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THE EXISTENCE OF SOLUTIONS OF INTEGRAL BOUNDARY VALUE PROBLEM FOR HILFER FRACTIONAL DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN AT RESONANCE*

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Abstract By using the extension of the continuous theorem of Ge and Ren, the solvability of integral boundary value problems for Hilfer fractional differential equations with p-Laplacian is investigated. In order to get this conclusion, we construct appropriate Banach spaces and define suitable operators. At the end of the article, an example is given to illustrate our main results.

Keywords Hilfer fractional derivative, continuous theorem, p-Laplacian, resonance, boundary value problem.

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1. Introduction

The fractional differential equations have become an important research field because of the in-depth development of fractional calculus theory and its wide applications in many sciences such as physics, engineering, biology and so on [1, 3, 5, 10, 14]. The boundary value problem of fractional differential equations with p-Laplacian plays an indispensable role in the theory and application of mathematics and physics, so it has been concerned by many experts and scholars [6–8, 13, 18, 19].

There are various definitions of fractional derivatives, such as our common Riemann-Liouville and Caputo fractional derivatives [2, 10]. On this basis, a more generalized fractional derivative "Hilfer" has been studied [5]. The Hilfer fractional derivative is an extension of the Riemann-Liouville and Caputo fractional derivatives. Therefore, fractional differential equations with Hilfer derivative have gradually become a research hotspot [11, 16, 17].

Recently, the existence of solutions for the p-Laplacian boundary value problem has been considered in [9, 20, 21].

In [20], the multiple positive solutions for nonlinear high-order Riemann-Liouville fractional differential equations boundary value problems with p-Laplacian operator

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has been studied by Leggett-Williams fixed point theorem:

$$\begin{cases} {}_0^R D_t^\alpha(\varphi_p({}_0^R D_t^\alpha u(t))) = f(t, u(t), {}_0^R D_t^\alpha u(t)),\ 0 \le t \le 1, \\ u^{(i)}(0) = 0, \quad [\varphi_p({}_0^R D_t^\alpha)]^{(i)}(0) = 0, \quad i = 0, 1, 2 \cdots, n-2, \\ [{}_0^R D_t^\beta]_{t=1} = 0, \quad 0 < \beta \le \alpha - 1, \\ [{}_0^R D_t^\beta(\varphi_p({}_0^R D_t^\alpha u(t)))]_{t=1} = 0, \end{cases}$$

where $n-1 < \alpha \le n$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, ${}_0^R D_t^\alpha$ is the standard Riemann-Liouville fractional derivative.

Zhang et al. [21] have obtained the solvability for a fractional p-Laplacian multi-point boundary value problem at resonance on infinite interval by Mawhin's continuation theorem:

$$\begin{cases} (\phi_p(D_{0+}^\alpha x(t)))' + f(t, x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^\alpha x(t)), \quad t \in (0, \infty), \\ x(0) = x'(0) = 0,\ \phi_p(D_{0+}^\alpha x(+\infty)) = \displaystyle\sum_{i=1}^n \alpha_i \phi_p(D_{0+}^\alpha x(\xi_i)), \end{cases}$$

where $1 < \alpha < 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$, $\sum_{i=1}^n \alpha_i = 1$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$.

Jiang [9] have considered the solvability of fractional differential equations with p-Laplacian:

$$\begin{cases} D_{0+}^\beta(\varphi_p(D_{0+}^\alpha u))(t) + f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^\alpha u(t)) = 0, \\ u(0) = D_{0+}^\alpha u(0) = 0,\ u(1) = \displaystyle\int_0^1 h(t)u(t)dt, \end{cases}$$

where $0 < \beta \le 1$, $1 < \alpha \le 2$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\int_0^1 h(t)t^{\alpha-1}dt = 1$, D_{0+}^α is the Riemann-Liouville fractional derivative.

In general, we use Mawhin's continuous theorem [15] to study the existence of solutions of abstract equation $Lx = Nx$, where L is a noninvertible linear operator. Ge and Ren generalized the Mawhin's continuous theorem and got the existence of solutions when L is a noninvertible nonlinear operator [4]. This is an effective tool to solve the p-Laplacian boundary value problems at resonance. Using this theorem, the author must define two operators P and Q, where P is a projection operator and Q is not a projection operator, but it is difficult to construct the operator Q in many p-Laplacian boundary value problems. In order to expand the basic theory of boundary value problems of fractional differential equations and obtain more generalized results, we will prove that when Q is not a projector but satisfies certain conditions, the solution of the equation $Lx = Nx$ exists. Next, we consider the existence of solutions for Hilfer fractional differential equations with p-Laplacian at resonance:

$$\begin{cases} D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)) = f\left(t, t^{2-\gamma_2}u(t)\right), \quad t \in (0, 1], \\ D_{0+}^{\alpha_2,\beta_2}u(0) = D_{0+}^{\gamma_2-1}u(0) = 0,\ u(1) = \displaystyle\int_0^1 h(t)u(t)dt, \end{cases} \tag{1.1}$$

where $D_{0+}^{\alpha_i,\beta_i}$ is Hilfer fractional derivative of order α_i and type β_i, $i-1 < \alpha_i < i$, $0 \le \beta_i \le 1$, $\gamma_i = \alpha_i + i\beta_i - \alpha_i\beta_i$, $i = 1, 2$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$f \in C\big((0,1] \times R, R\big)$ with the resonant condition $\int_0^1 h(t)t^{\gamma_2-2}dt = 1$, under this resonant condition the associated linear operator is uninvertible.

As far as we know, this is the first paper that uses the extension of the continuous theorem of Ge and Ren to investigate the boundary value problem of Hilfer fractional differential equations with p-Laplacian at resonance.

