.

$$H(\psi) := \{\phi \in C(\mathbb{R}, L_G^p(\Omega)) : \phi = \lim_{n\to\infty} \psi^{\sigma_n} \text{ for some sequence } \{\sigma_n\} \subset \mathbb{R}\}.$$

Remark 2.3. If ψ in $BC(\mathbb{R}, L_G^p(\Omega))$, then for any $\widetilde{\psi} \in H(\psi)$ one has $\left\|\widetilde{\psi}\right\|_p \le \|\psi\|_\infty$ for each $t \in \mathbb{R}$.

There are various notations about almost periodicity when they apply to stochastic process, such as almost periodicity in p-mean and almost periodicity in distribution. One refers the readers to Tudor [37] for more details.

2.3. p-distribution (pseudo) almost periodic stochastic process

From now on, let $\mathcal{L}(Z(t))$ be the distribution of the random variable $Z(t)$. $\mathbb{M}$ represents the set of all positive measures μ on $\mathcal{B}$ satisfying both $\mu(R) = +\infty$ and $\mu([a,b]) < +\infty$ for all $a, b \in \mathbb{R}(a \le b)$, where $\mathcal{B}$ is the Lebesgue σ-field of $\mathbb{R}$. Denote that $\mathcal{L}(L_G^p(\Omega))$ is the space of Borel probability measures on $L_G^p(\Omega)$ equipped with the metric

$$d_{BL}(\nu_1, \nu_2) := \sup\left\{\left|\int \psi d\nu_1 - \int \psi d\nu_2\right| : \|\psi\|_{BL} \le 1\right\},$$

$\nu_1, \nu_2 \in \mathcal{L}(L_G^p(\Omega))$, where ψ are Lipschitz continuous real-valued functions on $L_G^p(\Omega)$ and

$$\|\psi\|_{BL} = \max\{|\psi|_L, |\psi|_\infty\}$$

with

$$|\psi|_L = \sup_{x\neq y} \frac{|\psi(x) - \psi(y)|}{|x-y|}, \quad |\psi|_\infty = \sup_{x\in L_G^p(\Omega)} |\psi(x)|.$$

A sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(L_G^p(\Omega))$ is called weakly convergent to ν provided that there is a $\psi \in BC(L_G^p(\Omega))$ such that $\int \psi d\nu_n \to \int \psi d\nu$.

Definition 2.14 (Bedouhene et al. [7])**.** An L^p-continuous stochastic process Z is known as almost periodicity in p-distribution:
(i) If the mapping $t \to \mathcal{L}(Z(t+\cdot))$ from $\mathbb{R}$ to $\mathcal{L}(C(\mathbb{R}, L_G^p(\Omega)))$ is almost periodic.
(ii) If $p > 0$, the family $\{|Z(t)|^p : t \in \mathbb{R}\}$ is uniformly integrable.

Definition 2.15 (Bezandry et al. [8])**.** A continuous function $h : \mathbb{R} \to L_G^p(\Omega)$ is called p-mean almost periodic for $t \in \mathbb{R}$, if for any $\varepsilon > 0$, there is an $l = l_\varepsilon > 0$ such that any interval of length l contains at least a number τ satisfying

$$\sup_{t\in\mathbb{R}} E|h(t+\tau) - h(t)|^p < \epsilon.$$

Denote by $AP(\mathbb{R}, L_G^p(\Omega))$ the set of p-mean almost periodic functions, it is easy to check that $AP(\mathbb{R}, L_G^p(\Omega))$ is a Banach space endowed with supremum norm.

Definition 2.16 (Yoshizawa et al. [41]). A continuous function $h : \mathbb{R} \times L_G^p(\Omega) \to L_G^p(\Omega)$ is called almost periodic in t uniformly with respect to x in $L_G^p(\Omega)$ provided that for each compact set K in $L_G^p(\Omega)$, for any $\varepsilon > 0$, there is an $l = l_\varepsilon > 0$ such that any interval of length l contains at least a number τ with

$$\sup_{t\in\mathbb{R}} \sup_{x\in K} E\left|h(t+\tau,x) - h(t,x)\right|^p < \varepsilon.$$

Let $AP(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega))$ be the set of all such functions.

Definition 2.17 (Diop et al. [18]). For $\mu \in \mathbb{M}$, a stochastic process Y is defined as p-th μ-ergodic provided that $Y \in BC(\mathbb{R}, L_G^p(\Omega))$ and fulfills

$$\lim_{M\to\infty} \frac{1}{\mu([-M,M])} \int_{-M}^{M} E\left|Y(t)\right|^p d\mu(t) = 0.$$

Let $\varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$ represent the set formed by such stochastic processes. One can immediately verify that it forms a Banach space equipped with supremum norm.

Definition 2.18 (Diop et al. [18]). A continuous function $h : \mathbb{R} \times L_G^p(\Omega) \to L_G^p(\Omega)$ is called μ-ergodic in t uniformly with respect to x in $L_G^p(\Omega)$ provided the following conditions hold:

(a_1) for all $x \in X$, $h(\cdot, x) \in \varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$;

(a_2) h is uniformly continuous with respect to x on each compact set K of $L_G^p(\Omega)$.

Denote that $\varepsilon(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega), \mu)$ is the set of all such functions.

Definition 2.19 (Blot et al. [5]). Let $\mu \in \mathbb{M}$. The continuous function $g : \mathbb{R} \to L_G^p(\Omega)$ ($g : \mathbb{R} \times L_G^p(\Omega) \to L_G^p(\Omega)$) is said to be μ-pseudo almost periodic in p-mean sense if it can be expressed as: $g = g_1 + g_2$, where $g_1 \in AP(\mathbb{R}, L_G^p(\Omega))$ ($g_1 \in AP(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega))$) and $g_2 \in \varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$ ($g_2 \in \varepsilon(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega), \mu)$). Denote $PAP(\mathbb{R}, L_G^p(\Omega), \mu)$ ($PAP(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega), \mu)$) by the set formed by such functions, one can easily verify that $PAP(\mathbb{R}, L_G^p(\Omega), \mu)$ is a Banach space given the supremum norm.

2.4. Shcherbakov's comparability method, exponential dichotomy

Let BUC be the set consisting of all functions $h : \mathbb{R} \times L_G^p(\Omega) \to L_G^p(\Omega)$ that is continuous in t uniformly continuous concerning for ϖ on each bounded subset $Q \subseteq L_G^p(\Omega)$ and also bounded on each bounded subset of $\mathbb{R} \times L_G^p(\Omega)$. For given $h \in BUC(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega))$ and $\sigma \in \mathbb{R}$, let h^σ be the translation of h, i.e. $h^\sigma(t,x) := h(t+\sigma, x)$. For the more details of the space BUC and the properties of functions on space BUC, one can refer to paper [16]. Denote that $\mathbb{N}_\psi$ ($\mathbb{M}_\psi$) represents the family of all sequences $\{\tau_n\} \subset \mathbb{R}$ ($\{\tau_n\} \subset \mathbb{N}_\psi$) satisfying $\psi^{\tau_n} \to \psi$ (ψ^{τ_n} converges) in $C(\mathbb{R}, L_G^p(\Omega))$ as $n \to \infty$ and $\mathbb{N}_\psi^u$ ($\mathbb{M}_\psi^u$) represents the family of all sequences $\{\tau_n\} \subset \mathbb{R}$ ($\{\tau_n\} \subset \mathbb{N}_\psi$) such that ψ^{τ_n} converges ψ (ψ^{τ_n} converges) uniformly in $t \in \mathbb{R}$ as $n \to \infty$.

Definition 2.20 (Shcherbakov [35]). A function $\varphi \in C(\mathbb{R}, L_G^p(\Omega))$ is called comparable (respectively, strongly comparable) by the character of recurrence with $\psi \in C(\mathbb{R}, L_G^p(\Omega))$ provided that $\mathbb{N}_\psi \subseteq \mathbb{N}_\varphi$ (respectively, $\mathbb{M}_\psi \subseteq \mathbb{M}_\varphi$).

Definition 2.21. Let $\varpi(t)$ be a mild solution of equation (1.2). Then ϖ is said to be compatible (strongly compatible) in p-distribution provided that $\mathbb{N}_{G_1}\cap\mathbb{N}_{G_2}\cap\mathbb{N}_{G_3}\subseteq\widetilde{\mathbb{N}}_{\varpi}$ ($\mathbb{M}_{G_1}\cap\mathbb{M}_{G_2}\cap\mathbb{M}_{G_3}\subseteq\widetilde{\mathbb{M}}_{\varpi}$), where $\widetilde{\mathbb{N}}_{\varpi}$ ($\widetilde{\mathbb{M}}_{\varpi}$) represents the set of all sequences $\{t_n\}\subset\mathbb{R}$ such that the sequence $\varpi(\cdot+t_n)$ converges to $\varpi(\cdot)$ ($\varpi(\cdot+t_n)$ converges) in p-distribution uniformly on any compact interval.

Remark 2.4. The results of Definition 2.21 also hold for equation (1.1).

Lemma 2.1 (Engel and Nagel [20]). *A linear operator $A: D(A)\to L^p_G(\Omega)$ is called ω-sectorial of angle θ provided that there are constant $\widehat{M}>0$, $\omega\in\mathbb{R}$ and $\theta\in(\frac{\pi}{2},\pi)$ such that*

$$\rho(A)\supseteq S_{\theta,\omega}:=\{\lambda\in\mathbb{C}:\mu_0\neq\omega,|arg(\mu_0-\omega)|<\theta\},$$
$$\left\|(\lambda I-A)^{-1}\right\|\leq\frac{\widehat{M}}{|\mu_0-\omega|}\ \textit{for each}\ \mu_0\in S_{\theta,\omega}.$$

Provided that A is ω-sectorial of angle θ (see [28]), then it generates an analytic semigroup $\{S(t)\}_{t\geq0}$ in the sector $S_{\theta-\frac{\pi}{2},0}$, and there are two positive constants $\widetilde{M}$, $\overline{M}$ satisfying

$$\|S(t)\|\leq\widetilde{M}e^{\omega t},\ \textit{and}\ \|t(A-\omega)S(t)\|\leq\overline{M}e^{\omega t},t>0.$$

Definition 2.22. The semigroup $\{S(t):t\geq0\}$ on $L^p_G(\Omega)$ is called exponential stable provided that there exist $k,\delta>0$ satisfying $\|S(t)\|\leq ke^{-\delta t}$ for any $t\geq0$.

Definition 2.23 (Cao et al. [17]). The semigroup $\{S(t):t\in\mathbb{R}\}$ is said to have an exponential dichotomy, provided that there exists projection $\mathcal{P}$, $ker(\mathcal{P})$ is invariant with respect to $S(t)$, and two positive constants $\widehat{k},\delta'$ satisfying

(a_1) $\mathcal{P}S(t)=S(t)\mathcal{P}$;

(a_2) the restriction $S(t):\mathcal{Q}L^p_G(\Omega)\to\mathcal{Q}L^p_G(\Omega)$ of $S(t)$ is invertible;

(a_3) $\|S(t)\mathcal{P}\|\leq\widehat{k}e^{-\delta't}$ for $t\geq0$, and $\|S(t)\mathcal{Q}\|\leq\widehat{k}e^{\delta't}$, for $t<0$, where $\mathcal{Q}=I-\mathcal{P}$.

3. Asymptotic stability of Poisson stable solutions

Throughout this section, assume that the semigroup $\{S(t)\}_{t\geq0}$ generated by A of equations (1.1) and (1.2) satisfies exponential stability, and let $\iota_0=\left(\frac{p-2}{p\delta}\right)^{\frac{p}{2}-1}$ and $\iota_1=\left(\frac{2(p-1)}{p\delta}\right)^{p-1}$.

3.1. The existence of Poisson stable solutions

In this subsection, one considers the Poisson stable solutions for linear SDE (1.1) and semi-linear SDE (1.2) by the comparability method. One first investigates the nonhomogeneous linear equation (1.1).

Theorem 3.1. *Let coefficients $G_1,G_2,G_3\in BC(\mathbb{R},L^p_G(\Omega))$. Then for the linear SDE* (1.1), *one obtains the following four statements:*

(i) *there is a unique mild solution* $\varpi \in BC(\mathbb{R},\ L^p_G(\Omega))$ *of the linear SDE* (1.1) *given by*

$$\varpi(t) = \int_{-\infty}^{t} S(t-\varrho)G_1(\varrho)d\varrho + \int_{-\infty}^{t} S(t-\varrho)G_2(\varrho)dB_\varrho + \int_{-\infty}^{t} S(t-\varrho)G_3(\varrho)d\langle B\rangle_\varrho \quad (3.1)$$

with

$$\|\varpi\|_\infty \leq 3^{1-\frac{1}{p}} k \left(\frac{2}{\delta p}\right)^{\frac{1}{p}} \left(\iota_1 \|G_1\|_\infty^p + C_p \overline{\sigma}^p \iota_0 \|G_2\|_\infty^p + \overline{\sigma}^{2p} \iota_1 \|G_3\|_\infty^p\right)^{\frac{1}{p}}; \quad (3.2)$$

(ii) *for any given constants* $\mathfrak{l}, L_0$ *satisfying* $\mathfrak{l} > L_0 > 0$ *and* $t \in [-L_0, L_0]$, *one has*

$$\begin{aligned}\max_{|t|\leq L_0} E|\varpi(t)|^p \leq &\frac{2\cdot 3^{p-1}k^p}{\delta p}\Bigg\{\iota_1\left(\sup_{|\varrho|\leq \mathfrak{l}} E|G_1(\varrho)|^p + \overline{\sigma}^{2p}\sup_{|\varrho|\leq \mathfrak{l}} E|G_3(\varrho)|^p\right)\\ &+ \iota_0 C_p \overline{\sigma}^p \sup_{|\varrho|\leq \mathfrak{l}} E|G_2(\varrho)|^p\\ &+ e^{-\frac{\delta p(\mathfrak{l}-L_0)}{2}}\left[\iota_1\|G_1\|_\infty^p + C_p\iota_0\overline{\sigma}^p\|G_2\|_\infty^p + \overline{\sigma}^{2p}\iota_1\|G_3\|_\infty^p\right]\Bigg\};\end{aligned} \quad (3.3)$$

(iii) *if* $\mathbb{M}_{G_1} \cap \mathbb{M}_{G_2} \cap \mathbb{M}_{G_3} \neq \emptyset$, *then the solution* ϖ *is strongly compatible in* p-*distribution;*

(iv) *if* $\mathbb{M}^u_{G_1} \cap \mathbb{M}^u_{G_2} \cap \mathbb{M}^u_{G_3} \neq \emptyset$, *then* $\mathbb{M}^u_{G_1} \cap \mathbb{M}^u_{G_2} \cap \mathbb{M}^u_{G_3} \subseteq \widetilde{\mathbb{M}}^u_\varpi$, *where* $\widetilde{\mathbb{M}}^u_\varpi$ *is the set of all sequences* $\{t_n\}$ *satisfying* $\{\varpi(t+t_n)\}$ *uniform convergence in* p-*distribution for* $t \in \mathbb{R}$.

Proof. (i) From semigroup $\{S(t)\}_{t\geq 0}$ satisfying exponential stability and coefficients $G_1, G_2, G_3 \in BC(\mathbb{R}, L^p_G(\Omega))$, it follows that

$$\begin{aligned}&w_1(t) := \int_{-\infty}^{t} S(t-\varrho)G_1(\varrho)d\varrho,\ w_2(t) := \int_{-\infty}^{t} S(t-\varrho)G_2(\varrho)dB_\varrho,\\ &w_3(t) := \int_{-\infty}^{t} S(t-\varrho)G_3(\varrho)d\langle B\rangle_\varrho\end{aligned} \quad (3.4)$$

exist for $t \in \mathbb{R}$. Moreover, it is easy to see that solution of the linear SDE (1.1) satisfies

$$\begin{aligned}\varpi(t) =&S(t-t_0)\varpi(t_0) + \int_{t_0}^{t} S(t-\varrho)G_1(\varrho)d\varrho + \int_{t_0}^{t} S(t-\varrho)G_2(\varrho)dB_\varrho\\ &+ \int_{t_0}^{t} S(t-\varrho)G_3(\varrho)d\langle B\rangle_\varrho\end{aligned} \quad (3.5)$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$. As $t_0 \to -\infty$, it follows from $S(t-t_0)$ satisfying exponential stability that $S(t-t_0) \to 0$. By using (3.4) and (3.5), the equation (3.1) holds as $t_0 \to -\infty$. Therefore, the mapping ϖ defined as (3.1) in $BC(\mathbb{R}, L^p_G(\Omega))$ is a mild solution of (1.1).

