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DYNAMICAL ANALYSIS OF A FRACTIONAL ORDER HCV INFECTION MODEL WITH ACUTE AND CHRONIC AND GENERAL INCIDENCE RATE

Yu Yang[1,†], Jueyu Wang[1], Shengliang Zhang[2] and Tonghua Zhang[3]

Abstract This paper is concerned with a fractional order HCV infection model with acute and chronic and general incidence rate. We first give the positivity and boundedness of the solution for this model. Then, we establish the dynamical behavior of this model in terms of $\mathcal{R}_0^\alpha$. Numerical simulations are given to verify the obtained theoretical results.

Keywords Fractional order, HCV infection model, global stability, Lyapunov function.

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1. Introduction

The World Health Organization (WHO) estimated that in 2019, 58 million people were living with chronic hepatitis C virus (HCV) infection worldwide and 290000 people died from cirrhosis and hepatocellular carcinoma. About 30% (15-45%) of infected people spontaneously clear the virus within 6 months of infection without any treatment. The remaining 70% (55-85%) of people will develop chronic HCV infection. The natural feature of Hepatitis C is the existence of a chronic stage. Hence, it is difficult to characterize the natural history of this disease.

Reade et al. [21] proposed a model of disease with acute and chronic phases. Motivated by the work of [21], Martcheva and Chavez [18] considered an HCV model with chronic stage. Yuan and Zhang [32] then extended the work of [18]. They established the global stability of the endemic equilibrium. Cai and Li [3] further improved the main results of [32]. Different from [32], Zhang and Zhou [34] proposed a new HCV model to consider the moving from acute infection back to the susceptible. They also established the global behavior for this model. Notice that Cui et al. [7] recently studied the following SICR model with acute and chronic

†The corresponding author. Email: yangyu@lixin.edu.cn(Y. Yang)
1School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, 201209 Shanghai, China
2College of Economics and Management, Nanjing Forestry University, 210037 Nanjing, China
3Department of Mathematics, Swinburne University of Technology, Hawthorn, 3122 Victoria, Australia

HCV infections:

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)(I(t) + C(t)) - dS(t), \\ \dot{I}(t) = \beta S(t)(I(t) + C(t)) - (d + \gamma)I(t), \\ \dot{C}(t) = p\gamma I(t) - (d + \mu)C(t), \\ \dot{R}(t) = (1 - p)\gamma I(t) - dR(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$, $C(t)$ and $R(t)$ are the number of susceptible, acute infection, chronic infection and recovery of HCV, respectively. Λ is the birth rate, β is the transmission rate of acute or chronic hepatitis C cases, d is the natural death rate, p is the proportion of progressing to chronic stage, γ is the rate moving from acute stage to chronic stage, μ is the death rate induced by HCV. All parameters are positive and $p \in (0, 1)$. They investigated the global dynamical properties of system (1.1). Wang et al. [27] used (1.1) to fit the data in six districts in Xiamen City, China from 2004 to 2018 and predict the transmissibility of hepatitis C. Huang et al. [12] pointed out that incidence rate can play an important role in modeling of epidemic dynamics. More recently, Su and Yang [26] proposed a diffusive HCV infection model with nonlinear incidence and analyzed the stability of the two kinds of equilibria (see, for example, [19, 25]).

Since the fractional order derivative provides an excellent tool for describing the memory properties of various processes, fractional differential equations play a crucial role in modelling epidemiology properties. Recently, many researchers have begun to study the dynamical behavior of different fractional order epidemic models such as HIV and tuberculosis [1], HIV [24], SEIR [29], SIRI [15] and COVID-19 [10, 17]. For more review of epidemic models with frational order, we refer to [4]. However, the dimensions of most fractional order epidemic models in left-hand side and righ-hand side do not match. Such flaws have been found in [9, 11, 22, 29]. As far as we know, there is few work on the fractional order HCV infection models with acute and chronic and general incidence rate.

Inspired by [9, 11, 22, 29], in this paper, we propose the frational order HCV infection model as follows:

$$\begin{cases} {}_0^C D_t^\alpha S(t) = \Lambda^\alpha - \beta^\alpha f(S(t))(I(t) + C(t)) - d^\alpha S(t), \\ {}_0^C D_t^\alpha I(t) = \beta^\alpha f(S(t))(I(t) + C(t)) - (d^\alpha + \gamma^\alpha)I(t), \\ {}_0^C D_t^\alpha C(t) = p\gamma^\alpha I(t) - (d^\alpha + \mu^\alpha)C(t), \\ {}_0^C D_t^\alpha R(t) = (1 - p)\gamma^\alpha I(t) - d^\alpha R(t), \end{cases} \tag{1.2}$$

with $\alpha \in (0, 1]$ and initial conditions

$$S(0) > 0,\ I(0) > 0,\ C(0) > 0,\ R(0) > 0. \tag{1.3}$$

${}_0^C D_t^\alpha$ is the Caputo fractional derivative (see section 2). The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

(A) $f(0) = 0$, $f'(S) > 0$, for $S \geq 0$.