2. Preliminaries

Definition 2.1 ([4]). Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. A continuous operator L: $X \cap domL \to Y$ is said to be quasi-linear, if

(i) $ImL := L(X \cap domL)$ is a closed subset of Y,

(ii) $KerL := \{x \in X \cap domL : Lx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,

where $domL$ denotes the domain of the operator L.

Let $X_1 = KerL$ and X_2 be the complement space of X_1 in X, then $X = X_1 \bigoplus X_2$. Let $P : X \to X_1$ be projector and $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$.

Definition 2.2 ([4]). Suppose that $N_\lambda : \overline{\Omega} \to Y, \lambda \in [0,1]$, is a continuous and bounded operator. Denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} : Lx = N_\lambda x\}$. N_λ is said to be L-quasi-compact in $\overline{\Omega}$ if there exists a vector subspace Y_1 of Y satisfying $dimY_1 = dimX_1$ and two operators Q and R such that for $\lambda \in [0,1]$,

(a) $KerQ = ImL$,

(b) $QN_\lambda x = \theta, \lambda \in (0,1) \Leftrightarrow QNx = \theta$,

(c) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)\mid_{\Sigma_\lambda} = (I-P)\mid_{\Sigma_\lambda}$,

(d) $L[P + R(\cdot,\lambda)] = (I-Q)N_\lambda$,

where $Q : Y \to Y_1$, $QY = Y_1$ is continuous, bounded and satisfies $Q(I-Q) = 0$ and $R : \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact.

Lemma 2.1 (Theorem 2.1, [9]). *Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, and $\Omega \subset X$ be an open and bounded nonempty set. Suppose $L : domL \cap X \to Y$ is a quasi-linear operator and that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is L-quasi-compact. In addition, if the following conditions hold:*

(a) $Lx \neq N_\lambda x$, $\forall x \in \partial\Omega \cap domL$, $\lambda \in (0,1)$,

(b) $\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$,

then the abstract equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$, where $N = N_1$, $J : ImQ \to KerL$ is a homeomorphism with $J(\theta) = \theta$.

Definition 2.3 ([10]). The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,+\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds.$$

Definition 2.4 ([10]). The left-sided Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, +\infty) \to R$ is given by

$$D^{\alpha}_{0+} y(t) = \frac{d^n}{dt^n}(I^{n-\alpha}_{0+} y)(t),$$

where $n - 1 < \alpha < n$.

Definition 2.5 ([5]). The left-sided Hilfer fractional derivative of order α and type β for a function $y : (0, +\infty) \to R$ is given by

$$D^{\alpha,\beta}_{a+} y(t) = I^{\beta(n-\alpha)}_{a+} \frac{d^n}{dt^n}(I^{(1-\beta)(n-\alpha)}_{a+} y)(t),\ n - 1 < \alpha < n,\ 0 \leq \beta \leq 1.$$

Remark 2.1. (1) The operator $D^{\alpha,\beta}_{a+}$ can also be written as $D^{\alpha,\beta}_{a+} = I^{\beta(n-\alpha)}_{a+} D^{\gamma}_{a+}$, $\gamma = \alpha + n\beta - \alpha\beta$.

(2) If $\beta = 0$, then the left-sided Riemann-Liouville fractional derivative can be presented as $D^{\alpha}_{a+} = D^{\alpha,0}_{a+}$.

(3) If $\beta = 1$, then the left-sided Caputo fractional derivative can be presented as ${}^{C}D^{\alpha}_{a+} = D^{\alpha,1}_{a+}$.

Definition 2.6 ([10]). For $0 \leq \gamma < 1$, the weighted space of continuous functions y is defined by

$$C_{\gamma}(0, 1] := \{y \mid t^{\gamma} y(t) \in C[0, 1]\},$$

and the norm is $\| y \|_{c_{\gamma}} = \| t^{\gamma} y \|_{c}$. Then, $C_{\gamma}(0, 1]$ is the Banach space.

Lemma 2.2 ([10]). *If $\alpha > 0, \beta > 0$, and $y \in L^1[0, 1]$ for $t \in [0, 1]$, then*

$$I^{\alpha}_{0+} I^{\beta}_{0+} y(t) = I^{\alpha+\beta}_{0+} y(t), \quad D^{\alpha}_{0+} I^{\alpha}_{0+} y(t) = y(t).$$

Lemma 2.3 ([10]). *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $y \in C_{\gamma}(0, 1]$ and $I^{1-\alpha}_{0+} y \in C^{1}_{\gamma}(0, 1]$, then the following holds*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} y(t) = y(t) - \frac{I^{1-\alpha}_{0+} y(0)}{\Gamma(\alpha)} t^{\alpha-1},\ t \in (0, 1].$$

Lemma 2.4 ([10]). *For $n - 1 < \alpha \leq n$, $n \in N$, the general solution of the fractional differential equation $D^{\alpha}_{0+} u(t) = 0$ is given by*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, $n = [\alpha] + 1$.

Lemma 2.5 ([10]). *If $\alpha > 0$, $\beta > -1$, and $\beta \neq \alpha - i$, $i = 1, 2, \cdots, [\alpha] + 1$, then*

$$D^{\alpha}_{0+} t^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta-\alpha}, \quad D^{\alpha}_{0+} t^{\alpha-i} = 0.$$

Lemma 2.6 ([10]). *If $\alpha > \beta > 0$, and $y \in L_1(\mathbb{R}^+)$, then*

$$D^{\beta}_{0+} I^{\alpha}_{0+} y(t) = I^{\alpha-\beta}_{0+} f(t).$$

In particular, when $\beta = k \in \mathbb{N}$ and $\alpha > k$, then

$$\frac{d^k}{dt^k} I^{\alpha}_{0+} y(t) = I^{\alpha-k}_{0+} f(t).$$

Lemma 2.7 ([12]). *For any $u, v \geq 0$, then*

(1) $\varphi_p(u+v) \leq \varphi_p(u) + \varphi_p(v),\ 1 < p \leq 2,$

(2) $\varphi_p(u+v) \leq 2^{p-2}(\varphi_p(u) + \varphi_p(v)),\ p \geq 2,$

where $\varphi_p(s) = |s|^{p-2}s = s^{p-1},\ s \geq 0.$

Remark 2.2. $I_{0+}^{2-\gamma_2}u(0) = \lim\limits_{t\to 0^+} I_{0+}^{2-\gamma_2}u(t).$