Suppose that ϖ and $\overline{\varpi}$ are two mild solutions to (1.1) satisfying $\varpi(t_0) = \overline{\varpi}(t_0)$ for some $t_0 \leq t$. Letting $\omega(t) = \varpi(t) - \overline{\varpi}(t)$, one obtains by (1.1)

$$d\omega(t) = d[\varpi(t) - \overline{\varpi}(t)] = A\omega(t)dt, \quad \omega(t_0) = 0,$$

which implies

$$\omega(t) = \omega(t_0) \exp\left\{\int_{t_0}^{t} A ds\right\} = \omega(t_0) S(t-t_0) = 0.$$

Thus $\varpi = \overline{\varpi}$.

Now, one shows that (3.2) holds. According to (3.4), the solution $\varpi(t)$ of (1.1) can rewrite as

$$\varpi(t) = w_1(t) + w_2(t) + w_3(t).$$

Using the exponential stability of semigroup $S(t)$ and Hölder inequality with exponents $(p, \frac{p}{p-1})$, one has

$$\begin{aligned} E|w_1(t)|^p =& E\left|\int_{-\infty}^{t} S(t-\varrho)G_1(\varrho)d\varrho\right|^p \leq E\left(\int_{-\infty}^{t} ke^{-\delta(t-\varrho)}G_1(\varrho)d\varrho\right)^p \\ \leq& k^p \iota_1 \int_{-\infty}^{t} e^{-\frac{p\delta(t-\varrho)}{2}} E|G_1(\varrho)|^p d\varrho \leq \frac{2k^p\iota_1}{\delta p} \left\|G_1\right\|_\infty^p . \end{aligned} \tag{3.6}$$

According to Proposition 2.1 and Hölder inequality with exponents $(\frac{p}{2}, \frac{p}{p-2})$, it yields that

$$\begin{aligned} E|w_2(t)|^p =& E\left|\int_{-\infty}^{t} S(t-\varrho)G_2(\varrho)dB_\varrho\right|^p \leq E\left(\int_{-\infty}^{t} ke^{-\delta(t-\varrho)}G_2(\varrho)dB_\varrho\right)^p \\ \leq& k^p C_p \overline{\sigma}^p \iota_0 \int_{-\infty}^{t} e^{-\frac{p\delta(t-\varrho)}{2}} E|G_2(\varrho)|^p d\varrho \leq \frac{2k^p C_p \overline{\sigma}^p \iota_0}{p\delta} \left\|G_2\right\|_\infty^p . \end{aligned} \tag{3.7}$$

By Proposition 2.1 and Hölder inequality with exponents $(p, \frac{p}{p-1})$, one obtains

$$\begin{aligned} E|w_3(t)|^p =& E\left|\int_{-\infty}^{t} S(t-\varrho)G_3(\varrho)d\langle B\rangle_\varrho\right|^p \leq E\left(\int_{-\infty}^{t} ke^{-\delta(t-\varrho)}G_3(\varrho)d\langle B\rangle_\varrho\right)^p \\ \leq& k^p \overline{\sigma}^{2p} \iota_1 \int_{-\infty}^{t} e^{-\frac{p\delta(t-\varrho)}{2}} E|G_3(\varrho)|^p d\varrho \leq \frac{2k^p \overline{\sigma}^{2p}\iota_1}{\delta p} \left\|G_3\right\|_\infty^p . \end{aligned} \tag{3.8}$$

From (3.6)-(3.8), it follows that

$$\begin{aligned} & E|\varpi(t)|^p \\ \leq& 3^{p-1}(E|w_1(t)|^p + E|w_2(t)|^p + E|w_3(t)|^p) \\ \leq& 3^{p-1}k^p \left(\int_{-\infty}^{t} e^{-\frac{p\delta(t-\varrho)}{2}} (\iota_1 E\left|G_1(\varrho)\right|^p + C_p\overline{\sigma}^p\iota_0 E\left|G_2(\varrho)\right|^p + \overline{\sigma}^{2p}\iota_1 E\left|G_3(\varrho)\right|^p) d\varrho\right) \\ \leq& 3^{p-1}\frac{2k^p}{\delta p}(\iota_1 \left\|G_1\right\|_\infty^p + C_p\overline{\sigma}^p\iota_0 \left\|G_2\right\|_\infty^p + \overline{\sigma}^{2p}\iota_1 \left\|G_3\right\|_\infty^p), \end{aligned} \tag{3.9}$$

which implies that the inequality (3.2) holds.

(ii) Since coefficient $G_1 \in BC(\mathbb{R}, L_G^p(\Omega))$, it calculates that

$$\begin{aligned} \int_{-\infty}^{t} e^{-\frac{\delta p(t-\varrho)}{2}} E|G_1(\varrho)|^p d\varrho =& \int_{-\infty}^{-\mathfrak{l}} e^{-\frac{\delta p(t-\varrho)}{2}} E|G_1(\varrho)|^p d\varrho + \int_{-\mathfrak{l}}^{t} e^{-\frac{\delta p(t-\varrho)}{2}} E|G_1(\varrho)|^p d\varrho \\ \leq& \frac{2}{\delta p}\left(e^{-\frac{\delta p(t+\mathfrak{l})}{2}} \left\|G_1\right\|_\infty^p + \max_{|\varrho|\leq \mathfrak{l}} E|G_1(\varrho)|^p\right), \end{aligned} \tag{3.10}$$

for any $\mathfrak{l} > 0$ and $|t| \leq \mathfrak{l}$. Similar to (3.10),

$$\int_{-\infty}^{t} e^{-\frac{\delta p(t-\varrho)}{2}} E|G_2(\varrho)|^p d\varrho \leq \frac{2}{\delta p} e^{-\frac{\delta p(t+\mathfrak{l})}{2}} \|G_2\|_\infty^p + \frac{2}{\delta p} \max_{|\varrho|\leq \mathfrak{l}} E|G_2(\varrho)|^p, \tag{3.11}$$

and

$$\int_{-\infty}^{t} e^{-\frac{\delta p(t-\varrho)}{2}} E|G_3(\varrho)|^p d\varrho \leq \frac{2}{\delta p} e^{-\frac{\delta p(t+\mathfrak{l})}{2}} \|G_3\|_\infty^p + \frac{2}{\delta p} \max_{|\varrho|\leq \mathfrak{l}} E|G_3(\varrho)|^p. \tag{3.12}$$

By using inequalities (3.9)-(3.12), one has

$$\begin{aligned}
&\max_{|t|\leq L_0} E|\varpi(t)|^p \\
\leq& 3^{p-1} k^p \iota_1 \max_{|t|\leq L_0} \int_{-\infty}^{t} e^{-\frac{p\delta(t-\varrho)}{2}} E\left|G_1(\varrho)\right|^p d\varrho \\
&+ 3^{p-1} k^p C_p \overline{\sigma}^p \iota_0 \max_{|t|\leq L_0} \int_{-\infty}^{t} e^{-\frac{p\delta(t-\varrho)}{2}} E\left|G_2(\varrho)\right|^p d\varrho \\
&+ 3^{p-1} k^p \iota_1 \overline{\sigma}^{2p} \max_{|t|\leq L_0} \int_{-\infty}^{t} e^{-\frac{p\delta(t-\varrho)}{2}} E\left|G_3(\varrho)\right|^p d\varrho \\
\leq& 3^{p-1} k^p \left(\frac{2}{\delta p}\right) \left[\max_{|\varrho|\leq \mathfrak{l}} \left(\iota_1 E|G_1(\varrho)|^p + \overline{\sigma}^{2p} \iota_1 E|G_3(\varrho)|^p + C_p \overline{\sigma}^p \iota_0 E|G_2(\varrho)|^p\right)\right. \\
&\left. + e^{-\frac{\delta p(-L_0+\mathfrak{l})}{2}} \left(\iota_1 \|G_1\|_\infty^p + C_p \overline{\sigma}^p \iota_0 \|G_2\|_\infty^p + \overline{\sigma}^{2p} \iota_1 \|G_3\|_\infty^p\right)\right]
\end{aligned}$$

for any $\mathfrak{l} > L_0 > 0$. Hence the inequality (3.3) holds.

(*iii*) Since $\mathbb{M}_{G_1} \cap \mathbb{M}_{G_2} \cap \mathbb{M}_{G_3} \neq \emptyset$, by Definition 2.13 and the norm of space $C(\mathbb{R}, L_G^p(\Omega))$, there is $\widetilde{G}_j \in H(G_j)$ such that for $t_n \in \mathbb{M}_{G_1} \cap \mathbb{M}_{G_2} \cap \mathbb{M}_{G_3}$ and any $L_0 > 0$

$$\sup_{|t|\leq L_0} E|G_j(t+t_n) - \widetilde{G}_j(t)|^p \to 0, \quad j = 1, 2, 3 \tag{3.13}$$

as $n \to \infty$ in $C(\mathbb{R}, L_G^p(\Omega))$.

Let ϖ_n and $\widetilde{\varpi}$ be solutions to equations

$$d\varpi(t) = (A\varpi(t) + G_1(t+t_n))dt + G_2(t+t_n)dB_t + G_3(t+t_n)d\langle B\rangle_t$$

and

$$d\varpi(t) = (A\varpi(t) + \widetilde{G}_1(t))dt + \widetilde{G}_2(t)dB_t + \widetilde{G}_3(t)d\langle B\rangle_t$$

respectively. It follows from Remark 2.3 that

$$\sup_{t\in\mathbb{R}} (E|G_j(t+t_n) - \widetilde{G}_j(t)|^p)^{\frac{1}{p}} \leq 2\left\|G_j\right\|_\infty, \ j = 1, 2, 3. \tag{3.14}$$

It is easy to verify that $\varpi_n(t) - \widetilde{\varpi}(t)$ is the mild solution of the equation

$$\begin{aligned}
d\varpi(t) =& (A\varpi(t) + G_1(t+t_n) - \widetilde{G}_1(t))dt + (G_2(t+t_n) - \widetilde{G}_2(t))dB_t \\
&+ (G_3(t+t_n) - \widetilde{G}_3(t))d\langle B\rangle_t.
\end{aligned}$$

Note that one can find a common sufficiently large integer N such that $\mathfrak{l}_n \to \infty$ and $\sup\limits_{|\varrho|\le\iota_n} E|G_j(\varrho+t_n)-\widetilde{G}_j(\varrho)|^p \to 0, j=1,2,3$ simultaneously as $n>N$. Then by replacing $\mathfrak{l}$ with $\mathfrak{l}_n$ in the inequality (3.3), one obtains from (3.13) and (3.14) that

$$
\begin{aligned}
&\max_{|t|\le L_0} E|\varpi_n(t)-\widetilde{\widetilde{\varpi}}(t)|^p\\
\le&\frac{2\cdot 3^{p-1}k^p}{\delta p}\Bigg[\sup_{|\varrho|\le\mathfrak{l}_n}\bigg(\iota_1 E\left|G_1(\varrho+t_n)-\widetilde{G}_1(\varrho)\right|^p+\overline{\sigma}^{2p}\iota_1 E\left|G_3(\varrho+t_n)-\widetilde{G}_3(\varrho)\right|^p\\
&+C_p\overline{\sigma}^p\iota_0 E\left|G_2(\varrho+t_n)-\widetilde{G}_2(\varrho)\right|^p\bigg)+e^{-\frac{\delta p(-L_0+\mathfrak{l}_n)}{2}}\Bigg(\iota_1\left(\sup_{\varrho\in\mathbb{R}}(E|G_1(\varrho+t_n)-\widetilde{G}_1(\varrho)|^p)^{\frac{1}{p}}\right)^p\\
&+C_p\overline{\sigma}^p\iota_0\left(\sup_{\varrho\in\mathbb{R}}(E|G_2(\varrho+t_n)-\widetilde{G}_2(\varrho)|^p)^{\frac{1}{p}}\right)^p\\
&+\overline{\sigma}^{2p}\iota_1\left(\sup_{\varrho\in\mathbb{R}}(E|G_3(\varrho+t_n)-\widetilde{G}_3(\varrho)|^p)^{\frac{1}{p}}\right)^p\Bigg)\Bigg]\\
\le&\frac{2\cdot 3^{p-1}k^p}{\delta p}\Bigg[\sup_{|\varrho|\le\mathfrak{l}_n}\bigg(\iota_1 E\left|G_1(\varrho+t_n)-\widetilde{G}_1(\varrho)\right|^p+\overline{\sigma}^{2p}\iota_1 E\left|G_3(\varrho+t_n)-\widetilde{G}_3(\varrho)\right|^p\\
&+C_p\overline{\sigma}^p\iota_0 E\left|G_2(\varrho+t_n)-\widetilde{G}_2(\varrho)\right|^p\bigg)+e^{-\frac{\delta p(-L_0+\mathfrak{l}_n)}{2}}(\iota_1 2^p\left\|G_1\right\|_\infty^p\\
&+C_p\overline{\sigma}^p\iota_0 2^p\left\|G_2\right\|_\infty^p+\overline{\sigma}^{2p}\iota_1 2^p\left\|G_3\right\|_\infty^p)\Bigg]
\end{aligned}
$$

for any $L_0>0$. Then for any $L_0>0$, $\lim\limits_{n\to\infty}\max\limits_{|t|\le L_0} E|\varpi_n(t)-\widetilde{\widetilde{\varpi}}(t)|^p=0$, which indicates that $\{t_n\}\in\widetilde{\mathbb{M}}_\varpi$ and $\varpi_n(t)\to\widetilde{\widetilde{\varpi}}(t)$ in distribution uniformly for $t\in[-L_0,L_0]$ for all $L_0>0$.

By the transformation $s=\varrho-t_n$, the solution $\varpi(t)$ of (3.1) becomes

$$
\begin{aligned}
\varpi(t+t_n)=&\int_{-\infty}^t S(t-s)G_1(s+t_n)ds+\int_{-\infty}^t S(t-s)G_2(s+t_n)d\widetilde{B}_s\\
&+\int_{-\infty}^t S(t-s)G_3(s+t_n)d\langle\widetilde{B}\rangle_s,
\end{aligned}
$$

where $\widetilde{B}_s=B_{s+\varrho_n}-B_{\varrho_n}$ is a shifted G-Brownian motion and $\langle\widetilde{B}\rangle_s=\langle B\rangle_{s+\varrho_n}-\langle B\rangle_{\varrho_n}$ is a shifted second variance. Then $\varpi(t+t_n)$ and $\varpi_n(t)$ have the same distribution on $L_G^p(\Omega)$, which derives that $\varpi(t+t_n)\to\widetilde{\widetilde{\varpi}}(t)$ in distribution uniformly for $t\in[-L_0,L_0]$ for all $L_0>0$. On the other hand, it follows from conclusion (i) that the sequence $\{|\varpi_n(t)|^p : n\in\mathbb{N}, t\in\mathbb{R}\}$ is uniformly integrable, which implies that $\{|\varpi(t+t_n)|^p : n\in\mathbb{N}, t\in\mathbb{R}\}$ is also uniformly integrable. Then the result $\varpi(t+t_n)\to\widetilde{\widetilde{\varpi}}(t)$ holds in p-distribution uniformly for $t\in[-L_0\ L_0]$, $L_0>0$. Therefore, by Remark 2.4, one can conclude that ϖ is strongly compatible in p-distribution.