The paper is organized as follows. In Section 2, some basic results of fractional order calculus are given. In Section 3, we then establish the well-posedness of system

(1.2). The stability of equilibria for system (1.2) is analyzed in Section 4. Finally, we point out through numerical simulations the effects of fractional order on the dynamical behavior of system (1.2) and end up with a brief discussion.

2. Preliminaries

In this section, we give some notations, definitions and lemmas.

Definition 2.1 ([20]). The fractional integral of order $q > 0$ for a function $g : \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$J^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\eta)^{q-1} g(\eta) d\eta.$$

Definition 2.2 ([20]). The Caputo fractional derivative with order $q > 0$ $(n-1 < q < n)$ for a function $g \in C^n([a, +\infty), \mathbb{R})$ is defined as

$${}_0^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{g^{(n)}(\eta)}{(t-\eta)^{q+1-n}} d\eta.$$

Definition 2.3 ([20]). The two-parameters Mittag-Leffler function is defined as

$$E_{\eta_1,\eta_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\eta_1 + \eta_2)},\ \eta_1, \eta_2 > 0,\ z \in \mathbb{C}.$$

For short, set $E_{\eta_1}(z) := E_{\eta_1,1}(z)$.

The Laplace transform of the Caputo fractional derivative and the function $t^{\eta_2-1}E_{\eta_1,\eta_2}(\pm\lambda t^{\eta_1})$ are

$$\mathcal{L}\{{}_0^C D_t^q g(t)\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-k-1} g^{(k)}(0),\ n < q \leq n-1,$$

and

$$\mathcal{L}[t^{\eta_2-1}E_{\eta_1,\eta_2}(\pm\lambda t^{\eta_1})] = \frac{s^{\eta_1-\eta_2}}{s^{\eta_1} \mp \lambda},$$

where $F(s) = \mathcal{L}(g(t))$.

Lemma 2.1 ([8]). *If $\eta > 0$, $r > 0$, $\phi \in [-\pi, \pi]$ and $\varrho = re^{i\phi}$, then $\lim_{t\to\infty} E_\eta(-\varrho t^\eta) = 0$ for $|\phi| < \frac{\eta\pi}{2}$.*

Lemma 2.2 ([8]). *If $\eta_1 > 0$, $\eta_2 > 0$ and $z \in \mathbb{C}$. Then,*

$$E_{\eta_1,\eta_2}(z) = zE_{\eta_1,\eta_1+\eta_2}(z) + \frac{1}{\Gamma(\eta_2)}.$$

Lemma 2.3 ([35]). *For $\eta \in (0, 1]$ and $z \in \mathbb{R}$, we have $E_\eta(z) > 0$.*

Lemma 2.4 ([33]). *Let $\rho : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[t_0, +\infty)$ and $J^q|\rho(t)|^m \leq K$ for all $t > t_0$ with $0 < q < 1$, $m > 0$ and $K > 0$. Then $\lim_{t\to\infty} \rho(t) = 0$.*

Lemma 2.5 ([23]). *Suppose that g is continuous on $(0,\infty)$ and of exponential order q and that g' is piecewise continuous on $[0,\infty)$ and furthermore that $\lim_{t\to\infty} f(t)$ exists. Then, $\lim_{t\to\infty} g(t) = \lim_{\lambda\to 0} \lambda\mathcal{L}(g(t))$.*

Lemma 2.6 ([15]). *Let $g:(0,\infty)\to(0,\infty)$ be an increasing function of class C^1. For all $a,b>0$, we define*

$$\sigma(a,b) = \int_a^b \frac{g(\eta)-g(a)}{g(\eta)} d\eta.$$

Then, for any function $\mathcal{X}:(0,\infty)\to(0,\infty)$ of class C^1, we have

$${}_0^C D_t^q \left[\sigma(\mathcal{X}(t),a)\right] \leq \frac{g(\mathcal{X}(t))-g(a)}{g(\mathcal{X}(t))} \times {}_0^C D_t^q \mathcal{X}(t),$$

where $0<q\leq 1$.

3. Well-posedness

In this section, we prove the positivity and boundedness of the solution for system (1.2) with intial conditions (1.3).

Theorem 3.1. *System (1.2) with intial conditions (1.3) has a unique positive solution. Furthermore, the set*

$$\Omega = \left\{(S,I,C,R)\in\mathbb{R}_+^4 :\ S>0, I>0, C>0, R>0, S+I+C+R\leq S^0\right\}$$

is positively invariant.