3. Main results

Take $X = C_{2-\gamma_2}(0,1]$, $Y = C[0,1]$, with norms $\parallel u \parallel_X = \max\limits_{t\in[0,1]} |t^{2-\gamma_2}u(t)|$, $\parallel y \parallel_Y = \max\limits_{t\in[0,1]} |y(t)|$. We can easily get that $(X, \parallel\cdot\parallel)$ and $(Y, \parallel\cdot\parallel)$ are Banach spaces.

In order to obtain our main results, we always suppose that the following conditions holds:

(H_1) $h(t) \geq 0,\ t \in (0,1]$.

Define operators $L : domL \cap X \to Y$ and $N_\lambda : X \to Y$ as follows

$$Lu(t) = D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)), \quad N_\lambda u(t) = \lambda f\left(t, t^{2-\gamma_2}u(t)\right),\ t \in (0,1],\ \lambda \in [0,1],$$

where

$$\begin{aligned} domL = & \Big\{u(t) | u(t) \in X, D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)) \in Y, D_{0+}^{\alpha_2,\beta_2}u(0) = D_{0+}^{\gamma_2-1}u(0) = 0, \\ & u(1) = \int_0^1 h(t)u(t)dt\Big\}. \end{aligned}$$

Lemma 3.1. *Assuming the resonance condition holds, then L is a quasi-linear operator.*

Proof. It is easy to get that $KerL = \{u \in domL | u(t) = ct^{\gamma_2-2}, c \in R\}$.

For $y \in ImL$, there exists $u \in domL$ such that $D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)) = y(t)$. According to Remark 2.1, we get

$$I_{0+}^{\beta_1(1-\alpha_1)} D_{0+}^{\gamma_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)) = y(t). \tag{3.1}$$

Thus, applying $D_{0+}^{\beta_1(1-\alpha_1)}$ to the both sides of (3.1), and by Lemma 2.4, we have

$$D_{0+}^{\alpha_2,\beta_2}u(t) = \varphi_q(I_{0+}^{\alpha_1}y(t) + c_1t^{\gamma_1-1}).$$

Since $D_{0+}^{\alpha_2,\beta_2}u(0) = 0$, we can get

$$D_{0+}^{\alpha_2,\beta_2}u(t) = \varphi_q(I_{0+}^{\alpha_1}y(t)). \tag{3.2}$$

Applying $D_{0+}^{\beta_2(2-\alpha_2)}$ to the both sides of (3.2), and by Lemma 2.4, we obtain

$$u(t) = I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t)) + c_2t^{\gamma_2-1} + c_3t^{\gamma_2-2}.$$

Since $D_{0+}^{\gamma_2-1}u(0)=0$, and $u(1)=\int_0^1 h(t)u(t)dt$, we can get

$$I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))|_{t=1}-\int_0^1 h(t)I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))dt=0. \tag{3.3}$$

Consequently,

$$ImL\subseteq\Big\{y\in Y|I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))|_{t=1}-\int_0^1 h(t)I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))dt=0\Big\}.$$

On the other hand, if $y\in Y$ satisfies (3.3), take $u(t)=I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))+ct^{\gamma_2-2}$. It is easy to prove that u satisfies the boundary conditions of the problem (1.1), and we have

$$\begin{aligned}Lu(t)&=I_{0+}^{\beta_1(1-\alpha_1)}D_{0+}^{\gamma_1}\varphi_p\Big(I_{0+}^{\beta_2(2-\alpha_2)}D_{0+}^{\gamma_2}I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))+cI_{0+}^{\beta_2(2-\alpha_2)}D_{0+}^{\gamma_2}t^{\gamma_2-2}\Big)\\&=I_{0+}^{\beta_1(1-\alpha_1)}D_{0+}^{\beta_1(1-\alpha_1)}y(t)\\&=y(t).\end{aligned}$$

Therefore,

$$ImL\supseteq\Big\{y\in Y|I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))|_{t=1}-\int_0^1 h(t)I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))dt=0\Big\}.$$

In summary, we get

$$ImL=\Big\{y\in Y|I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))|_{t=1}-\int_0^1 h(t)I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))dt=0\Big\}.$$

Obviously, $ImL\subset Y$ is closed. So, L is quasi-linear. The proof is completed. □

Define the operator $P:X\to KerL$ by

$$Pu(t)=\frac{I_{0+}^{2-\gamma_2}u(0)}{\Gamma(\gamma_2-1)}t^{\gamma_2-2}.$$

It is clear that $P^2u=Pu$ and $ImP=KerL$, $X=KerL\oplus KerP$. So, $P:X\to KerL$ is a projector.

Define the operator $Q:Y\to R$ by

$$Qy(t)=c,$$

where c satisfies

$$\begin{aligned}&\int_0^1(1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y(u)-c)du\Big)ds\\&-\int_0^1 h(t)\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y(u)-c)du\Big)dsdt=0.\end{aligned} \tag{3.4}$$

We will prove that c is the unique constant satisfying (3.4).