(iv) Since $\mathbb{M}^u_{G_1}\cap\mathbb{M}^u_{G_2}\cap\mathbb{M}^u_{G_3}$ is non-empty, letting $t_n\in\mathbb{M}^u_{G_1}\cap\mathbb{M}^u_{G_2}\cap\mathbb{M}^u_{G_3}$, then by Definition 2.13 there is a $\widetilde{G}_j\in H(G_j)$ such that $G_j(t+t_n)\to\widetilde{G}_j(t)$ uniformly in $t\in\mathbb{R}$ as $n\to\infty$, that is,

$$\sup_{t\in\mathbb{R}} E|G_j(t+t_n)-\widetilde{G}_j(t)|^p\to 0 \ as\ n\to\infty,\ j=1,2,3 \tag{3.15}$$

in the space $C(\mathbb{R}, L_G^p(\Omega))$. According to inequality (3.9), one has

$$\begin{aligned}&\sup_{t\in\mathbb{R}} E|\varpi_n(t)-\widetilde{\widetilde{\varpi}}(t)|^p\\ \le& 3^{p-1}k^p\frac{2}{p\delta}\left(\iota_1\sup_{\varrho\in\mathbb{R}}E|G_1(\varrho+t_n)-\widetilde{G}_1(\varrho)|^p+C_p\overline{\sigma}^p\iota_0\sup_{\varrho\in\mathbb{R}}E|G_2(\varrho+t_n)-\widetilde{G}_2(\varrho)|^p\right.\\ &\left.+\overline{\sigma}^{2p}\iota_1\sup_{\varrho\in\mathbb{R}}E|G_3(\varrho+t_n)-\widetilde{G}_3(\varrho)|^p\right).\end{aligned}$$

Moreover, it follows from (3.15) that $\varpi_n \to \widetilde{\widetilde{\varpi}}$ uniformly on $\mathbb{R}$ in L^p-norm as $n\to\infty$. Hence, $\{t_n\}\in\widetilde{\mathbb{M}}^u_{\varpi}$. In conclusion, by the same to the proof of (iii), one can derive $\{t_n\}\in\widetilde{\mathbb{M}}^u_{\varpi}$ in p-distribution sense with respect to $t\in\mathbb{R}$. This completes the proof. □

Now, one investigates Poisson stable solutions for the semi-linear SDE (1.2).

Theorem 3.2. *Suppose that there are two positive constants l, A_0 such that coefficients G_1, G_2 and G_3 satisfy*

$$|G_1(t,0)|\vee|G_2(t,0)|\vee|G_3(t,0)|\le A_0, \tag{3.16}$$

$$|G_1(t,\varpi_1)-G_1(t,\varpi_2)|\vee|G_2(t,\varpi_1)-G_2(t,\varpi_2)|\vee|G_3(t,\varpi_1)-G_3(t,\varpi_2)|\le l\,|\varpi_1-\varpi_2| \tag{3.17}$$

for $G_1,G_2,G_3\in C(\mathbb{R}\times L_G^p(\Omega), L_G^p(\Omega))$, $\varpi_1,\varpi_2\in L_G^p(\Omega)$ and $t\in\mathbb{R}$. Then for the semi-linear SDE (1.2)*, the following statements hold:*

(i) if $l<l_0=\frac{(\delta p)^{\frac{1}{p}}}{2^{\frac{1}{p}}3^{1-\frac{1}{p}}k(\iota_1+C_p\overline{\sigma}^p\iota_0+\overline{\sigma}^{2p}\iota_1)^{\frac{1}{p}}}$, then equation (1.2) *possesses a unique mild solution $\varpi\in C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r))$, where*

$$B_{L_G^p(\Omega)}(0,r)=\{\varpi\in L_G^p(\Omega):\|\varpi\|_p\le r\}$$

with

$$r=\frac{2^{\frac{1}{p}}3^{1-\frac{1}{p}}k(\iota_1+C_p\overline{\sigma}^p\iota_0+\overline{\sigma}^{2p}\iota_1)^{\frac{1}{p}}A_0}{(p\delta)^{\frac{1}{p}}-2^{\frac{1}{p}}3^{1-\frac{1}{p}}k(\iota_1+C_p\overline{\sigma}^p\iota_0+\overline{\sigma}^{2p}\iota_1)^{\frac{1}{p}}l}.$$

(ii) if $l<2^{\frac{1}{p}-1}l_0$ and the coefficient G_j, $j=1,2,3$ is continuous in t uniformly for ϖ on every bounded collection $Q\subset L_G^p(\Omega)$, then

(b_1) $\mathbb{M}^u_{G_1}\cap\mathbb{M}^u_{G_2}\cap\mathbb{M}^u_{G_3}\subseteq\widetilde{\mathbb{M}}^u_{\varpi}$ provided that $\mathbb{M}^u_{G_1}\cap\mathbb{M}^u_{G_2}\cap\mathbb{M}^u_{G_3}\neq\emptyset$;

(b_2) the solution ϖ is strongly compatible in p-distribution provided that $\mathbb{M}_{G_1}\cap\mathbb{M}_{G_2}\cap\mathbb{M}_{G_3}\neq\emptyset$.

Proof. (i) It is easy to check that $(C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r)),d)$ is a complete metric space. Since $S(t-\varrho)$ satisfies exponential stability and coefficient $G_j(t,\varpi(t))$, $j=1,2,3$ fulfills the conditions (3.16) and (3.17), it follows that

$$\int_{-\infty}^t S(t-\varrho)G_1(\varrho,\varpi(\varrho))d\varrho,\quad \int_{-\infty}^t S(t-\varrho)G_2(\varrho,\varpi(\varrho))dB_\varrho,\quad \int_{-\infty}^t S(t-\varrho)G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho$$

exist for $t\in\mathbb{R}$. Moreover, one can verify that the solution of the semi-linear SDE (1.2) satisfies

$$\varpi(t)=S(t-t_0)\varpi(t_0)+\int_{t_0}^t S(t-\varrho)G_1(\varrho,\varpi(\varrho))d\varrho+\int_{t_0}^t S(t-\varrho)G_2(\varrho,\varpi(\varrho))dB_\varrho$$

$$+\int_{t_0}^{t} S(t-\varrho)G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho$$

for all $t \geq t_0$ with each $t_0 \in \mathbb{R}$. As $t_0 \to -\infty$, according to $S(t-t_0)$ satisfying exponential stability that $S(t-t_0) \to 0$. Then one can define an operator $\widehat{\Phi}$: $C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r)) \to C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r))$ by

$$\begin{aligned}\widehat{\Phi}(\varpi)(t) :=& \int_{-\infty}^{t} S(t-\varrho)G_1(\varrho,\varpi(\varrho))d\varrho + \int_{-\infty}^{t} S(t-\varrho)G_2(\varrho,\varpi(\varrho))dB_\varrho \\ &+ \int_{-\infty}^{t} S(t-\varrho)G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho.\end{aligned}$$

For any $\varpi \in C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r))$, because of the conditions (3.16) and (3.17), one has

$$\|G_j(t,\varpi(t))\|_p \leq \|G_j(t,0)\|_p + l\,\|\varpi(t)\|_p \leq A_0 + lr, \quad t \in \mathbb{R}, \tag{3.18}$$

where A_0 and r are independent of ϖ. Moreover, it follows from (3.2) and (3.18) that

$$\left\|\widehat{\Phi}(\varpi)\right\|_\infty \leq 3^{1-\frac{1}{p}} k \left(\frac{2}{\delta p}\right)^{\frac{1}{p}} \left(\iota_1 + C_p\overline{\sigma}^p\iota_0 + \overline{\sigma}^{2p}\iota_1\right)^{\frac{1}{p}} (A_0 + lr) \leq r.$$

Therefore, the self-mapping operator $\widehat{\Phi}(\varpi)$ is well-defined on $C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r))$.

Next one shows that $\widehat{\Phi}(\varpi)$ is a contraction. Since each fixed point of $\widehat{\Phi}$ gives a mild solution of (1.2), the function $\widehat{\Phi}(\varpi_1) - \widehat{\Phi}(\varpi_2)$ is the mild solution on $BC(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega))$ of the equation

$$\begin{aligned}d\varpi(t) =& (A\varpi(t) + G_1(t,\varpi_1(t)) - G_1(t,\varpi_2(t)))dt + (G_2(t,\varpi_1(t)) - G_2(t,\varpi_2(t)))dB_t \\ &+ (G_3(t,\varpi_1(t)) - G_3(t,\varpi_2(t)))d\langle B\rangle_t.\end{aligned}$$

By condition (3.17), it derives from (3.9) that

$$\begin{aligned}\left\|\widehat{\Phi}(\varpi_1) - \widehat{\Phi}(\varpi_2)\right\|_\infty^p \leq& \frac{2\cdot 3^{p-1}k^p}{\delta p}\Big(\iota_1 \sup_{t\in\mathbb{R}} E\left|G_1(t,\varpi_1(t)) - G_1(t,\varpi_2(t))\right|^p \\ &+ C_p\overline{\sigma}^p\iota_0 \sup_{t\in\mathbb{R}} E|G_2(t,\varpi_1(t)) - G_2(t,\varpi_2(t))|^p \\ &+ \overline{\sigma}^{2p}\iota_1 \sup_{t\in\mathbb{R}} E\left|G_3(t,\varpi_1(t)) - G_3(t,\varpi_2(t))\right|^p\Big) \\ \leq& \frac{2\cdot 3^{p-1}k^p}{\delta p}\left(\iota_1 + C_p\overline{\sigma}^p\iota_0 + \overline{\sigma}^{2p}\iota_1\right) l^p \left\|\varpi_1 - \varpi_2\right\|_\infty^p,\end{aligned}$$

which indicates that $\widehat{\Phi}$ is a contraction due to $l < l_0$. Then by the Banach fixed point theorem, there is a unique fixed point $\varpi \in C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r))$ satisfying $\widehat{\Phi}(\varpi) = \varpi$, which means that (1.2) has a unique solution ϖ in $C(\mathbb{R}, B_{L_G^p(\Omega)}(0,r))$.

Now, one proves the conclusions in (ii).

(b_1) Since $\mathbb{M}_{G_1}^u \cap \mathbb{M}_{G_2}^u \cap \mathbb{M}_{G_3}^u$ is non-empty, setting $t_n \in \mathbb{M}_{G_1}^u \cap \mathbb{M}_{G_2}^u \cap \mathbb{M}_{G_3}^u$, then combining with Definition 2.13, it follows that there exist $\widetilde{G}_j \in H(G_j)$, $j = 1,2,3$ such that for any $r > 0$,

$$\sup_{\varrho\in\mathbb{R}, \|\varpi\|_p \leq r} A_{j,\varrho,\varpi} \to 0 \ \text{ as } n \to \infty, \quad j = 1,2,3, \tag{3.19}$$

where $A_{j,\varrho,\varpi} = |G_j(\varrho + t_n, \varpi(\varrho)) - \widetilde{G}_j(\varrho, \varpi(\varrho))|^p$ with

$$\begin{aligned} A_{j,\varrho,\varpi} &\leq 2^{p-1}|G_j(\varrho + t_n, \varpi(\varrho))|^p + 2^{p-1}|\widetilde{G}_j(\varrho, \varpi(\varrho))|^p \\ &\leq 2^{2p-1}l^p|\varpi(\varrho)|^p + 2^{2p-1}A_0^p, \ j = 1, 2, 3. \end{aligned} \tag{3.20}$$

It is easy to check that $G_j(t+t_n, \varpi(t)), j = 1, 2, 3, n \in \mathbb{N}$ has Lipschitz continuity and boundedness, then $\widetilde{G}_j(t, \varpi(t))$, $j = 1, 2, 3$ also has the these two properties with same constants l, A_0. Therefore, the following two equations

$$d\varpi(t) = (A\varpi(t) + G_1(t+t_n, \varpi(t)))dt + G_2(t+t_n, \varpi(t))dB_t + G_3(t+t_n, \varpi(t))d\langle B\rangle_t, \tag{3.21}$$

$$d\varpi(t) = (A\varpi(t) + \widetilde{G}_1(t, \varpi(t)))dt + \widetilde{G}_2(t, \varpi(t))dB_t + \widetilde{G}_3(t, \varpi(t))d\langle B\rangle_t \tag{3.22}$$

possess unique solutions $\varpi_n, \widetilde{\varpi}$ in $C(\mathbb{R}, B_{L_G^p(\Omega)}(0, r))$, respectively, which imply that $\zeta_n = \varpi_n - \widetilde{\varpi}$ is the unique solution to equation

$$\begin{aligned} d\zeta_n(t) =&(A\zeta_n(t) + (G_1(t + t_n, \varpi_n(t)) - \widetilde{G}_1(t, \widetilde{\varpi}(t))))dt \\ &+ (G_2(t + t_n, \varpi_n(t)) - \widetilde{G}_2(t, \widetilde{\varpi}(t)))dB_t \\ &+ (G_3(t + t_n, \varpi_n(t)) - \widetilde{G}_3(t, \widetilde{\varpi}(t)))d\langle B\rangle_t \end{aligned} \tag{3.23}$$

in $C(\mathbb{R}, B_{L_G^p(\Omega)}(0, 2r))$ with $G_j(t+t_n, \varpi_n(t)) - \widetilde{G}_j(t, \widetilde{\varpi}(t)) \in BC(\mathbb{R}\times L_G^p(\Omega), L_G^p(\Omega))$, $j = 1, 2, 3$.

One now shows that $\{\varpi_n(t)\}$ converges to $\widetilde{\varpi}(t)$ uniformly for $t \in \mathbb{R}$ with L^p-norm. From (3.9) and (3.23), it follows that

$$\begin{aligned} \sup_{t\in\mathbb{R}} E|\zeta_n(t)|^p \leq& 3^{p-1}\frac{2k^p}{\delta p} \sup_{\varrho\in\mathbb{R}, \max\{\|\varpi_n\|_p, \|\widetilde{\varpi}\|_p\}\leq r} \Big(\iota_1 E|G_1(\varrho + t_n, \varpi_n(\varrho)) \\ &- \widetilde{G}_1(\varrho, \widetilde{\varpi}(\varrho))|^p + C_p\overline{\sigma}^p\iota_0 E|G_2(\varrho + t_n, \varpi_n(\varrho)) - \widetilde{G}_2(\varrho, \widetilde{\varpi}(\varrho))|^p \\ &+ \overline{\sigma}^{2p}\iota_1 E|G_3(\varrho + t_n, \varpi_n(\varrho)) - \widetilde{G}_3(\varrho, \widetilde{\varpi}(\varrho))|^p\Big). \end{aligned} \tag{3.24}$$

By the Lipschitz continuity of $G_j(\varrho, \varpi)$, one has

$$\begin{aligned} &E|G_j(\varrho + t_n, \varpi_n(\varrho)) - \widetilde{G}_j(\varrho, \widetilde{\varpi}(\varrho))|^p \\ \leq& 2^{p-1}E\,|G_j(\varrho + t_n, \varpi_n(\varrho)) - G_j(\varrho + t_n, \widetilde{\varpi}(\varrho))|^p \\ &+ 2^{p-1}E\left|G_j(\varrho + t_n, \widetilde{\varpi}(\varrho)) - \widetilde{G}(\varrho, \widetilde{\varpi}(\varrho))\right|^p \\ \leq& 2^{p-1}\left(l^pE|\zeta_n|^p + \sup_{\varrho\in\mathbb{R}, \|\widetilde{\varpi}\|_p\leq r} EA_{j,\varrho,\widetilde{\varpi}}\right), \ j = 1, 2, 3. \end{aligned} \tag{3.25}$$

Since $\sup_{\varrho\in\mathbb{R}} E|\zeta_n(\varrho)|^p = \sup_{t\in\mathbb{R}} E|\zeta_n(t)|^p$ as $\varrho \in (-\infty, t]$, then substituting (3.25) into (3.24), one has

$$\begin{aligned} \sup_{\varrho\in\mathbb{R}} E|\zeta_n(\varrho)|^p \leq& \frac{2\cdot 6^{p-1}k^p}{\delta p}\left(\iota_1\left(l^p\sup_{\varrho\in\mathbb{R}} E|\zeta_n(\varrho)|^p + \sup_{\varrho\in\mathbb{R}, \|\widetilde{\varpi}\|_p\leq r} EA_{1,\varrho,\widetilde{\varpi}}\right)\right. \\ &+ C_p\overline{\sigma}^p\iota_0\left(l^p\sup_{\varrho\in\mathbb{R}} E|\zeta_n(\varrho)|^p + \sup_{\varrho\in\mathbb{R}, \|\widetilde{\varpi}\|_p\leq r} EA_{2,\varrho,\widetilde{\varpi}}\right) \end{aligned}$$

$$+\overline{\sigma}^{2p}\iota_1\left(l^p\sup_{\varrho\in\mathbb{R}}E|\zeta_n(\varrho)|^p+\sup_{\varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\le r}EA_{3,\varrho,\widetilde{\varpi}}\right)\Bigg),$$

which together with $l<2^{\frac{1}{p}-1}l_0$ deduces that

$$\sup_{\varrho\in\mathbb{R}}|\zeta_n(\varrho)|^p \le\frac{2k^p\left(\iota_1\sup\limits_{\varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\le r}EA_{1,\varrho,\widetilde{\varpi}}+C_p\overline{\sigma}^p\iota_0\sup\limits_{\varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\le r}EA_{2,\varrho,\widetilde{\varpi}}+\overline{\sigma}^{2p}\iota_1\sup\limits_{\varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\le r}EA_{3,\varrho,\widetilde{\varpi}}\right)}{\delta p6^{1-p}-2k^pl^p(\iota_1+C_p\overline{\sigma}^p\iota_0+\overline{\sigma}^{2p}\iota_1)}. \tag{3.26}$$

From conclusion (i), one knows that $\widetilde{\varpi}(t)$ is L^p-bounded solution of equation (3.22), and then the families $\{|\widetilde{\varpi}(t)|^p : t\in\mathbb{R}\}$ and $\{A_{j,\varrho,\widetilde{\varpi}} : \varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\le r\}, j=1,2,3$ are uniformly integrable by (3.20). Therefore, by taking the limit in (3.26) and (3.19), one obtains $\varpi_n(t)\to\widetilde{\varpi}(t)$ uniformly in $t\in\mathbb{R}$ with L^p norm.