For $y\in Y$, let

$$F(c)=\int_0^1(1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y(u)-c)du\Big)ds$$

$$-\int_0^1 h(t)\int_0^t (t-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y(u)-c)du\Big)dsdt.$$

Therefore,

$$\begin{aligned}
F(c) &= \int_0^1 h(t)t^{\gamma_2-2}\int_0^1 (1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y(u)-c)du\Big)dsdt\\
&\quad-\int_0^1 h(t)\int_0^t (t-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y(u)-c)du\Big)dsdt\\
&=\int_0^1 h(t)\int_0^t [t^{\gamma_2-2}(1-s)^{\alpha_2-1}-(t-s)^{\alpha_2-1}]\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y(u)-c)du\Big)dsdt\\
&\quad+\int_0^1 h(t)t^{\gamma_2-2}\int_t^1 (1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y(u)-c)du\Big)dsdt. \quad (3.5)
\end{aligned}$$

Obviously, $F(c)$ is continuous and strictly decreasing in R. We make $c_1=\min\limits_{t\in(0,1]} y(t)$, $c_2=\max\limits_{t\in(0,1]} y(t)$. It is easy to see that $F(c_1)\geq 0$, $F(c_2)\leq 0$, then, there exists a unique constant $c\in[c_1,c_2]$ such that $F(c)=0$. Furthermore, $Q(\Omega)$ is bounded if $\Omega\subset Y$ is bounded, *i.e.* Q is bounded.

By the definition of Q, we can easily know that Q is not a projector but satisfies $Q(I-Q)Y=Q(Y-QY)=0$, $y\in Y$.

Lemma 3.2. *Q is continuous in Y.*

Proof. For $y_1,y_2\in Y$, assume $Qy_1=c_1$, $Qy_2=c_2$. Noticing $h(t)\geq 0$ and that φ_q is strictly increasing. If $c_2-c_1>\max\limits_{t\in(0,1]}(y_2(t)-y_1(t))$, then

$$\begin{aligned}
0&=\int_0^1 h(t)\int_0^t [t^{\gamma_2-2}(1-s)^{\alpha_2-1}-(t-s)^{\alpha_2-1}]\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y_2(u)-c_2)du\Big)dsdt\\
&\quad+\int_0^1 h(t)t^{\gamma_2-2}\int_t^1 (1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y_2(u)-c_2)du\Big)dsdt\\
&=\int_0^1 h(t)\int_0^t [t^{\gamma_2-2}(1-s)^{\alpha_2-1}-(t-s)^{\alpha_2-1}]\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y_1(u)-c_1\\
&\quad+(y_2(u)-y_1(u))-(c_2-c_1))du\Big)dsdt\\
&\quad+\int_0^1 h(t)t^{\gamma_2-2}\int_t^1 (1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y_1(u)-c_1\\
&\quad+(y_2(u)-y_1(u))-(c_2-c_1))du\Big)dsdt\\
&<\int_0^1 h(t)\int_0^t [t^{\gamma_2-2}(1-s)^{\alpha_2-1}-(t-s)^{\alpha_2-1}]\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y_1(u)-c_1)du\Big)dsdt\\
&\quad+\int_0^1 h(t)t^{\gamma_2-2}\int_t^1 (1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y_1(u)-c_1)du\Big)dsdt=0.
\end{aligned}$$

A contradiction. On the other hand, if $c_2-c_1<\min\limits_{t\in(0,1]}(y_2(t)-y_1(t))$, then

$$0=\int_0^1 h(t)\int_0^t [t^{\gamma_2-2}(1-s)^{\alpha_2-1}-(t-s)^{\alpha_2-1}]\varphi_q\Big(\int_0^s (s-u)^{\alpha_1-1}(y_2(u)-c_2)du\Big)dsdt$$

$$
\begin{aligned}
&+\int_0^1 h(t)t^{\gamma_2-2}\int_t^1(1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y_2(u)-c_2)du\Big)dsdt\\
=&\int_0^1 h(t)\int_0^t[t^{\gamma_2-2}(1-s)^{\alpha_2-1}-(t-s)^{\alpha_2-1}]\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y_1(u)-c_1\\
&+(y_2(u)-y_1(u))-(c_2-c_1))du\Big)dsdt\\
&+\int_0^1 h(t)t^{\gamma_2-2}\int_t^1(1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y_1(u)-c_1\\
&+(y_2(u)-y_1(u))-(c_2-c_1))du\Big)dsdt\\
>&\int_0^1 h(t)\int_0^t[t^{\gamma_2-2}(1-s)^{\alpha_2-1}-(t-s)^{\alpha_2-1}]\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y_1(u)-c_1)du\Big)dsdt\\
&+\int_0^1 h(t)t^{\gamma_2-2}\int_t^1(1-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s(s-u)^{\alpha_1-1}(y_1(u)-c_1)du\Big)dsdt=0.
\end{aligned}
$$

A contradiction, too. So, we can get

$$
\min_{t\in(0,1]}(y_2(t)-y_1(t))\le c_2-c_1\le\max_{t\in(0,1]}(y_2(t)-y_1(t)), i.e.\ |c_2-c_1|\le\| y_2-y_1\|_c .
$$

Therefore, Q is continuous in Y. The proof is completed. □

Lemma 3.3. *Define an operator* $R : X\times[0,1]\to X_2$ *as*

$$
R(u,\lambda)(t)=\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}(N_\lambda u(r)-QN_\lambda u(r))dr\Big)ds,
$$

where $KerL\bigoplus X_2=X$.

Then $R:\overline{\Omega}\times[0,1]\to X_2$ *is continuous and compact, where* $\Omega\subset X$ *is an open bounded set.*

Proof. Obviously, R is continuous. Let A be any bounded set in X, for $\forall u\in A$, $\lambda\in[0,1]$, by the continuity of f and the boundedness of Q, we can get that there exist constants $k_1>0$, $k_2>0$ such that $|f(t,t^{2-\gamma_2}u(t))|\le k_1$, $|Qf|\le k_2$.