Since L^p convergence means convergence in distribution, one thus has $\varpi_n(t)\to\widetilde{\varpi}(t)$ in distribution with respect to $\mathbb{R}$ uniformly. By transformation $s=\varrho-t_n$, the solution $\varpi(t)$ of equation (1.2) becomes

$$\begin{aligned}\varpi(t+t_n)=&\int_{-\infty}^t S(t-s)G_1(s+t_n,\varpi(s+t_n))ds+\int_{-\infty}^t S(t-s)G_2(s+t_n,\varpi(s+t_n))d\widetilde{B}_s\\&+\int_{-\infty}^t S(t-s)G_3(s+t_n,\varpi(s+t_n))d\langle\widetilde{B}\rangle_s,\end{aligned}$$

where $\widetilde{B}_s=B_{s+\varrho_n}-B_{\varrho_n}$ is also a G-Brownian motion with the same distribution as B_s and $\langle\widetilde{B}\rangle_s=\langle B\rangle_{s+\varrho_n}-\langle B\rangle_{\varrho_n}$ is also a second variation with the same distribution as $\langle B\rangle_s$, which imply that $\varpi_n(t)$ and $\varpi(t+t_n)$ share the same distributions on $L_G^p(\Omega)$. Note that $\{|\varpi(t+t_n)|^p : n\in\mathbb{N},t\in\mathbb{R}\}$ is uniformly integrable. This manifests $\varpi(t+t_n)\to\widetilde{\varpi}(t)$ in p-distribution uniformly with respect to $t\in\mathbb{R}$. Then according to Definition 2.21, ϖ is strongly compatible in p-distribution with respect to $t\in\mathbb{R}$.

(b_2) Because $\mathbb{M}_{G_1}\cap\mathbb{M}_{G_2}\cap\mathbb{M}_{G_3}$ is non-empty, letting $t_n\in\mathbb{M}_{G_1}\cap\mathbb{M}_{G_2}\cap\mathbb{M}_{G_3}$, then it follows from Definition 2.13 that there is a $\widetilde{G}_j\in H(G_j)$ such that for any $r,\mathfrak{l}>0$,

$$\sup_{|\varrho|\le\mathfrak{l},\|\varpi\|_p\le r}A_{j,\varrho,\varpi}\to 0 \text{ as } n\to\infty,\ j=1,2,3, \tag{3.27}$$

where $A_{j,\varrho,\varpi}=|G_j(\varrho+t_n,\varpi)-\widetilde{G}_j(\varrho,\varpi)|^p, j=1,2,3$.

Let ϖ_n and $\widetilde{\varpi}$ be the unique bounded solutions corresponding to equation (3.21) and equation (3.22), respectively. Then $\zeta_n=\varpi_n-\widetilde{\varpi}, n\in\mathbb{N}$ is the unique bounded solution corresponding to equation (3.23). From (3.9), one gains

$$\begin{aligned}E|\zeta_n(t)|^p\le&3^{p-1}k^p\Bigg[\int_{-\infty}^t e^{-\frac{\delta p(t-\varrho)}{2}}\Big(\iota_1E|G_1(\varrho+t_n,\varpi_n(\varrho))-\widetilde{G}_1(\varrho,\widetilde{\varpi}(\varrho))|^p\\&+C_p\overline{\sigma}^p\iota_0E|G_2(\varrho+t_n,\varpi_n(\varrho))-\widetilde{G}_2(\varrho,\widetilde{\varpi}(\varrho))|^p\\&+\overline{\sigma}^{2p}\iota_1E|G_3(\varrho+t_n,\varpi_n(\varrho))-\widetilde{G}_3(\varrho,\widetilde{\varpi}(\varrho))|^p\Big)d\varrho\Bigg],\end{aligned} \tag{3.28}$$

and similar to (3.25), one has

$$E|G_j(\varrho+t_n,\varpi_n(\varrho))-\widetilde{G}_j(\varrho,\widetilde{\varpi}(\varrho))|^p \leq 2^{p-1}\left(l^p E\left|\zeta_n\right|^p+EA_{j,\varrho,\widetilde{\varpi}}\right), j=1,2,3. \tag{3.29}$$

Proceeding like (3.18), for any $\varrho\in\mathbb{R}$, it yields

$$E|G_j(\varrho+t_n,\varpi_n(\varrho))-\widetilde{G}_j(\varrho,\widetilde{\varpi}(\varrho))|^p \leq 2^p(A_0+lr)^p, j=1,2,3,$$

which implies that

$$\sup_{\varrho\in\mathbb{R},\max\{\|\varpi_n\|_p,\|\widetilde{\varpi}\|_p\}\leq r} E|G_j(\varrho+t_n,\varpi_n(\varrho))-\widetilde{G}_j(\varrho,\widetilde{\varpi}(\varrho))|^p \leq 2^p(A_0+lr)^p, j=1,2,3. \tag{3.30}$$

Therefore, combining inequalities (3.28) and (3.29), one has

$$\begin{aligned} E\left|\zeta_n(t)\right|^p \leq \int_{-\infty}^t & e^{-\frac{\delta p(t-\varrho)}{2}}\left(a_0 E\left|\zeta_n(\varrho)\right|^p+k^p 6^{p-1}(\iota_1 EA_{1,\varrho,\widetilde{\varpi}}\right.\\ &\left.+C_p\overline{\sigma}^p\iota_0 EA_{2,\varrho,\widetilde{\varpi}}+\overline{\sigma}^{2p}\iota_1 EA_{3,\varrho,\widetilde{\varpi}})\right)d\varrho, \end{aligned} \tag{3.31}$$

where $a_0=k^p6^{p-1}\iota_1 l^p+k^pC_p\overline{\sigma}^p6^{p-1}\iota_0 l^p+k^p\overline{\sigma}^{2p}6^{p-1}\iota_1 l^p$. By $l<2^{\frac{1}{p}-1}l_0$ and [16, Lemma 4.3], one further gains

$$\begin{aligned} &\max_{|t|\leq L_0} E|\zeta_n(t)|^p\\ \leq & B_{11}\left(\iota_1\sup_{\varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\leq r} EA_{1,\varrho,\widetilde{\varpi}}+C_p\overline{\sigma}^p\iota_0\sup_{\varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\leq r} EA_{2,\varrho,\widetilde{\varpi}}+\overline{\sigma}^{2p}\iota_1\sup_{\varrho\in\mathbb{R},\|\widetilde{\varpi}\|_p\leq r} EA_{3,\varrho,\widetilde{\varpi}}\right)\\ &+B_{12}\left(1-e^{-k_0(L_0+\mathfrak{l})}\right)\sup_{|\varrho|\leq\mathfrak{l},\|\widetilde{\varpi}\|_p\leq r}\left(\iota_1 EA_{1,\varrho,\widetilde{\varpi}}+C_p\overline{\sigma}^p\iota_0 EA_{2,\varrho,\widetilde{\varpi}}+\overline{\sigma}^{2p}\iota_1 EA_{3,\varrho,\widetilde{\varpi}}\right) \end{aligned} \tag{3.32}$$

for any $\mathfrak{l}>L_0>0$ and $t\in[-L_0,L_0]$, where

$$B_{11}=\frac{6^{p-1}k^p}{k_0}e^{-k_0(-L_0+\mathfrak{l})}, B_{12}=\frac{k^p\cdot 6^{p-1}}{k_0}, k_0=\frac{p\delta}{2}-a_0.$$

By the definition of the limit, one can find a common sufficiently large integer N such that $\mathfrak{l}_n\to\infty$ and $\sup_{|\varrho|\leq\mathfrak{l}_n,\|\widetilde{\varpi}\|_p\leq r} A_{j,\varrho,\widetilde{\varpi}}\to 0, j=1,2,3$ simultaneously as $n>N$. Then by (3.30) and replacing $\mathfrak{l}$ with $\mathfrak{l}_n$ in the inequality (3.32), one has

$$\begin{aligned} &\max_{|t|\leq L_0} E|\zeta_n(t)|^p\\ \leq & \frac{6^{p-1}}{k_0}(2k)^p e^{-k_0(-L_0+\mathfrak{l}_n)}(\iota_1+C_p\overline{\sigma}^p\iota_0+\overline{\sigma}^{2p}\iota_1)(A_0+lr)^p+\frac{k^p\cdot 6^{p-1}}{k_0}\left(1-e^{-k_0(L_0+\mathfrak{l}_n)}\right)\\ &\times\sup_{|\varrho|\leq\mathfrak{l}_n,\|\widetilde{\varpi}\|_p\leq r}\left(\iota_1 EA_{1,\varrho,\widetilde{\varpi}}+C_p\overline{\sigma}^p\iota_0 EA_{2,\varrho,\widetilde{\varpi}}+\overline{\sigma}^{2p}\iota_1 EA_{3,\varrho,\widetilde{\varpi}}\right). \end{aligned} \tag{3.33}$$

Due to [16, Remark 2.2-(iii)] and (3.27), it yields from (3.33) that

$$\lim_{n\to\infty}\max_{|t|\leq L_0} E|\zeta_n(t)|^p=0.$$

This indicates that $\varpi_n(t) \to \widetilde{\varpi}(t)$ in distribution uniformly for $t \in [-L_0, L_0]$ for all $L_0 > 0$. Since $\varpi_n(t)$ and $\varpi(t+t_n)$ have the same distribution, $\varpi(t+t_n) \to \widetilde{\varpi}(t)$ in distribution uniformly for $t \in [-L_0, L_0]$ for all $L_0 > 0$. On the other hand, the uniform integral of $\{|\varpi(t+t_n)|^p : n \in \mathbb{N}, t \in \mathbb{R}\}$ is similar to the proof of (b_1) in the conclusion (ii). Hence, one can conclude that ϖ is strongly compatible in p-distribution. This completes the proof. □

Theorem 3.3. *Provided that all conditions of Theorem 3.2 hold, then*

(i) if the functions G_1, G_2, G_3 are jointly Poisson stable (respectively, almost recurrent, Levitan almost periodic, Lagrange stable, Birkhoff recurrent, Bohr almost automorphic, Bohr almost periodic, quasi-periodic with a limited spectrum, τ-periodic, stationary) for $t \in \mathbb{R}$ uniformly for the second argument on each bounded set $Q \subset L_G^p(\Omega)$, then the unique bounded solution ϖ corresponding to (1.2) *heritages the same property as functions G_1, G_2, G_3 in p-distribution sense.*

(ii) if G_1, G_2, G_3 are jointly pseudo-recurrent (respectively, pseudo-periodic) and are jointly Lagrange stable for $t \in \mathbb{R}$ uniformly for the second argument on each bounded set Q, then the unique bounded solution ϖ of (1.2) *is pseudo-recurrent (respectively, pseudo-periodic) in p-distribution.*

Proof. It is a straightforward result of Theorem 3.2 and [16, Theorem 2.26, Remark 2.30]. □

Remark 3.1. The results for Theorem 3.3 also hold for equation (1.1) provided that all conditions of Theorem 3.1 hold.

3.2. Asymptotic stability of bounded solutions to (1.2)

In this subsection, one begins to establish the theorem of asymptotically stable solution in global for the semi-linear SDE (1.2) with coefficient $G_j \in C(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega))$ for $j = 1, 2, 3$.

Theorem 3.4. *Suppose that coefficients G_1, G_2 and G_3 satisfy the conditions* (3.16) *and* (3.17) *with $l < (\frac{3}{4})^{1-\frac{1}{p}} l_0$, where l_0 are given by Theorem* 3.2 (*i*). *Then the unique solution $\varpi(t; t_0, \varpi_0)$ of equation* (1.2) *is asymptotically stable in global, i.e.*

$$\lim_{t\to\infty} E|\varpi(t; t_0, \varpi_1) - \varpi(t; t_0, \varpi_0)|^p = 0,$$

where $\varpi(t, t_0, \varpi_1)$ is the solution of the semi-linear SDE (1.2) *with initial point $\varpi_1 \in L_G^p(\Omega)$ at t_0.*

Proof. From Theorem 3.2 (*i*), the semi-linear SDE (1.2) has a unique global solution $\varpi(t; t_0, \varpi_0)$ with initial point ϖ_0 at t_0 given by

$$\begin{aligned}\varpi(t; t_0, \varpi_0) =& S(t-t_0)\varpi_0 + \int_{t_0}^t S(t-\varrho) G_1(\varrho, \varpi(\varrho; t_0, \varpi_0)) d\varrho \\ &+ \int_{t_0}^t S(t-\varrho) G_2(\varrho, \varpi(\varrho; t_0, \varpi_0)) dB_\varrho \\ &+ \int_{t_0}^t S(t-\varrho) G_3(\varrho, \varpi(\varrho; t_0, \varpi_0)) d\langle B\rangle_\varrho, \quad t \geq t_0.\end{aligned}$$

Then, one has

$$
\begin{aligned}
&\varpi(t;t_0,\varpi_1)-\varpi(t;t_0,\varpi_0)\\
=&S(t-t_0)(\varpi_1-\varpi_0)+\int_{t_0}^{t}S(t-\varrho)(G_1(\varrho,\varpi(\varrho;t_0,\varpi_1))-G_1(\varrho,\varpi(\varrho;t_0,\varpi_0)))d\varrho\\
&+\int_{t_0}^{t}S(t-\varrho)(G_2(\varrho,\varpi(\varrho;t_0,\varpi_1))-G_2(\varrho,\varpi(\varrho;t_0,\varpi_0)))dB_\varrho\\
&+\int_{t_0}^{t}S(t-\varrho)(G_3(\varrho,\varpi(\varrho;t_0,\varpi_1))-G_3(\varrho,\varpi(\varrho;t_0,\varpi_0)))d\langle B\rangle_\varrho.
\end{aligned}
$$