For $u\in\overline{\Omega}$,

$$
\begin{aligned}
&\Big|t^{2-\gamma_2}R(u,\lambda)(t)\Big|\\
=&\Big|\frac{t^{2-\gamma_2}}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}(N_\lambda u(r)-QN_\lambda u(r))dr\Big)ds\Big|\\
\le&\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}|N_\lambda u(r)-QN_\lambda u(r)|dr\Big)ds\\
\le&\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\frac{\lambda(k_1+k_2)}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}dr\Big)ds\\
\le&\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)ds\\
\le&\frac{1}{\Gamma(\alpha_2+1)}\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big).
\end{aligned}
$$

So, R is bounded in $\overline{\Omega}\times[0,1]$.

For $(u,\lambda)\in\overline{\Omega}\times[0,1]$, $0<t_1<t_2\le 1$, we have

$$
\begin{aligned}
&\Big|t_2^{2-\gamma_2}R(u,\lambda)(t_2)-t_1^{2-\gamma_2}R(u,\lambda)(t_1)\Big|\\
=&\Big|\frac{t_2^{2-\gamma_2}}{\Gamma(\alpha_2)}\int_0^{t_2}(t_2-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}(N_\lambda u(r)-QN_\lambda u(r))dr\Big)ds\\
&-\frac{t_1^{2-\gamma_2}}{\Gamma(\alpha_2)}\int_0^{t_1}(t_1-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}(N_\lambda u(r)-QN_\lambda u(r))dr\Big)ds\Big|\\
\le&\frac{t_2^{2-\gamma_2}}{\Gamma(\alpha_2)}\int_0^{t_2}(t_2-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}|N_\lambda u(r)-QN_\lambda u(r)|dr\Big)ds\\
&+\frac{t_1^{2-\gamma_2}}{\Gamma(\alpha_2)}\int_0^{t_1}(t_1-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}|N_\lambda u(r)-QN_\lambda u(r)|dr\Big)ds\\
\le&\frac{1}{\Gamma(\alpha_2)}\int_0^{t_1}[t_2^{2-\gamma_2}(t_2-s)^{\alpha_2-1}-t_1^{2-\gamma_2}(t_1-s)^{\alpha_2-1}]\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)ds\\
&+\frac{1}{\Gamma(\alpha_2)}\int_{t_1}^{t_2}t_2^{2-\gamma_2}(t_2-s)^{\alpha_2-1}\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)ds\\
\le&\frac{\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)}{\Gamma(\alpha_2)}\int_0^{t_1}\Big[t_2^{2-\gamma_2}(t_2-s)^{\alpha_2-1}-t_1^{2-\gamma_2}(t_2-s)^{\alpha_2-1}+t_1^{2-\gamma_2}(t_2-s)^{\alpha_2-1}\\
&-t_1^{2-\gamma_2}(t_1-s)^{\alpha_2-1}\Big]ds+\frac{\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)}{\Gamma(\alpha_2)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}ds\\
\le&\frac{\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)}{\Gamma(\alpha_2)}\Big[(t_2^{2-\gamma_2}-t_1^{2-\gamma_2})\int_0^{t_1}(t_2-s)^{\alpha_2-1}ds\\
&+t_1^{2-\gamma_2}\int_0^{t_1}(t_2-s)^{\alpha_2-1}-(t_1-s)^{\alpha_2-1}ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}ds\Big]\\
\le&\frac{\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)}{\Gamma(\alpha_2+1)}\Big[t_2^{\alpha_2}(t_2^{2-\gamma_2}-t_1^{2-\gamma_2})-(t_2-t_1)^{\alpha_2}(t_2^{2-\gamma_2}-t_1^{2-\gamma_2})+t_2^{\alpha_2}-(t_2-t_1)^{\alpha_2}\\
&-t_1^{\alpha_2}+(t_2-t_1)^{\alpha_2}\Big]\\
\le&\frac{\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)}{\Gamma(\alpha_2+1)}\Big[(t_2^{2-\gamma_2}-t_1^{2-\gamma_2})+(t_2^{\alpha_2}-t_1^{\alpha_2})\Big].
\end{aligned}
$$

So, $\{R(u,\lambda)\mid(u,\lambda)\in\overline{\Omega}\times[0,1]\}$ is equicontinuous. By Arzela-Ascoli Theorem, we get that $R:\Omega\times[0,1]\to X_2$ is compact. The proof is completed. □

Lemma 3.4. *Assume that $\Omega\subset X$ is an open and bounded set. Then N_λ is L-quasi-compact in $\overline{\Omega}$.*

Proof. It is clear that $ImP=KerL$, $dimKerL=dimImQ$, $Q(I-Q)=0$, $KerQ=ImL$, $R(\cdot,0)=0$ and that Definition 2.2(b) holds.

For $u\in\Sigma_\lambda=\{u\in\overline{\Omega}\mid Lu=N_\lambda u\}$, we can get $N_\lambda u\in ImL=KerQ$. Thus, we have $QN_\lambda u=0$ and $N_\lambda u=D_{0^+}^{\alpha_1,\beta_1}\varphi_p\Big(D_{0^+}^{\alpha_2,\beta_2}u\Big)$. It follows from $D_{0^+}^{\alpha_2,\beta_2}u(0)=$