Further, by condition (3.17), it yields

$$
\begin{aligned}
&E|\varpi(t;t_0,\varpi_1)-\varpi(t;t_0,\varpi_0)|^p\\
\le&4^{p-1}E\left|S(t-t_0)(\varpi_1-\varpi_0)\right|^p\\
&+4^{p-1}E\left|\int_{t_0}^{t}S(t-\varrho)(G_1(\varrho,\varpi(\varrho,t_0,\varpi_1))-G_1(\varrho,\varpi(\varrho,t_0,\varpi_0)))d\varrho\right|^p\\
&+4^{p-1}E\left|\int_{t_0}^{t}S(t-\varrho)(G_2(\varrho,\varpi(\varrho,t_0,\varpi_1))-G_2(\varrho,\varpi(\varrho,t_0,\varpi_0)))dB_\varrho\right|^p\\
&+4^{p-1}E\left|\int_{t_0}^{t}S(t-\varrho)(G_3(\varrho,\varpi(\varrho,t_0,\varpi_1))-G_3(\varrho,\varpi(\varrho,t_0,\varpi_0)))d\langle B\rangle_\varrho\right|^p\\
\le&4^{p-1}k^pe^{-\delta p(t-t_0)}E\left|\varpi_1-\varpi_0\right|^p+4^{p-1}k^p\left(\int_{t_0}^{t}e^{-\frac{p\delta(t-\varrho)}{2(p-1)}}d\varrho\right)^{p-1}\\
&\times\int_{t_0}^{t}e^{-\frac{p\delta(t-\varrho)}{2}}E\left|G_1(\varrho,\varpi(\varrho,t_0,\varpi_1))-G_1(\varrho,\varpi(\varrho,t_0,\varpi_0))\right|^pd\varrho\\
&+4^{p-1}k^pC_p\overline{\sigma}^p\left(\int_{t_0}^{t}e^{-\frac{p\delta(t-\varrho)}{p-2}}d\varrho\right)^{\frac{p}{2}-1}\\
&\times\int_{t_0}^{t}e^{-\frac{p\delta(t-\varrho)}{2}}E\left|G_2(\rho,\varpi(\varrho,t_0,\varpi_1))-G_2(\varrho,\varpi(\rho,t_0,\varpi_0))\right|^pd\rho\\
&+4^{p-1}k^p\overline{\sigma}^{2p}\left(\int_{t_0}^{t}e^{-\frac{p\delta(t-\varrho)}{2(p-1)}}d\varrho\right)^{p-1}\\
&\times\int_{t_0}^{t}e^{-\frac{p\delta(t-\rho)}{2}}E\left|G_3(\varrho,\varpi(\varrho,t_0,\varpi_1))-G_3(\varrho,\varpi(\varrho,t_0,\varpi_0))\right|^pd\varrho\\
\le&4^{p-1}k^pe^{-\frac{p\delta(t-t_0)}{2}}E|\varpi_1-\varpi_0|^p+k_0\int_{t_0}^{t}e^{-\frac{p\delta(t-\varrho)}{2}}E|\varpi(t;t_0,\varpi_1)-\varpi(t;t_0,\varpi_0)|^pd\varrho,
\end{aligned}
$$

where $k_0=4^{p-1}k^pl^p(\iota_1+C_p\overline{\sigma}^p\iota_0+\overline{\sigma}^{2p}\iota_1)$. Then the Gronwall inequality gives that

$$
E|\varpi(t;t_0,\varpi_1)-\varpi(t;t_0,\varpi_0)|^p\le 4^{p-1}k^pe^{-(\frac{p\delta}{2}-k_0)(t-t_0)}E|\varpi_1-\varpi_0|^p, t\ge t_0,
$$

which implies that

$$
\lim_{t\to\infty}E|\varpi(t;t_0,\varpi_1)-\varpi(t;t_0,\varpi_0)|^p=0. \tag{3.34}
$$

Hence $\varpi(t;t_0,\varpi_0)$ is asymptotically stable. This completes the proof. $\square$

Corollary 3.1. *Suppose that the conditions of Theorem* 3.4 *hold, then equation* (1.2) *admits a unique L^p-bounded solution $\varpi(t;t_0,\varpi_0)\in C(\mathbb{R},B_{L_G^p(\Omega)}(0,r))$. Moreover, for any $\varpi_0\in L_G^p(\Omega)$,*

$$\limsup_{t\to\infty} E|\varpi(t;t_0,\varpi_0)|^p < r^p+1,$$

where r is given by Theorem 3.2 (*i*).

4. (Pseudo) almost periodic solutions of (1.2) with exponential dichotomy

In paper [44], almost periodic solutions were investigated for SDEs driven by G-Brownian motion satisfying exponential stability. However, it seems that no literatures on p-distribution (pseudo) almost periodic solutions for (1.2) satisfying exponential dichotomy. This section will solve these questions. In addition, throughout this section, one denotes $\overline{\iota}_0=\left(\frac{p-2}{p\delta'}\right)^{\frac{p}{2}-1}$, $\overline{\iota}_1=\left(\frac{2(p-1)}{p\delta'}\right)^{p-1}$.

4.1. The existence of p-distribution almost periodic solutions

In this subsection, the existence of p-distribution almost periodic solutions to the semi-linear SDE (1.2) with exponential dichotomy will be discussed. Some necessary assumptions are:

(H_1) for every $\varpi\in L_G^p(\Omega)$, $S(h)\varpi\to\varpi$ as $h\to 0^+$ uniformly for $t\in\mathbb{R}$. Moreover, $S(t)$ is compact for $t\geq 0$;

(H_2) $S(t)$ is exponential dichotomy on $L_G^p(\Omega)$, i.e., there are positive numbers $\widehat{k}$ and δ' satisfying

$$\|S(t)\mathcal{P}\|\leq \widehat{k}e^{-\delta' t}, t\geq 0,\quad \|S(t)\mathcal{Q}\|\leq \widehat{k}e^{\delta' t}, t\leq 0,$$

where $\mathcal{Q}=I-\mathcal{P}$;

(H_3) there exists $L_j>0$, $j=1,2,3$ such that

$$E|G_j(t,\varpi_1)-G_j(t,\varpi_2)|^p\leq L_jE\left|\varpi_1-\varpi_2\right|^p \text{ and } \sup_{t\in\mathbb{R}}E\left|G_j(t,\varpi)\right|^p\leq M_j(\|\varpi\|_\infty)$$

for any $t\in\mathbb{R}$, $\varpi_1,\varpi_2\in L_G^p(\Omega)$, where nondecreasing continuous function $M_j:\mathbb{R}^+\to\mathbb{R}^+$ satisfies

$$\lim_{\varsigma\to\infty}\frac{M_j(\varsigma)}{\varsigma}=0;$$

(H_4) the coefficient $G_j(t,\varpi):\mathbb{R}\times L_G^p(\Omega)\to L_G^p(\Omega)$, $j=1,2,3$ is p-mean almost periodic on $t\in\mathbb{R}$, uniformly for ϖ on any bounded subset of $L_G^p(\Omega)$.

Theorem 4.1. *Let assumptions* (H_1)-(H_3) *hold. If*

$$6^{p-1}\widehat{k}^p\frac{4}{p\delta'}\left(\overline{\iota}_1L_1+\overline{\sigma}^{2p}\overline{\iota}_1L_3+C_p\overline{\sigma}^pL_2\overline{\iota}_0\right)<1, \tag{4.1}$$

then the semi-linear SDE (1.2) *has a unique L^p-bounded solution.*

Proof. Note that one can check that $\varpi(t)$ is a mild solution of the semi-linear SDE (1.2) since it fulfills integral equation below

$$\begin{aligned}\varpi(t) =& \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1(\varrho,\varpi(\varrho))d\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1(\varrho,\varpi(\varrho))d\varrho \\ &+ \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2(\varrho,\varpi(\varrho))dB_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2(\varrho,\varpi(\varrho))dB_\varrho \\ &+ \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho.\end{aligned}$$

Consider the operator $\mathcal{T}$ given by

$$\begin{aligned}\mathcal{T}(\varpi(t)) :=& \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1(\varrho,\varpi(\varrho))d\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1(\varrho,\varpi(\varrho))d\varrho \\ &+ \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2(\varrho,\varpi(\varrho))dB_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2(\varrho,\varpi(\varrho))dB_\varrho \\ &+ \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho.\end{aligned} \tag{4.2}$$

To show the existence of the solution of (1.2), one divides it into two steps.

Step 1. The operator $\mathcal{T} : BUC(\mathbb{R}, L_G^p(\Omega)) \to BUC(\mathbb{R}, L_G^p(\Omega))$ is well defined. In fact, $\mathcal{T}\varpi$ is L^p-bounded. It follows from (H_2) and (H_3) that

$$\begin{aligned}& E\left|\mathcal{T}\varpi(t)\right|^p \\ \leq& 3^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1(\varrho,\varpi(\varrho))d\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1(\varrho,\varpi(\varrho))d\varrho\right|^p \\ &+3^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2(\varrho,\varpi(\varrho))dB_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2(\varrho,\varpi(\varrho))dB_\varrho\right|^p \\ &+ 3^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho\right|^p \\ \leq& 6^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1(\varrho,\varpi(\varrho))d\varrho\right|^p + 6^{p-1}E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1(\varrho,\varpi(\varrho))d\varrho\right|^p \\ &+6^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2(\varrho,\varpi(\varrho))dB_\varrho\right|^p + 6^{p-1}E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2(\varrho,\varpi(\varrho))dB_\varrho\right|^p \\ &+6^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho\right|^p + 6^{p-1}E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho\right|^p \\ \leq& 6^{p-1}\widehat{k}^p \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2(p-1)}} d\varrho\right)^{p-1} \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} E\left|G_1(\varrho,\varpi(\varrho))\right|^p d\varrho\right) \\ &+ 6^{p-1}\widehat{k}^p \left(\int_{t}^{+\infty} e^{\frac{p\delta'(t-\varrho)}{2(p-1)}} d\varrho\right)^{p-1} \left(\int_{t}^{+\infty} e^{\frac{p\delta'(t-\varrho)}{2}} E\left|G_1(\varrho,\varpi(\varrho))\right|^p d\varrho\right) \\ &+ 6^{p-1}\widehat{k}^p C_p \overline{\sigma}^p \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{p-2}} d\varrho\right)^{\frac{p}{2}-1} \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} E\left|G_2(\varrho,\varpi(\varrho))\right|^p d\varrho\right)\end{aligned}$$

$$
\begin{aligned}
&+6^{p-1}\widehat{k}^p C_p \overline{\sigma}^p \left(\int_t^{+\infty} e^{\frac{p\delta'(t-\varrho)}{p-2}} d\varrho\right)^{\frac{p}{2}-1} \left(\int_t^{+\infty} e^{\frac{p\delta'(t-\varrho)}{2}} E\left|G_2(\varrho,\varpi(\varrho))\right|^p d\varrho\right)\\
&+6^{p-1}\widehat{k}^p \overline{\sigma}^{2p} \left(\int_{-\infty}^t e^{-\frac{p\delta'(t-\varrho)}{2(p-1)}} d\varrho\right)^{p-1} \left(\int_{-\infty}^t e^{-\frac{p\delta'(t-\varrho)}{2}} E\left|G_3(\varrho,\varpi(\varrho))\right|^p d\varrho\right)\\
&+6^{p-1}\widehat{k}^p \overline{\sigma}^{2p} \left(\int_t^{+\infty} e^{\frac{p\delta'(t-\varrho)}{2(p-1)}} d\varrho\right)^{p-1} \left(\int_t^{+\infty} e^{\frac{p\delta'(t-\varrho)}{2}} E\left|G_3(\varrho,\varpi(\varrho))\right|^p d\varrho\right)\\
\le&\frac{4\cdot 6^{p-1}}{p\delta'}\widehat{k}^p \overline{\iota}_1 \mathcal{M}_1(\|\varpi\|_\infty) + \frac{4\cdot 6^{p-1}}{p\delta'} C_p \overline{\sigma}^p \widehat{k}^p \overline{\iota}_0 \mathcal{M}_2(\|\varpi\|_\infty)\\
&+\overline{\sigma}^{2p}\frac{4\cdot 6^{p-1}}{p\delta'}\widehat{k}^p \overline{\iota}_1 \mathcal{M}_3(\|\varpi\|_\infty).
\end{aligned}
$$

One now proves that $\mathcal{T}\varpi$ is continuous in t. By (H_1), for any $\varepsilon > 0$, there exists a $\widetilde{\xi} > 0$ satisfying

$$\widetilde{\xi} < \min\{h_1, h_2, h_3\}$$

such that when $0 < t' - t'' < \widetilde{\xi}$, one has

$$\|S(t'-t'') - I\|^p \le \min\{r_1, r_2, r_3\},$$

where

$$h_1 = \frac{\varepsilon^{\frac{1}{p}}}{12\widehat{k}\,\|G_1\|_\infty}, \quad h_2 = \frac{\varepsilon^{\frac{2}{p}}}{144 C_p^{\frac{2}{p}} \overline{\sigma}^2 \widehat{k}^2 \|G_2\|_\infty^2}, \quad h_3 = \frac{h_1 \|G_1\|_\infty}{\overline{\sigma}^2 \|G_3\|_\infty},$$

and

$$r_1 = \frac{\varepsilon p \delta'}{2\cdot 12^p \widehat{k}^p \overline{\iota}_1 \|G_1\|_\infty^p}, \quad r_2 = \frac{\varepsilon p\delta'}{2\cdot 12^p C_p \overline{\sigma}^p \widehat{k}^p \overline{\iota}_0 \|G_2\|_\infty^p}, \quad r_3 = \frac{r_1 \|G_1\|_\infty^p}{\overline{\sigma}^{2p}\|G_3\|_\infty^p}.$$

Then, it follows that

$$
\begin{aligned}
&E\left|(\mathcal{T}\varpi)(t') - (\mathcal{T}\varpi)(t'')\right|^p\\
\le& 12^{p-1} E\left|\int_{-\infty}^{t''} [S(t'-t'')-I] S(t''-\varrho)\mathcal{P}G_1(\varrho,\varpi(\varrho))d\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t''}^{t'} S(t'-\varrho)\mathcal{P}G_1(\varrho,\varpi(\varrho))d\varrho\right|^p\\
&+12^{p-1}E\left|\int_{-\infty}^{t''} [S(t'-t'')-I] S(t''-\varrho)\mathcal{P}G_2(\varrho,\varpi(\varrho))dB_\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t''}^{t'} S(t'-\varrho)\mathcal{P}G_2(\varrho,\varpi(\varrho))dB_\varrho\right|^p\\
&+12^{p-1}E\left|\int_{-\infty}^{t''} [S(t'-t'')-I] S(t''-\varrho)\mathcal{P}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t''}^{t'} S(t'-\varrho)\mathcal{P}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho\right|^p
\end{aligned}
$$

$$
\begin{aligned}
&+12^{p-1}E\left|\int_{t''}^{+\infty}[S(t'-t'')-I]S(t''-\varrho)\mathcal{Q}G_1(\varrho,\varpi(\varrho))d\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t'}^{t''}S(t''-\varrho)\mathcal{Q}G_1(\varrho,\varpi(\varrho))d\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t''}^{+\infty}[S(t'-t'')-I]S(t''-\varrho)\mathcal{Q}G_2(\varrho,\varpi(\varrho))dB_\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t'}^{t''}S(t''-\varrho)\mathcal{Q}G_2(\varrho,\varpi(\varrho))dB_\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t''}^{+\infty}[S(t'-t'')-I]S(t''-\varrho)\mathcal{Q}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho\right|^p\\
&+12^{p-1}E\left|\int_{t'}^{t''}S(t''-\varrho)\mathcal{Q}G_3(\varrho,\varpi(\varrho))d\langle B\rangle_\varrho\right|^p\\
=&\Upsilon_{11}+\Upsilon_{12}+\Upsilon_{21}+\Upsilon_{22}+\Upsilon_{31}+\Upsilon_{32}+\Upsilon_{41}+\Upsilon_{42}+\Upsilon_{51}+\Upsilon_{52}+\Upsilon_{61}+\Upsilon_{62}.
\end{aligned}
$$

By applying the Hölder inequality and (H_2), one obtains that

$$
\begin{aligned}
&\Upsilon_{11}+\Upsilon_{12}\\
\leq&12^{p-1}\widehat{k}^p\left\|S(t'-t'')-I\right\|^p\left[\int_{-\infty}^{t''}e^{-\frac{p\delta'}{2(p-1)}(t''-\varrho)}d\varrho\right]^{p-1}\\
&\times\left[\int_{-\infty}^{t''}e^{-\frac{\delta' p}{2}(t''-\varrho)}E\left|G_1(\varrho,\varpi(\varrho))\right|^p d\varrho\right]\\
&+12^{p-1}\widehat{k}^p\left[\int_{t''}^{t'}e^{-\frac{p\delta'}{2(p-1)}(t'-\varrho)}d\varrho\right]^{p-1}\left[\int_{t''}^{t'}e^{-\frac{p\delta'}{2}(t'-\varrho)}E\left|G_1(\varrho,\varpi(\varrho))\right|^p d\varrho\right]\\
\leq&\frac{2\cdot 12^{p-1}\widehat{k}^p}{\delta' p}\left\|S(t'-t'')-I\right\|^p\bar{\iota}_1\left\|G_1\right\|_\infty^p+12^{p-1}\widehat{k}^p(t'-t'')^p\left\|G_1\right\|_\infty^p\\
\leq&\frac{2\cdot 12^{p-1}\widehat{k}^p}{\delta' p}\bar{\iota}_1\left\|G_1\right\|_\infty^p r_1+12^{p-1}\widehat{k}^p h_1^p\left\|G_1\right\|_\infty^p<\frac{\varepsilon}{6}.
\end{aligned}
$$

By Proposition 2.1, Hölder inequality and (H_2), it yields that

$$
\begin{aligned}
&\Upsilon_{21}+\Upsilon_{22}\\
\leq&12^{p-1}\widehat{k}^pC_pE\left[\int_{-\infty}^{t''}e^{-2\delta'(t''-\varrho)}\left\|S(t'-t'')-I\right\|^2\left|G_2(\varrho,\varpi(\varrho))\right|^2 d\langle B\rangle_\varrho\right]^{\frac{p}{2}}\\
&+12^{p-1}\widehat{k}^pC_pE\left[\int_{t''}^{t'}e^{-2\delta'(t'-\varrho)}\left|G_2(\varrho,\varpi(\varrho))\right|^2 d\langle B\rangle_\varrho\right]^{\frac{p}{2}}\\
\leq&12^{p-1}\widehat{k}^pC_p\overline{\sigma}^pE\left[\int_{-\infty}^{t''}e^{-2\delta'(t''-\varrho)}\left\|S(t',t'')-I\right\|^2\left|G_2(\varrho,\varpi(\varrho))\right|^2 d\varrho\right]^{\frac{p}{2}}\\
&+12^{p-1}\widehat{k}^pC_p\overline{\sigma}^pE\left[\int_{t''}^{t'}e^{-2\delta'(t'-\varrho)}\left|G_2(\varrho,\varpi(\varrho))\right|^2 d\varrho\right]^{\frac{p}{2}}
\end{aligned}
$$

$$\leq 12^{p-1}\widehat{k}^p C_p\overline{\sigma}^p \|S(t'-t'')-I\|^p \left[\int_{-\infty}^{t''} e^{-\frac{p\delta'(t''-\varrho)}{p-2}}d\varrho\right]^{\frac{p}{2}-1}$$
$$\times \left[\int_{-\infty}^{t''} e^{-\frac{p\delta'}{2}(t''-\varrho)}E\left|G_2(\varrho,\varpi(\varrho))\right|^p d\varrho\right]$$
$$+ 12^{p-1}\widehat{k}^p C_p\overline{\sigma}^p \left[\int_{t''}^{t'} e^{-\frac{p\delta'(t'-\varrho)}{p-2}}\right]^{\frac{p}{2}-1}\int_{t''}^{t'} e^{-\frac{p\delta'}{2}(t'-\varrho)}\|G_2\|_\infty^p d\varrho$$
$$\leq \frac{2\cdot 12^{p-1}\widehat{k}^p C_p\overline{\sigma}^p}{p\delta'}\|S(t'-t'')-I\|^p \bar{\iota}_0\|G_2\|_\infty^p + 12^{p-1}\widehat{k}^p C_p\overline{\sigma}^p\|G_2\|_\infty^p (t^{'}-t^{''})^{\frac{p}{2}}$$
$$\leq \frac{2\cdot 12^{p-1}\widehat{k}^p C_p\overline{\sigma}^p}{p\delta'}\bar{\iota}_0\|G_2\|_\infty^p r_2 + 12^{p-1}\widehat{k}^p C_p\overline{\sigma}^p\|G_2\|_\infty^p h_2^{\frac{p}{2}} < \frac{\varepsilon}{6}.$$

Similarly, from Proposition 2.1, Hölder inequality and (H_2), one obtains some results for $\Upsilon_{31}+\Upsilon_{32}$, $\Upsilon_{41}+\Upsilon_{42}$, $\Upsilon_{51}+\Upsilon_{52}$ and $\Upsilon_{61}+\Upsilon_{62}$ as follows:

$$\Upsilon_{31}+\Upsilon_{32}\leq \frac{2\cdot 12^{p-1}\overline{\sigma}^{2p}\widehat{k}^p}{\delta' p}\|S(t'-t'')-I\|^p\bar{\iota}_1\|G_3\|_\infty^p + 12^{p-1}\widehat{k}^p\overline{\sigma}^{2p}(t'-t'')^p\|G_3\|_\infty^p < \frac{\varepsilon}{6},$$
$$\Upsilon_{41}+\Upsilon_{42}\leq \frac{2\cdot 12^{p-1}\widehat{k}^p}{p\delta'}\|S(t'-t'')-I\|^p\bar{\iota}_1\|G_1\|_\infty^p + 12^{p-1}\widehat{k}^p(t'-t'')^p\|G_1\|_\infty^p < \frac{\varepsilon}{6},$$
$$\Upsilon_{51}+\Upsilon_{52}\leq \frac{2\cdot 12^{p-1}C_p\overline{\sigma}^p\widehat{k}^p}{p\delta'}\|S(t'-t'')-I\|^p\bar{\iota}_0\|G_2\|_\infty^p + 12^{p-1}C_p\overline{\sigma}^p\widehat{k}^p(t'-t'')^{\frac{p}{2}}\|G_2\|_\infty^p < \frac{\varepsilon}{6},$$
$$\Upsilon_{61}+\Upsilon_{62}\leq \frac{2\cdot 12^{p-1}\overline{\sigma}^{2p}\widehat{k}^p}{p\delta'}\|S(t'-t'')-I\|^p\bar{\iota}_1\|G_3\|_\infty^p + 12^{p-1}\widehat{k}^p\overline{\sigma}^{2p}(t'-t'')^p\|G_3\|_\infty^p < \frac{\varepsilon}{6}.$$

Hence $\mathcal{T}\varpi$ is continuous in t.

Step 2. One shows that $\mathcal{T}$ is a contraction mapping.

$$E\left|(\mathcal{T}\varpi_1)(t)-(\mathcal{T}\varpi_2)(t)\right|^p$$
$$=E\Bigg|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}(G_1(\varrho,\varpi_1(\varrho))-G_1(\varrho,\varpi_2(\varrho)))d\varrho$$
$$-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_1(\varrho,\varpi_1(\varrho))-G_1(\varrho,\varpi_2(\varrho)))d\varrho$$
$$+\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}(G_2(\varrho,\varpi_1(\varrho))-G_2(\varrho,\varpi_2(\varrho)))dB_\varrho$$
$$-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_2(\varrho,\varpi_1(\varrho))-G_2(\varrho,\varpi_2(\varrho)))dB_\varrho$$
$$+\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}(G_3(\varrho,\varpi_1(\varrho))-G_3(\varrho,\varpi_2(\varrho)))d\langle B\rangle_\varrho$$
$$-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_3(\varrho,\varpi_1(\varrho))-G_3(\varrho,\varpi_2(\varrho)))d\langle B\rangle_\varrho\Bigg|^p$$
$$\leq 6^{p-1}\widehat{k}^p\Bigg[E\left(\int_{-\infty}^{t} e^{-\delta'(t-\varrho)}(G_1(\varrho,\varpi_1(\varrho))-G_1(\varrho,\varpi_2(\varrho)))d\varrho\right)^p$$
$$+E\left(\int_{t}^{+\infty} e^{\delta'(t-\varrho)}(G_1(\varrho,\varpi_1(\varrho))-G_1(\varrho,\varpi_2(\varrho)))d\varrho\right)^p$$

$$
\begin{aligned}
&+E\left(\int_{-\infty}^{t}e^{-\delta'(t-\varrho)}(G_2(\varrho,\varpi_1(\varrho))-G_2(\varrho,\varpi_2(\varrho)))dB_\varrho\right)^p\\
&+E\left(\int_{t}^{+\infty}e^{\delta'(t-\varrho)}(G_2(\varrho,\varpi_1(\varrho))-G_2(\varrho,\varpi_2(\varrho)))dB_\varrho\right)^p\\
&+E\left(\int_{-\infty}^{t}e^{-\delta'(t-\varrho)}(G_3(\varrho,\varpi_1(\varrho))-G_3(\varrho,\varpi_2(\varrho)))d\langle B\rangle_\varrho\right)^p\\
&+E\left(\int_{t}^{+\infty}e^{\delta'(t-\varrho)}(G_3(\varrho,\varpi_1(\varrho))-G_3(\varrho,\varpi_2(\varrho)))d\langle B\rangle_\varrho\right)^p\Bigg]\\
\le&6^{p-1}\widehat{k}^p\frac{4}{p\delta'}\left(\bar{\iota}_1L_1+\overline{\sigma}^{2p}\bar{\iota}_1L_3+C_p\overline{\sigma}^pL_2\bar{\iota}_0\right)E|\varpi_1(s)-\varpi_2(s)|^p\\
\le&\theta\sup_{s\in\mathbb{R}}E|\varpi_1(s)-\varpi_2(s)|^p.
\end{aligned}
$$

This implies that $\mathcal{T}\varpi$ is a contraction mapping. Therefore, the semi-linear SDE (1.2) has a unique L^p-bounded solution by (4.1). This completes the proof. ☐

Theorem 4.2. *Let assumptions* (H_1)*-*(H_4) *hold. Then the unique* L^p*-bounded solution of the semi-linear SDE* (1.2) *is almost periodic in p-distribution provided that*

$$\bar{\iota}_1(L_1+\overline{\sigma}^{2p}L_3)+\bar{\iota}_0C_p\overline{\sigma}^pL_2<\frac{p\delta'}{6\cdot 12^{p-1}\widehat{k}^p}. \tag{4.3}$$

Proof. Let $\varpi^\infty(t)$ and $\varpi_n(t)$, $n\in\mathbb{N}$ satisfy the the following integral equations

$$
\begin{aligned}
\varpi^\infty(t)=&\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_1^\infty(\varrho,\varpi^\infty(\varrho))d\varrho-\int_{t}^{+\infty}S(t-\varrho)\mathcal{Q}G_1^\infty(\varrho,\varpi^\infty(\varrho))d\varrho\\
&+\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_2^\infty(\varrho,\varpi^\infty(\varrho))dB_\varrho-\int_{t}^{+\infty}S(t-\varrho)\mathcal{Q}G_2^\infty(\varrho,\varpi^\infty(\varrho))dB_\varrho\\
&+\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_3^\infty(\varrho,\varpi^\infty(\varrho))d\langle B\rangle_\varrho-\int_{t}^{+\infty}S(t-\varrho)\mathcal{Q}G_3^\infty(\varrho,\varpi^\infty(\varrho))d\langle B\rangle_\varrho
\end{aligned}
$$

and

$$
\begin{aligned}
\varpi_n(t)=&\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_1(\varrho+\sigma_n,\varpi_n(\varrho))d\varrho-\int_{t}^{+\infty}S(t-\varrho)\mathcal{Q}G_1(\varrho+\sigma_n,\varpi_n(\varrho))d\varrho\\
&+\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_2(\varrho+\sigma_n,\varpi_n(\varrho))dB_\varrho-\int_{t}^{+\infty}S(t-\varrho)\mathcal{Q}G_2(\varrho+\sigma_n,\varpi_n(\varrho))dB_\varrho\\
&+\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_3(\varrho+\sigma_n,\varpi_n(\varrho))d\langle B\rangle_\varrho-\int_{t}^{+\infty}S(t-\varrho)\mathcal{Q}G_3(\varrho+\sigma_n,\varpi_n(\varrho))d\langle B\rangle_\varrho,
\end{aligned}
$$

respectively. Notice that (4.3) can infer (4.1), then like $\varpi^\infty(t)$, such $\varpi_n(t)$ is unique and L^p-bounded by Theorem 4.1. Taking the transformation of $s+\sigma_n=\varrho$, one can derive that

$$
\begin{aligned}
\varpi(t+\sigma_n)=&\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_1^\infty(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\varrho\\
&-\int_{t}^{+\infty}S(t-\varrho)\mathcal{Q}G_1^\infty(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\varrho
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2^{\infty}(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\widehat{B}_\varrho \\
&-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2^{\infty}(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\widehat{B}_\varrho \\
&+\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3^{\infty}(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\langle\widehat{B}\rangle_\varrho \\
&-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3^{\infty}(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\langle\widehat{B}\rangle_\varrho,
\end{aligned}
$$

where $\widehat{B}_\varrho = B_{\varrho+\varrho_n} - B_{\varrho_n}$ is a G-Brownian motion with the same distribution as B_ϱ, and $\langle\widehat{B}\rangle_\varrho$ has the same distributions as $\langle B\rangle_\varrho$.

Now let us show that $\varpi_n(t)$ converges in p-distribution to $\varpi^\infty(t)$ for every fixed $t\in\mathbb{R}$. It is easy to see that

$$
\begin{aligned}
&E\left|\varpi_n(t)-\varpi^{\infty}(t)\right|^p \\
&= E\bigg|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\varrho-\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\varrho \\
&\quad-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\varrho+\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\varrho \\
&\quad+\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))dB_\varrho-\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2^{\infty}(\varrho,\varpi^{\infty}(\varrho))dB_\varrho \\
&\quad-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))dB_\varrho+\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2^{\infty}(\varrho,\varpi^{\infty}(\varrho))dB_\varrho \\
&\quad+\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\langle B\rangle_\varrho-\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\langle B\rangle_\varrho \\
&\quad-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\langle B\rangle_\varrho+\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\langle B\rangle_\varrho\bigg|^p \\
&\le \Sigma_{11}+\Sigma_{12}+\Sigma_{21}+\Sigma_{22}+\Sigma_{31}+\Sigma_{32},
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_{11}&=6^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\varrho-\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\varrho\right|^p, \\
\Sigma_{12}&=6^{p-1}E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\varrho-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\varrho\right|^p, \\
\Sigma_{21}&=6^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))dB_\varrho-\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_2^{\infty}(\varrho,\varpi^{\infty}(\varrho))dB_\varrho\right|^p, \\
\Sigma_{22}&=6^{p-1}E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))dB_\varrho-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_2^{\infty}(\varrho,\varpi^{\infty}(\varrho))dB_\varrho\right|^p, \\
\Sigma_{31}&=6^{p-1}E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\langle B\rangle_\varrho-\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_3^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\langle B\rangle_\varrho\right|^p, \\
\Sigma_{32}&=6^{p-1}E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3(\varrho+\sigma_n,\varpi(\varrho+\sigma_n))d\langle B\rangle_\varrho-\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_3^{\infty}(\varrho,\varpi^{\infty}(\varrho))d\langle B\rangle_\varrho\right|^p.
\end{aligned}
$$

Herein, the calculation of Σ_{11} is given by

$$\begin{aligned}
\Sigma_{11} \leq\, & 12^{p-1} E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}[G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n)) - G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho))]d\varrho\right|^p \\
& + 12^{p-1} E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}[G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho)) - G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))]d\varrho\right|^p \\
\leq\, & 12^{p-1}\widehat{k}^p \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2(p-1)}} d\varrho\right)^{p-1} \\
& \times \left(\int_{-\infty}^{t} e^{-\frac{\delta' p(t-\varrho)}{2}} E|G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n)) - G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho))|^p d\varrho\right) \\
& + 12^{p-1}\widehat{k}^p \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2(p-1)}} d\varrho\right)^{p-1} \\
& \times \int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} E\left|G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho)) - G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|^p d\varrho \\
\leq\, & 12^{p-1}\widehat{k}^p L_1\bar{\iota}_1 \int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} E\left|\varpi(\varrho+\sigma_n) - \varpi^{\infty}(\varrho)\right|^p d\varrho + I_{11}^n
\end{aligned}$$

with

$$I_{11}^n = 12^{p-1}\widehat{k}^p\bar{\iota}_1 \int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} E\left|G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho)) - G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|^p d\varrho.$$

Note that $\|G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho)) - G_1(\varrho+\sigma_n,\varpi^{\infty}(0))\|_p \leq L_1^{\frac{1}{p}}\|\varpi^{\infty}(\varrho) - \varpi^{\infty}(0))\|_p$ and $\varpi^{\infty}(\varrho)$ is L^p-bounded, it yields that

$$\begin{aligned}
& \sup_{\varrho\in\mathbb{R}} \|G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho))\|_p \\
\leq & \sup_{\varrho\in\mathbb{R}} \|G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho)) - G_1(\varrho+\sigma_n,0)\|_p + \sup_{\varrho\in\mathbb{R}} \|G_1(\varrho+\sigma_n,0)\|_p \\
\leq & L_1^{\frac{1}{p}} \sup_{\varrho\in\mathbb{R}} \|\varpi^{\infty}(\varrho)\|_p + \sup_{\varrho\in\mathbb{R}} \|G_1(\varrho+\sigma_n,0)\|_p < \infty,
\end{aligned}$$

and hence

$$\sup_{\varrho\in\mathbb{R}} \|G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))\|_p < \infty.$$

Since G_1 is p-mean almost periodic in t by (H_4) and $\varpi^{\infty}(\cdot)$ is bounded in $L_G^p(\Omega)$, one gains $I_{11}^n \to 0$ as $n \to \infty$ by the Lebesgue dominated convergence theorem and the arbitrary of ε.