$D_{0+}^{\gamma_2-1}u(0) = D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(0) = R(u,\lambda)(0) = 0$ that

$$\begin{aligned}
R(u,\lambda) &= I_{0+}^{\alpha_2}\varphi_q\Big(I_{0+}^{\alpha_1}(N_\lambda u(t) - QN_\lambda u(t))\Big) \\
&= I_{0+}^{\alpha_2}\varphi_q\Big(I_{0+}^{\alpha_1}I_{0+}^{\beta_1(1-\alpha_1)}D_{0+}^{\gamma_1}\varphi_p(I_{0+}^{\beta_2(2-\alpha_2)}D_{0+}^{\gamma_2}u(t))\Big) \\
&= I_{0+}^{\gamma_2}D_{0+}^{\gamma_2}u(t) = \frac{d}{dt}\{\frac{1}{\Gamma(\gamma_2+1)}\int_0^t (t-s)^{\gamma_2}(\frac{d}{ds})^2 I_{0+}^{2-\gamma_2}u(s)ds\} \\
&= \frac{d}{dt}\{\frac{1}{\Gamma(\gamma_2)}\int_0^t (t-s)^{\gamma_2-1}(\frac{d}{ds})I_{0+}^{2-\gamma_2}u(s)ds - \frac{D_{0+}^{\gamma_2-1}u(0)}{\Gamma(\gamma_2+1)}t^{\gamma_2}\} \\
&= \frac{d}{dt}\{\frac{1}{\Gamma(\gamma_2-1)}\int_0^t (t-s)^{\gamma_2-2}I_{0+}^{2-\gamma_2}u(s)ds - \frac{I_{0+}^{2-\gamma_2}u(0)}{\Gamma(\gamma_2)}t^{\gamma_2-1}\} \\
&= u(t) - \frac{I_{0+}^{2-\gamma_2}u(0)}{\Gamma(\gamma_2-1)}t^{\gamma_2-2} \\
&= (I-P)u,
\end{aligned}$$

i.e. Definition 2.2(c) holds.

For $u \in \overline{\Omega}$, we have

$$\begin{aligned}
&L[Pu(t) + R(u,\lambda)(t)] \\
=&D_{0+}^{\alpha_1,\beta_1}\varphi_p\Big(I_{0+}^{\beta_2(2-\alpha_2)}D_{0+}^{\gamma_2}\frac{I_{0+}^{2-\gamma_2}u(0)}{\Gamma(\gamma_2-1)}t^{\gamma_2-2}\Big) \\
&+ I_{0+}^{\beta_1(1-\alpha_1)}D_{0+}^{\gamma_1}\varphi_p\Big(I_{0+}^{\beta_2(2-\alpha_2)}D_{0+}^{\gamma_2}I_{0+}^{\alpha_2}\varphi_q\Big(I_{0+}^{\alpha_1}(N_\lambda u(t) - QN_\lambda u(t))\Big)\Big) \\
=&I_{0+}^{\beta_1(1-\alpha_1)}D_{0+}^{\gamma_1}I_{0+}^{\alpha_1}(N_\lambda u(t) - QN_\lambda u(t)) \\
=&N_\lambda u(t) - QN_\lambda u(t) = (I-Q)N_\lambda u(t),
\end{aligned}$$

i.e. Definition 2.2(d) holds. Therefore, N_λ is L-quasi-compact in $\overline{\Omega}$. The proof is completed. □

Theorem 3.1. *Suppose* (H_1) *and the following conditions hold:*

(H_2) *There exists a constant* $M > 0$ *such that one of the following inequalities holds:*

(1) $(t^{2-\gamma_2}u(t))f(t, t^{2-\gamma_2}u(t)) > 0,\ t \in (0,1],\ | t^{2-\gamma_2}u(t) |> M,$

(2) $(t^{2-\gamma_2}u(t))f(t, t^{2-\gamma_2}u(t)) < 0,\ t \in (0,1],\ | t^{2-\gamma_2}u(t) |> M.$

(H_3) *There exist nonnegative functions* $a(t), b(t) \in Y$*, such that*

$$\Big|f(t, t^{2-\gamma_2}u(t))\Big| \le a(t)\varphi_p(|t^{2-\gamma_2}u(t)|) + b(t),\ \ t \in (0,1],$$

where $\Gamma(\alpha_2+1) > \max\limits_{q\in(0,+\infty)}\{2^{q-1}\varphi_q(\frac{\|a\|_c}{\Gamma(\alpha_1+1)}), 2\varphi_q(\frac{\|a\|_c}{\Gamma(\alpha_1+1)})\}.$

Then the problem (1.1) *has at least one solution in* X*.*

Before we prove theorem 3.1, we show two Lemmas.

Lemma 3.5. *Let* $\Omega_1 = \{u | u \in domL\backslash KerL, Lu = N_\lambda u, \lambda \in (0,1)\}$*. Assume* (H_1)–(H_3) *hold. Then* Ω_1 *is bounded in* X*.*

Proof. Let $u \in \Omega_1$, we have $Lu = N_\lambda u$, $N_\lambda u \in ImL = KerQ$, we get $QN_\lambda u(t) = 0$. It follows from (H_2) that there exists a constant $t_0 \in (0, 1]$, such that $\left| t_0^{2-\gamma_2} u(t_0) \right| \leq M$.

By $Lu(t) = \lambda Nu(t)$ and boundary condition $D_{0+}^{\alpha_2,\beta_2} u(0) = D_{0+}^{\gamma_2 - 1} u(0) = 0$, we have

$$u(t) = I_{0+}^{\alpha_2} \varphi_q \Big(\lambda I_{0+}^{\alpha_1} Nu(t) \Big) + ct^{\gamma_2 - 2}. \tag{3.6}$$

Taking $t = t_0$ into equation (3.6), we have

$$u(t_0) = \frac{1}{\Gamma(\alpha_2)} \int_0^{t_o} (t_0 - s)^{\alpha_2 - 1} \varphi_q \Big(\frac{\lambda}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1 - 1} f(r, r^{2-\gamma_2} u(r)) dr \Big) ds + ct_0^{\gamma_2 - 2}.$$