In the light of (H_3) and almost periodicity of G_1, one has

$$\begin{aligned}
\Sigma_{12} \leq & 12^{p-1} E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n)) - G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho)))d\varrho\right|^p \\
& + 12^{p-1} E\left|\int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho)) - G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho)))d\varrho\right|^p \\
\leq & 12^{p-1} E\left(\int_{t}^{+\infty} \widehat{k}e^{\delta'(t-\varrho)} \left|G_1(\varrho+\sigma_n,\varpi(\varrho+\sigma_n)) - G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho))\right| d\varrho\right)^p
\end{aligned}$$

$$
\begin{aligned}
&+12^{p-1}E\left(\int_t^{+\infty}\widehat{k}e^{\delta'(t-\varrho)}\left|G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho))-G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|d\varrho\right)^p\\
\leq &12^{p-1}\widehat{k}^pL_1\bar{\iota}_1\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho+I_{12}^n
\end{aligned}
$$

with

$$
I_{12}^n=12^{p-1}\widehat{k}^p\bar{\iota}_1\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|G_1(\varrho+\sigma_n,\varpi^{\infty}(\varrho))-G_1^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|^p d\varrho.
$$

Similar to I_{11}^n, one obtains that $I_{12}^n\to 0$ as $n\to\infty$.

By the same way, one derives

$$
\begin{aligned}
\Sigma_{21}&\leq 12^{p-1}C_p\overline{\sigma}^p\widehat{k}^pL_2\bar{\iota}_0\int_{-\infty}^t e^{-\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho+I_{21}^n,\\
\Sigma_{22}&\leq 12^{p-1}\widehat{k}^pC_p\overline{\sigma}^pL_2\bar{\iota}_0\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho+I_{22}^n\\
\Sigma_{31}&\leq 12^{p-1}\widehat{k}^pL_3\overline{\sigma}^{2p}\bar{\iota}_1\int_{-\infty}^t e^{-\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho+I_{31}^n,\\
\Sigma_{32}&\leq 12^{p-1}\widehat{k}^pL_3\overline{\sigma}^{2p}\bar{\iota}_1\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho+I_{32}^n,
\end{aligned}
$$

with

$$
\begin{aligned}
I_{21}^n&=12^{p-1}C_p\overline{\sigma}^p\widehat{k}^p\bar{\iota}_0\int_{-\infty}^t e^{-\frac{p\delta'(t-\varrho)}{2}}E\left|G_2(\varrho+\sigma_n,\varpi^{\infty}(\varrho))-G_2^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|^p d\varrho,\\
I_{22}^n&=12^{p-1}\widehat{k}^pC_p\overline{\sigma}^p\bar{\iota}_0\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|G_2(\varrho+\sigma_n,\varpi^{\infty}(\varrho))-G_2^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|^p d\varrho,\\
I_{31}^n&=12^{p-1}\widehat{k}^p\overline{\sigma}^{2p}\bar{\iota}_1\int_{-\infty}^t e^{-\frac{p\delta'(t-\varrho)}{2}}E\left|G_3(\varrho+\sigma_n,\varpi^{\infty}(\varrho))-G_3^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|^p d\varrho,\\
I_{32}^n&=12^{p-1}\widehat{k}^p\overline{\sigma}^{2p}\bar{\iota}_1\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|G_3(\varrho+\sigma_n,\varpi^{\infty}(\varrho))-G_3^{\infty}(\varrho,\varpi^{\infty}(\varrho))\right|^p d\varrho.
\end{aligned}
$$

Similar to the processes of I_{11}^n and I_{12}^n, one has $I_{21}^n\to 0$, $I_{22}^n\to 0$, $I_{31}^n\to 0$ and $I_{32}^n\to 0$.

Based on above estimations, one obtains that

$$
\begin{aligned}
&E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p\\
\leq &I^n+12^{p-1}\widehat{k}^pC_p\overline{\sigma}^p\bar{\iota}_0L_2\int_{-\infty}^t e^{\frac{-p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho\\
&+12^{p-1}\widehat{k}^pC_p\overline{\sigma}^p\bar{\iota}_0L_2\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho\\
&+12^{p-1}\widehat{k}^p(L_1+\overline{\sigma}^{2p}L_3)\bar{\iota}_1\int_{-\infty}^t e^{-\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho\\
&+12^{p-1}\widehat{k}^p(L_1+\overline{\sigma}^{2p}L_3)\bar{\iota}_1\int_t^{+\infty}e^{\frac{p\delta'(t-\varrho)}{2}}E\left|\varpi(\varrho+\sigma_n)-\varpi^{\infty}(\varrho)\right|^p d\varrho
\end{aligned}
$$

with $I^n = I_{11}^n + I_{12}^n + I_{21}^n + I_{22}^n + I_{31}^n + I_{32}^n \to 0$ as $n \to \infty$. By [19, Lemma 2.5], one concludes that

$$E\,|\varpi(\varrho + \sigma_n) - \varpi^\infty(\varrho)|^p \to 0 \quad \text{ as } n \to \infty, \text{ for every } t \in \mathbb{R}.$$

Since $\varpi(\varrho + \sigma_n)$ share the same distribution as $\varpi_n(\varrho)$, it could be easily checked that $\varpi(\varrho + \sigma_n) \to \varpi^\infty(\varrho)$ as $n \to \infty$. Hence, ϖ is almost periodic in distribution. On the other hand, the sequence $\{|\varpi_n(\varrho)|^p, t \in \mathbb{R}, n \in \mathbb{N}\}$ is uniformly integrable, so $\{|\varpi(\varrho + \sigma_n)|^p, \varrho \in \mathbb{R}, n \in \mathbb{N}\}$ is also uniformly integrable. Thus, it is immediate to obtain that $\varpi(t)$ is almost periodic in p-distribution. □

4.2. The existence of p-distribution pseudo almost periodic solutions

In this subsection, one investigates the p-distribution pseudo almost periodic solutions of the semi-linear SDE (1.2) and gives the following assumptions:

(H_5) the coefficient $G_j(t, \varpi) : \mathbb{R} \times L_G^p(\Omega) \to L_G^p(\Omega)$, $j = 1, 2, 3$ is p-mean pseudo almost periodic on $t \in \mathbb{R}$ and uniformly for ϖ on any bounded subset of $L_G^p(\Omega)$;

(H_6) let $\mu \in \mathbb{M}$. For every $\tau \in \mathbb{R}$, there exist a bounded interval D and a positive constant β satisfying $\mu(J + \tau) \le \beta\mu(J)$ whenever J is a Borel subset of $\mathbb{R}$ satisfying $J \cap D = \emptyset$.

From (H_5), G_j can be decomposed as $G_j = G_{j,1} + G_{j,2} \in PAP(\mathbb{R} \times L_G^p(\Omega), L_G^p(\Omega), \mu)$, where $G_{j,1} \in AP(R \times L_G^p(\Omega), L_G^p(\Omega))$, $G_{j,2} \in \varepsilon(R \times L_G^p(\Omega), L_G^p(\Omega), \mu), j = 1, 2, 3$ with μ satisfying (H_6).

Theorem 4.3. *Let* (H_1)-(H_3), (H_5)-(H_6) *hold, and* $G_{j,1}$, $j = 1, 2, 3$ *satisfy* (H_3). *Then there is a p-distribution pseudo almost periodic solution of the semi-linear SDE* (1.2) *provided that* (4.3) *holds.*

Proof. According to Theorem 4.2, one can see that the equation

$$d\varpi(t) = (A\varpi(t) + G_{1,1}(t, \varpi(t)))dt + G_{2,1}(t, \varpi(t))dB_t + G_{3,1}(t, \varpi(t))d\langle B\rangle_t$$

possesses a unique p-distribution almost periodic solution $\varpi \in BUC(\mathbb{R}, L_G^p(\Omega))$. Thus $K := \{\overline{\varpi(t)|t \in \mathbb{R}}\}$ is a compact set in $L_G^p(\Omega)$, which means that for any $\varepsilon > 0$, there are $\varpi_1, \varpi_2, \cdots, \varpi_m$ satisfying

$$K \subset \bigcup_{i=1}^{m} \left\{\varpi \in K : E\,|\varpi - \varpi_i|^p \le \frac{\varepsilon}{2^{p-1}}\right\}.$$

On the other hand, by the proof Theorem 4.2, the operator $\mathcal{T}$ given in (4.2) has a unique fixed point $\xi \in BUC(\mathbb{R}, L_G^p(\Omega))$, which is the solution to the semi-linear SDE (1.2). Since the operator $\mathcal{T}$ is contractive, ξ is just the limit of sequence $(\xi_n)_{n\in\mathbb{N}}$ satisfying $\xi_{n+1} = \mathcal{T}(\xi_n)$ with arbitrary ξ_0.

Now one chooses a special sequence to show that ξ is pseudo almost periodic in p-distribution sense. Let

$$\xi_0 = \varpi, \xi_{n+1} = \mathcal{T}(\xi_n), \theta_n = \xi_n - \varpi.$$

Before proving ξ being pseudo almost periodic, one needs to show $\theta_{n+1}(t) \in \varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$ by the induction method. For $n = 0$, $\theta_0 = 0 \in \varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$. Now assume that $\theta_n(t) \in \varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$. In fact, for each $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\begin{aligned}
\theta_{n+1}(t) =& \mathcal{T}\xi_n(t) - \varpi(t) \\
=& \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}(G_1(\varrho, \xi_n(\varrho)) - G_1(\varrho, \varpi(\varrho)))d\varrho \\
& - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_1(\varrho, \xi_n(\varrho)) - G_1(\varrho, \varpi(\varrho)))d\varrho \\
& + \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_{1,2}(\varrho, \varpi(\varrho))d\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_{1,2}(\varrho, \varpi(\varrho))d\varrho \\
& + \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}(G_2(\varrho, \xi_n(\varrho)) - G_2(\varrho, \varpi(\varrho)))dB_\varrho \\
& - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_2(\varrho, \xi_n(\varrho)) - G_2(\varrho, \varpi(\varrho)))dB_\varrho \\
& + \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_{2,2}(\varrho, \varpi(\varrho))dB_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_{2,2}(\varrho, \varpi(\varrho))dB_\varrho \\
& + \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}(G_3(\varrho, \xi_n(\varrho)) - G_3(\varrho, \varpi(\varrho)))d\langle B\rangle_\varrho \\
& - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}(G_3(\varrho, \xi_n(\varrho)) - G_3(\varrho, \varpi(\varrho)))d\langle B\rangle_\varrho \\
& + \int_{-\infty}^{t} S(t-\varrho)\mathcal{P}G_{3,2}(\varrho, \varpi(\varrho))d\langle B\rangle_\varrho - \int_{t}^{+\infty} S(t-\varrho)\mathcal{Q}G_{3,2}(\varrho, \varpi(\varrho))d\langle B\rangle_\varrho \\
=& \Theta^{11}+\Theta^{12}+\Theta^{13}+\Theta^{14}+\Theta^{21}+\Theta^{22}+\Theta^{23}+\Theta^{24}+\Theta^{31}+\Theta^{32}+\Theta^{33}+\Theta^{34}.
\end{aligned}$$

It follows from (H_3) that

$$E\left|G_j(t, \xi_n(t)) - G_j(t, \varpi(t))\right|^p \leq L_j E\left|\theta_n(t)\right|^p, \quad j = 1, 2, 3. \tag{4.4}$$

By Hölder inequality and (H_2), one has

$$\begin{aligned}
& \frac{1}{\mu([-M, M])} \int_{[-M,M]} E|\Theta^{11}(t)|^p d\mu(t) \\
=& \frac{1}{\mu([-M, M])} \int_{[-M,M]} E\left|\int_{-\infty}^{t} S(t-\varrho)\mathcal{P}(G_1(\varrho, \xi_n(\varrho)) - G_1(\varrho, \varpi(\varrho)))d\varrho\right|^p d\mu(t) \\
\leq& \frac{\widehat{k}^p \overline{\iota}_1}{\mu([-M, M])} \int_{[-M,M]} \int_{-\infty}^{t} e^{-\frac{\delta' p(t-\varrho)}{2}} E\left|G_1(\varrho, \xi_n(\varrho)) - G_1(\varrho, \varpi(\varrho))\right|^p d\varrho d\mu(t).
\end{aligned} \tag{4.5}$$

Due to Lebesgue dominated convergence theorem and (4.4), further together with $\theta_n \in \varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$, it follows that (4.5) converges to 0 as $M \to \infty$. With the similar method, one can derive that

$$\frac{1}{\mu([-M, M])} \int_{[-M,M]} E|\Theta^{12}(t)|^p d\mu(t) \to 0, \quad \text{as } M \to \infty.$$

By the compactness of K, one can gain

$$
\begin{aligned}
&\frac{1}{\mu([-M,M])}\int_{[-M,M]}E|\Theta^{13}(t)|^p d\mu(t)\\
=&\frac{1}{\mu([-M,M])}\int_{[-M,M]}E\left|\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_{1,2}(\varrho,\varpi(\varrho))d\varrho\right|^p d\mu(t)\\
\leq&\frac{2^{p-1}}{\mu([-M,M])}\int_{[-M,M]}E\left|\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}(G_{1,2}(\varrho,\varpi(\varrho))-G_{1,2}(\varrho,\varpi_i))d\varrho\right|^p d\mu(t)\\
&+\frac{2^{p-1}}{\mu([-M,M])}\int_{[-M,M]}E\left|\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_{1,2}(\varrho,\varpi_i)d\varrho\right|^p d\mu(t)\\
\leq&\frac{2^p L_1\widehat{k}^p\bar{\iota}_1\varepsilon}{p\delta'}+2^{p-1}\widehat{k}^p\bar{\iota}_1\Sigma_{i=1}^m\int_0^{+\infty}e^{-\frac{p\delta'\varrho}{2}}\frac{1}{\mu([-M,M])}\int_{[-M,M]}E\left|G_{1,2}(\varrho,\varpi_i)\right|^p d\varrho d\mu(t).
\end{aligned}
$$

Since $G_{1,2}\in\varepsilon(\mathbb{R}\times L_G^p(\Omega),L_G^p(\Omega),\mu)$ from (H_5) and $\varepsilon(\mathbb{R}\times L_G^p(\Omega),L_G^p(\Omega),\mu)$ is translation invariant from (H_6) and [5, Section 3], one achieves the consequence

$$\frac{1}{\mu([-M,M])}\int_{[-M,M]}E|\Theta^{13}(t)|^p d\mu(t)\to 0,\quad \text{as } M\to\infty$$

by the arbitrariness of ε and the Lebesgue dominated convergence theorem. Similarly, one can deduce that

$$\frac{1}{\mu([-M,M])}\int_{[-M,M]}E|\Theta^{14}(t)|^p d\mu(t)\to 0,\ \text{as } M\to\infty.$$

Additionally, by Proposition 2.1, $\theta_n\in\varepsilon(\mathbb{R},L_G^p(\Omega),\mu)$ and (4.4), it holds that

$$
\begin{aligned}
&\frac{1}{\mu([-M,M])}\int_{[-M,M]}E|\Theta^{21}(t)|^p d\mu(t)\\
=&\frac{1}{\mu([-M,M])}\int_{[-M,M]}E\left|\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}(G_2(\varrho,\xi_n(\varrho))-G_2(\varrho,\varpi(\varrho)))dB_\varrho\right|^p d\mu(t)\\
\leq&C_p\overline{\sigma}^p\widehat{k}^p\bar{\iota}_0\int_0^{+\infty}e^{-\frac{p\delta'\varrho}{2}}\frac{1}{\mu([-M,M])}\int_{[-M,M]}E\left|G_2(\varrho,\xi_n(\varrho))-G_2(\varrho,\varpi(\varrho))\right|^p d\mu(t)d\varrho\to 0,\\
&\text{as } M\to\infty.
\end{aligned}
$$

Similarly,

$$\frac{1}{\mu([-M,M])}\int_{[-M,M]}E|\Theta^{22}(t)|^p d\mu(t)\to 0,\ \text{as } M\to\infty.$$

Again by the compactness of K, $G_{2,2}\in\varepsilon(\mathbb{R}\times L_G^p(\Omega),L_G^p(\Omega),\mu)$ and the translation invariance of $\varepsilon(\mathbb{R}\times L_G^p(\Omega),\ L_G^p(\Omega),\mu)$, one obtains that

$$
\begin{aligned}
&\frac{1}{\mu([-M,M])}\int_{[-M,M]}E|\Theta^{23}(t)|^p d\mu(t)\\
=&\frac{1}{\mu([-M,M])}\int_{[-M,M]}E\left|\int_{-\infty}^{t}S(t-\varrho)\mathcal{P}G_{2,2}(\varrho,\varpi(\varrho))dB_\varrho\right|^p d\mu(t)
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{2^{p-1} C_p \overline{\sigma}^p \widehat{k}^p \overline{\iota}_0}{\mu([-M,M])} \int_{[-M,M]} \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} E \left| G_{2,2}(\varrho, \varpi(\varrho)) - G_{2,2}(\varrho, \varpi_i) \right|^p d\varrho \right) d\mu(t) \\
& + \frac{2^{p-1} C_p \overline{\sigma}^p \widehat{k}^p \overline{\iota}_0}{\mu([-M,M])} \int_{[-M,M]} \left(\int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} E \left| G_{2,2}(\varrho, \varpi_i) \right|^p d\varrho \right) d\mu(t), \\
\leq & \frac{2^{p} C_p \overline{\sigma}^p \widehat{k}^p \overline{\iota}_0}{p\delta'} \varepsilon + \frac{2^{p-1} C_p \overline{\sigma}^p \widehat{k}^p \overline{\iota}_0}{\mu([-M,M])} \int_{-\infty}^{t} e^{-\frac{p\delta'(t-\varrho)}{2}} \int_{[-M,M]} E \left| G_{2,2}(\varrho, \varpi_i) \right|^p d\varrho d\mu(t) \to 0, \\
& \text{as } M \to \infty.
\end{aligned}
$$

Similarly, one can deduce

$$
\frac{1}{\mu([-M,M])} \int_{[-M,M]} E \left| \Theta^{24}(t) \right|^p d\mu(t) \to 0 \text{ as } M \to \infty.
$$

Processing like $\Theta^{21}, \Theta^{22}, \Theta^{23}, \Theta^{24}$, the results

$$
\frac{1}{\mu([-M,M])} \int_{[-M,M]} E \left| \Theta^{3j}(t) \right|^p d\mu(t) \to 0 \text{ as } M \to \infty, \ j = 1,2,3,4
$$

can be easily deduced, so we omit it here for simplicity.

Note that the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ converges to ξ in $BUC(\mathbb{R}, L_G^p(\Omega))$. Set $\theta = \xi - \varpi$. Then $\theta_n \to \theta$. As a consequence, for any $\varepsilon > 0$, there is a positive integer n_0 such that

$$
\sup_{t\in\mathbb{R}} E \left| \theta(t) - \theta_n(t) \right|^p < \varepsilon, \quad \text{for} \quad n \geq n_0.
$$

Therefore, one can conclude that

$$
\begin{aligned}
& \frac{1}{\mu([-M,M])} \int_{[-M,M]} E \left| \theta(t) \right|^p d\mu(t) < 2^{p-1} \varepsilon + \frac{2^{p-1}}{\mu([-M,M])} \int_{[-M,M]} E \left| \theta_n(t) \right|^p d\mu(t), \\
& n \geq n_0,
\end{aligned}
$$

which means that $\theta \in \varepsilon(\mathbb{R}, L_G^p(\Omega), \mu)$. Hence $\xi = \varpi + \theta$ is the pseudo almost periodic solution in distribution sense. Further, similar to the proof of p-distribution almost periodic solution in Theorem 4.2, one obtains that the distribution pseudo almost periodic solution ξ is p-distribution pseudo almost periodic. So one completes the proof. $\square$

5. Examples

In this section, some examples are provided to illustrate obtained results.

Example 5.1. Consider the following SDE:

$$
\begin{aligned}
d\phi = & \left(-3\phi + \frac{\cos\tau + \cos\sqrt{3}\tau}{5} \cdot \frac{\phi}{\phi^2 + 3} \right) d\tau + \frac{1}{3} \phi \sin(\sin\sqrt{2}\tau + \cos\tau) dB_\tau \\
& + \frac{1}{2} \phi \cos \left(\frac{\sin\tau + \sin\sqrt{2}\tau}{2} \right) d\langle B \rangle_\tau \\
= & (A\phi + Q_1(\tau,\phi)) d\tau + Q_2(\tau,\phi) dB_\tau + Q_3(\tau,\phi) d\langle B \rangle_\tau, \qquad (5.1)
\end{aligned}
$$

where B_τ is a one-dimensional G-Browian motion, and $\langle B\rangle_\tau$ is a second variation. Then system (5.1) admits a globally asymptotically stable quasi-periodic solution in p-distribution sense.

Actually, the semigroup generated by A is exponentially stable on $L^p_G(\Omega)$ with $k=1$ and $\delta=3$. Q_1, Q_2, Q_3 are also quasi-periodic in τ uniformly for ϕ on each bounded subset of $L^p_G(\Omega)$, hence they are jointly quasi-periodic. Additionally, $\max\{Lip(Q_1), Lip(Q_2), Lip(Q_3)\} \le \frac{1}{2}$. In conclusion, all conditions of Theorem 3.2 and Theorem 3.4 are met. Hence (5.1) has a unique global L^p-bounded solution by Theorem 3.2. Furthermore, this solution is quasi-periodic in p-distribution sense by Corollary 3.3, and is globally asymptotically stable by Theorem 3.4.

Example 5.2. Consider the following system on the interval $[0,1]$:

$$\begin{aligned}
&dZ(\tau,\varphi)\\
=&F(\tau,\varphi)Z(\tau,\varphi)d\tau + \frac{\sin\tau+\sin\sqrt{2}\tau}{4}d\tau + \frac{\varphi}{\varphi^2+3}.\sin\left(\frac{1}{2+cos\tau+cos\sqrt{3}\tau}\right)dB_\tau\\
&+\frac{\varphi}{\varphi^2+1}\cos\left(\frac{1}{2+\sin\tau+\sin\sqrt{3}\tau}\right)d\langle B\rangle_\tau\\
=&F(\tau,\varphi)Z(\tau,\varphi)d\tau + \Sigma_1(\tau,Z(\tau,\varphi))d\tau + \Sigma_2(\tau,Z(\tau,\varphi))dB_\tau + \Sigma_3(\tau,Z(\tau,\varphi))d\langle B\rangle_\tau,
\end{aligned} \tag{5.2}$$

$Z(\tau,0)=Z(\tau,1)=0$, $\tau>0$, where B_τ is a one-dimensional G-Brownian motion. Set F the Laplace operator, so $F: D(F)=H^2(0,1)\cap H^1_0(0,1)\to L^2(0,1)$. Denote $H=L^2(0,1)$ and by $\|\cdot\|$ the norm on H. Then in global sense, system (5.2) admits a asymptotically stable p-distribution Levitan almost periodic solution.

In fact, the system (5.2) can be expressed as follows:

$$d\beta(\tau) = (F\beta(\tau)+\Xi_1(\tau,\beta(\tau)))d\tau + \Xi_2(\tau,\beta(\tau))dB_\tau + \Xi_3(\tau,\beta(\tau))d\langle B\rangle_\tau \tag{5.3}$$

on the Hilbert space $L^p_G(\Omega)$, where $\beta(\tau) := Z(\tau,\cdot)$, $\Xi_1(\tau,\beta(\tau)) := \Sigma_1(\tau,Z(\tau,\cdot))$, $\Xi_2(\tau,\beta(\tau)) := \Sigma_2(\tau,Z(\tau,\cdot))$, $\Xi_3(\tau,\beta(\tau)) := \Sigma_3(\tau,Z(\tau,.))$. The operator F possesses eigenvalues $\{-n^2\pi^2\}_{n=1}^\infty$ and produces a semigroup $S(\tau)$ on $L^p_G(\Omega)$ fulfilling $\|S(\tau)\|\le e^{-\pi^2\tau}$ for $\tau\ge 0$. Since

$$\max\{Lip(\Xi_1), Lip(\Xi_2), Lip(\Xi_3)\}\le 1,$$

it is immediate to check that $\Xi_j, j=1,2,3$ satisfy linear growth condition and Lipschtiz condition. Besides, one needs to verify that Ξ_2 is continuous for τ uniformly on φ in every bounded subset of $L^p_G(\Omega)$. Indeed, because Ξ_2 is bounded, for given $\alpha>0$ one obtains

$$\sup_{\tau\in\mathbb{R},\|Z\|\le M}\int_{[0,1]}|\Xi_2(\tau,Z(\varphi))|^{p+\alpha}d\varphi<\infty$$

for each $M>0$. Hence the family $\{|\Xi_2(\tau,Z(\varphi))|^p : \tau\in\mathbb{R}, \|Z\|\le M\}$ of functions is uniformly integrable for φ on $[0,1]$, that is for $\tau_n\to\tau$,

$$\int_{[0,1]}|\Xi_2(\tau_n,Z(\varphi))-\Xi_2(\tau,Z(\varphi))|^p d\varphi$$

$$\leq 2^{p-1}\int_{[0,1]\cap\mathcal{N}_k}|\Xi_2(\tau_n,Z(\varphi))-\Xi_2(\tau,Z(\varphi))|^p d\varphi$$
$$+\,2^{p-1}\int_{[0,1]\setminus\mathcal{N}_k}|\Xi_2(\tau_n,Z(\varphi))-\Xi_2(\tau,Z(\varphi))|^p d\varphi$$

is small enough for large enough $k>0$, where $\mathcal{N}_k := \{\varphi\in[0,1] : \|Z(\varphi)\|\leq k\}$. Ξ_1 is quasi periodic, Ξ_2 and Ξ_3 are Levitan almost periodic for τ uniformly concerning for φ. Thus by Theorem 3.2 and Corollary 3.3, the system (5.3) possesses a unique p-distribution Levitan periodic; By Theorem 3.4, this p-distribution solution is is globally asymptotical stable and bounded globally.

Example 5.3. Take into account the evolution equation with G-Brownian motion:

$$d\psi = (D\psi+\Lambda_1(\tau,\psi))d\tau + \Lambda_2(\tau,\psi)dB_\tau + \Lambda_3(\tau,\psi)d\langle B\rangle_\tau, \tag{5.4}$$

where B_τ is a two-dimensional G-Browian motion, $\langle B\rangle_\tau$ is a second variation, and

$$D=\begin{bmatrix}8 & 0\\ 0 & -6\end{bmatrix},\quad \Lambda_1(\tau,\psi)=\begin{bmatrix}0\\ \frac{\sin 2\pi\tau+\sin\sqrt{2}\tau}{4}\psi+\frac{1}{2}e^{-|\tau|}\sin\psi\end{bmatrix},$$
$$\Lambda_2(\tau,\psi)=\begin{bmatrix}0\\ (\sin\tau+\sin 2\sqrt{2}\pi\tau)\psi+e^{-|\tau|}\cos\psi\end{bmatrix},$$
$$\Lambda_3(\tau,\psi)=\begin{bmatrix}0\\ \alpha_2(\cos\tau+\cos\pi\tau)\psi+\nu_2 e^{-|\tau|}\cos\psi^2\end{bmatrix}.$$

It is easy to see that $\frac{\sin 2\pi\tau+\sin\sqrt{2}\tau}{4}\psi$ is almost periodic. Note that

$$\lim_{M\to\infty}\frac{1}{\mu([-M,M])}\int_{-M}^{0}E\left|\frac{1}{2}e^{-|\tau|}\sin\psi\right|^p d\mu(\tau)\leq \lim_{M\to\infty}\frac{1-e^{-(p+1)M}}{2^p(1+p)(1-e^{-M}+M)}=0$$

and

$$\lim_{M\to\infty}\frac{1}{\mu([-M,M])}\int_{0}^{M}E\left|\frac{1}{2}e^{-|\tau|}\sin\psi\right|^p d\mu(\tau)\leq \lim_{M\to\infty}\frac{1-e^{-Mp}}{2^p p(1-e^{-M}+M)}=0.$$

Hence $\frac{1}{2}e^{-|\tau|}\sin\psi\in\varepsilon(\mathbb{R},L_G^p(\Omega),\mu)$. As a result, $\Lambda_1\in PAP(\mathbb{R}\times L_G^p(\Omega),L_G^p(\Omega),\mu)$. Similarly, $\Lambda_2,\Lambda_3\in PAP(\mathbb{R}\times L_G^p(\Omega),L_G^p(\Omega),\mu)$. By Theorem 4.3, system (5.4) possesses a p-distribution pseudo almost periodic solution belonging to $L_G^p(\Omega)$.

6. Conclusion and discussion

As we know, almost all results about Poisson stable solutions in distribution are restricted to deterministic equations or SDEs with linear expectation; and in the framework of sublinear expectation, the almost periodic solutions and almost automorphic solutions are just studied in the case that SDEs satisfy exponential stability. However, in the framework of sublinear expectation, few results on the Poisson stable solutions for SDEs and (pseudo) almost periodic solutions for SDEs with exponential dichotomy. This paper is devoted to Poisson stable (in particular, pseudo-recurrent, pseudo-periodic, almost recurrent, Levitan almost periodic, Birkhoff recurrent, Bohr almost automorphic, Bohr almost periodic, quasi-periodic with a limited spectrum, τ-periodic, stationary) solutions in p-distribution for SDEs with G-Brownian motion. Further, some sufficient criteria for the asymptotic stability of the corresponding Poisson stable solutions are obtained. When the semi-linear SDEs satisfy exponential dichotomy, some existence theorems of (pseudo) almost periodic solutions in p-distribution are set up. All results obtained above generalize the consequences of paper [16, 44] in some sense. In the future, a lemma different from [19, Lemma 2.5] is expected to achieve so that Poisson stable solutions can be discussed in the case that SDEs satisfy exponential dichotomy.

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