That means

$$\begin{aligned}
| c | \leq & \frac{t_0^{2-\gamma_2}}{\Gamma(\alpha_2)} \int_0^{t_o} (t_0 - s)^{\alpha_2 - 1} \varphi_q \Big(\frac{\lambda}{\Gamma(\alpha_1)} \int_0^s (s - r)^{\alpha_1 - 1} \mid f(r, r^{2-\gamma_2} u(r)) \mid dr \Big) ds \\
& + \mid t_0^{2-\gamma_2} u(t_0) \mid \\
\leq & \frac{1}{\Gamma(\alpha_2)} \int_0^{t_o} (t_0 - s)^{\alpha_2 - 1} \varphi_q \Big(\frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1 - 1} [a(r) \varphi_p(\mid r^{2-\gamma_2} u(r) \mid) + b(r)] dr \Big) ds \\
& + K \\
\leq & K + \frac{1}{\Gamma(\alpha_2 + 1)} \varphi_q \Big(\frac{\| a \|_c \varphi_p(\| u \|_X)}{\Gamma(\alpha_1 + 1)} + \frac{\| b \|_c}{\Gamma(\alpha_1 + 1)} \Big).
\end{aligned}$$

So, we have

$$\begin{aligned}
& \mid t^{2-\gamma_2} u(t) \mid \\
\leq & \frac{1}{\Gamma(\alpha_2)} \int_0^t (t - s)^{\alpha_2 - 1} \varphi_q \Big(\frac{\lambda}{\Gamma(\alpha_1)} \int_0^s (s - r)^{\alpha_1 - 1} \mid f(r, r^{2-\gamma_2} u(r)) \mid dr \Big) ds + \mid c \mid \\
\leq & \frac{1}{\Gamma(\alpha_2)} \int_0^t (t - s)^{\alpha_2 - 1} \varphi_q \Big(\frac{1}{\Gamma(\alpha_1)} \int_0^s (s - r)^{\alpha_1 - 1} [a(r) \varphi_p(\mid r^{2-\gamma_2} u(r) \mid) + b(r)] dr \Big) ds \\
& + K + \frac{1}{\Gamma(\alpha_2 + 1)} \varphi_q \Big(\frac{\| a \|_c \varphi_p(\| u \|_X)}{\Gamma(\alpha_1 + 1)} + \frac{\| b \|_c}{\Gamma(\alpha_1 + 1)} \Big) \\
\leq & K + \frac{2}{\Gamma(\alpha_2 + 1)} \varphi_q \Big(\frac{\| a \|_c \varphi_p(\| u \|_X)}{\Gamma(\alpha_1 + 1)} + \frac{\| b \|_c}{\Gamma(\alpha_1 + 1)} \Big).
\end{aligned}$$

If $1 < p \leq 2$, then

$$\begin{aligned}
\| u \|_X \leq & K + \frac{2}{\Gamma(\alpha_2 + 1)} \varphi_q \Big(\frac{\| a \|_c \varphi_p(\| u \|_X)}{\Gamma(\alpha_1 + 1)} + \frac{\| b \|_c}{\Gamma(\alpha_1 + 1)} \Big) \\
\leq & K + \frac{2^{q-1}}{\Gamma(\alpha_2 + 1)} \varphi_q \Big(\frac{\| a \|_c}{\Gamma(\alpha_1 + 1)} \Big) \| u \|_X + \frac{2^{q-1}}{\Gamma(\alpha_2 + 1)} \varphi_q \Big(\frac{\| b \|_c}{\Gamma(\alpha_1 + 1)} \Big).
\end{aligned}$$

By sorting out the above formula, we get

$$\| u \|_X \leq \frac{K\Gamma(\alpha_2 + 1) + 2^{q-1} \varphi_q(\frac{\|b\|_c}{\Gamma(\alpha_1 + 1)})}{\Gamma(\alpha_2 + 1) - 2^{q-1} \varphi_q(\frac{\|a\|_c}{\Gamma(\alpha_1 + 1)})}.$$

If $p \geq 2$, then

$$\begin{aligned}\| u \|_X &\leq K + \frac{2}{\Gamma(\alpha_2+1)}\varphi_q\Big(\frac{\| a \|_c \varphi_p(\| u \|_X)}{\Gamma(\alpha_1+1)} + \frac{\| b \|_c}{\Gamma(\alpha_1+1)}\Big)\\ &\leq K + \frac{2}{\Gamma(\alpha_2+1)}\varphi_q\Big(\frac{\| a \|_c}{\Gamma(\alpha_1+1)}\Big) \| u \|_X + \frac{2}{\Gamma(\alpha_2+1)}\varphi_q\Big(\frac{\| b \|_c}{\Gamma(\alpha_1+1)}\Big).\end{aligned}$$

Therefore,

$$\| u \|_X \leq \frac{K\Gamma(\alpha_2+1) + 2\varphi_q(\frac{\|b\|_c}{\Gamma(\alpha_1+1)})}{\Gamma(\alpha_2+1) - 2\varphi_q(\frac{\|a\|_c}{\Gamma(\alpha_1+1)})}.$$

We can conclude that Ω_1 is bounded in X. □

Lemma 3.6. *Let $\Omega_2 = \{u|u \in KerL, QNu = 0\}$. Suppose $(H_1)-(H_2)$ hold. Then Ω_2 is bounded in X.*

Proof. Let $u \in \Omega_2$, we have

$$u(t) = ct^{\gamma_2-2},\ c \in R. \tag{3.7}$$

Since $QNu(t) = 0$, according to (H_2), there exists $t_0 \in (0, 1]$ such that $\left|t_0^{2-\gamma_2}u(t_0)\right| \leq M$. Taking $t = t_0$ into equation (3.7), we have $|c| = |t_0^{2-\gamma_2}u(t_0)| \leq M$.

Therefore, Ω_2 is bounded in X. □

Proof of Theorem 3.1. Let $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \{x|x \in X, \| x \| \leq M\}$ be an open and bounded set of X. By Lemma 3.5 and Lemma 3.6, we can get $Lu \neq N_\lambda u$, $u \in domL \cap \partial\Omega$ and $QNu \neq 0$, $u \in KerL \cap \partial\Omega$.

Let $H(u, \delta) = \rho\delta u + (1-\delta)JQNu$, $\delta \in [0, 1]$, $u \in KerL \cap \overline{\Omega}$, where $J : ImQ \to KerL$ is a homeomorphism with $Jk = kt^{\gamma_2-2}$,

$$\rho = \begin{cases} 1, & \text{if } \ (H_3)(1) \ \text{ holds,} \\ -1, & \text{if } \ (H_3)(2) \ \text{ holds.}\end{cases}$$

For $u \in KerL \cap \partial\Omega$, we have $u(t) = k_0t^{\gamma_2-2}$ and $|t^{2-\gamma_2}u(t)| = |k_0| > M$. Therefore

$$H(u, \delta) = \rho\delta k_0t^{\gamma_2-2} + (1-\delta)(Qf)t^{\gamma_2-2}.$$

If $\delta = 1$, then $H(u, 1) = \rho k_0t^{\gamma_2-2} \neq 0$. If $\delta = 0$, then $H(u, 0) = (Qf)t^{\gamma_2-2} \neq 0$. If $0 < \delta < 1$, suppose $H(u, \delta) = 0$, then $\rho\delta k_0t^{\gamma_2-2} = -(1-\delta)(Qf)t^{\gamma_2-2}$. So, $k_0 = -\frac{(1-\delta)(Qf)}{\delta\rho}$. By (H_2), we get

$$k_0^2 = -\frac{(1-\delta)(k_0Qf)}{\delta\rho} < 0.$$

A contradiction. So, $H(u, \delta) \neq 0$, $u \in KerL \cap \partial\Omega$, $\delta \in [0, 1]$.

Therefore, via the homotopy property of degree, we obtain

$$\begin{aligned}deg(JQN, \Omega \cap KerL, 0) &= deg(H(\cdot, 0), \Omega \cap KerL, 0)\\ &= deg(H(\cdot, 1), \Omega \cap KerL, 0)\\ &= deg(\rho I, \Omega \cap KerL, 0) \neq 0.\end{aligned}$$

Applying Lemma 2.1, we conclude that problem (1.1) has at least one solution in X. The proof is completed. □

4. Example

Consider the following boundary value problem at resonance:

$$\begin{cases} D_{0+}^{\frac{1}{2},\frac{1}{3}}\varphi_p(D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t)) = \dfrac{1}{8}t^2\sin(t^{\frac{1}{4}}u(t))^2 + 2 - t^3,\ t \in (0,1], \\ D_{0+}^{\frac{3}{2},\frac{1}{2}}u(0) = D_{0+}^{\frac{3}{4}}u(0) = 0,\ u(1) = \displaystyle\int_0^1 2t^{\frac{5}{4}}u(t)dt. \end{cases} \tag{4.1}$$

Corresponding to problem (1.1), we have $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{3}{2}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{2}$, $\gamma_1 = \frac{2}{3}$, $\gamma_2 = \frac{7}{4}$, $h(t) = 2t^{\frac{5}{4}}$ and

$$f(t, t^{2-\gamma_2}u(t)) = \frac{1}{8}t^2\sin(t^{\frac{1}{4}}u(t))^2 + 2 - t^3.$$

Take $a(t) = \frac{1}{8}t^2$, $b(t) = 2 - t^3$.

If $p = 3$, we can obtain

$$\Gamma(\alpha_2 + 1) \approx 1.3296 > 2\varphi_q\Big(\frac{\| a \|_c}{\Gamma(\alpha_1 + 1)}\Big) \approx 0.75$$

and

$$|f(t, t^{2-\gamma_2}u(t))| \leq \frac{1}{8}t^2|(t^{\frac{1}{4}}u(t))^2| + 2 - t^3 = a(t)\varphi_p(|t^{\frac{1}{4}}u(t)|) + b(t).$$

That means condition (H_3) holds.

Next, we show that condition (H_2) holds. Let $M = 2$, if $t^{\frac{1}{4}}u(t) > M$ holds for any $t \in (0,1]$, then

$$\begin{aligned} (t^{\frac{1}{4}}u(t))f(t, t^{\frac{1}{4}}u(t)) &= (t^{\frac{1}{4}}u(t))\Big[\frac{1}{8}t^2\sin(t^{\frac{1}{4}}u(t))^2 + 2 - t^3\Big] \\ &> M(-\frac{1}{8}t^2 + 2 - t^3) \\ &> \frac{7M}{8} > 0. \end{aligned}$$

If $t^{\frac{1}{4}}u(t) < -M$ holds for any $t \in (0,1]$, then

$$\begin{aligned} (t^{\frac{1}{4}}u(t))f(t, t^{\frac{1}{4}}u(t)) &= (t^{\frac{1}{4}}u(t))\Big[\frac{1}{8}t^2\sin(t^{\frac{1}{4}}u(t))^2 + 2 - t^3\Big] \\ &< -M(-\frac{1}{8}t^2 + 2 - t^3) \\ &< -\frac{7M}{8} < 0. \end{aligned}$$

Hence, condition (H_2) holds. Therefore, by an application of Theorem 3.1, we obtain that problem (4.1) has at least one solution.

If $p = \frac{5}{3}$, then

$$\Gamma(\alpha_2 + 1) \approx 1.3296 > 2^{q-1}\varphi_q\Big(\frac{\| a \|_c}{\Gamma(\alpha_1 + 1)}\Big) \approx 1.062,$$

and

$$|f(t, t^{2-\gamma_2}u(t))| \leq \frac{1}{8}t^2|(t^{\frac{1}{4}}u(t))^2| + 2 - t^3 = a(t)\varphi_p(|t^{\frac{1}{4}}u(t)|) + b(t).$$

That means the condition (H_3) holds. Let $M = 2$, by simple calculations, we can get that the condition (H_2) holds. By Theorem 3.1, we obtain that problem (4.1) has at least one solution.

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