Proof. Define

$$X(t) = \begin{pmatrix} S(t) \\ I(t) \\ C(t) \\ R(t) \end{pmatrix}, \quad X_0 = \begin{pmatrix} S(0) \\ I(0) \\ C(0) \\ R(0) \end{pmatrix}$$

and

$$h(X(t)) = \begin{pmatrix} h_1(X(t)) \\ h_2(X(t)) \\ h_3(X(t)) \\ h_4(X(t)) \end{pmatrix} = \begin{pmatrix} \Lambda^\alpha - \beta^\alpha f(S(t))(I(t)+C(t)) - d^\alpha S(t) \\ \beta^\alpha f(S(t)(I(t)+C(t)) - (d^\alpha+\gamma^\alpha)I(t) \\ p\gamma^\alpha I(t) - (d^\alpha+\mu^\alpha)C(t) \\ (1-p)\gamma^\alpha I(t) - d^\alpha R(t) \end{pmatrix}.$$

System (1.2) with (1.3) can be rewritten as

$$\begin{cases} {}_0^C D_t^\alpha X(t) = h(X(t)), \\ X(0) = X_0. \end{cases}$$

One derives that the Jacobian matrix $\dfrac{\partial h}{\partial x} = \dfrac{\partial(h_1, h_2, h_3, h_4)}{\partial(S, I, C, R)}$ is continuous on $\mathbb{R}^4_+$. Thanks to [14, Remark 1.2.1], h is locally Lipschitz on $\mathbb{R}^4_+$. Using [16, Remark 3.8], we conclude that system (1.2) with (1.3) has a unique solution .

For all $t \in [0, \infty)$, we now show the solution $X(t)$ is positive. From the first equation of system (1.2), one can get ${}_0^C D_t^\alpha S(t)\big|_{S=0} = \Lambda^\alpha > 0$. Using [13, Remark 1], we have $S(t) > 0$ for all $t \in [0, \infty)$. We next prove that $I(t) > 0$ for all $t \in [0, \infty)$. If not, then there exists $T_0 > 0$ such that $I(t) > 0$ for $t \in [0, T_0)$ and $I(T_0) = 0$. By the third equation of system (1.2), one has

$$ {}_0^C D_t^\alpha C(t) \geq -(d^\alpha + \mu^\alpha)C(t), \text{ for } t \in [0, T_0]. $$

Applying Laplace transform to the above inequality and Lemma 2.3, we get

$$ C(t) \geq C(0)E_\alpha(-(d^\alpha + \mu^\alpha)t^\alpha) > 0, \text{ for } t \in [0, T_0]. \tag{3.1} $$

From (3.1), one has

$$ {}_0^C D_t^\alpha I(t) \geq -(d^\alpha + \gamma^\alpha)I(t), \text{ for } t \in [0, T_0]. $$

Thus, $I(t) \geq I(0)E_\alpha(-(d^\alpha + \gamma^\alpha)t^\alpha) > 0$, for $t \in [0, T_0]$, contracting to $I(T_0) = 0$. Hence, $I(t) > 0$ for $t \in [0, \infty)$. Simlarly, we can show that $C(t) > 0$ and $R(t) > 0$ for $t \in [0, \infty)$.

Define $N(t) = S(t) + I(t) + C(t) + R(t)$. By system (1.2), we derive

$$ \begin{aligned} {}_0^C D_t^\alpha N(t) &= \Lambda^\alpha - d^\alpha S(t) - d^\alpha I(t) - (d^\alpha + \mu^\alpha)C(t) - d^\alpha R(t) \\ &\leq \Lambda^\alpha - d^\alpha N(t). \end{aligned} $$

Laplace transform is used for the above inequality. Then,

$$ N(t) \leq \Lambda^\alpha t^\alpha E_{\alpha,\alpha+1}(-d^\alpha t^\alpha) + E_{\alpha,1}(-d^\alpha t^\alpha)N(0). $$

By Lemma 2.2 and $N(0) \leq S^0$ with $S^0 = \Lambda^\alpha/d^\alpha$, we have

$$ N(t) \leq S^0(d^\alpha t^\alpha E_{\alpha,\alpha+1}(-d^\alpha t^\alpha) + E_{\alpha,1}(-d^\alpha t^\alpha)) = \frac{S^0}{\Gamma(1)} = S^0. $$

Therefore, Ω is positively invariant. □

4. Stability analysis

Clearly, the disease-free equilibrium of system (1.2) is $E^0 = (S^0, 0, 0, 0)$. By [17, Theorem 3.2], the basic reproduction number for system (1.2) is

$$ \mathcal{R}_0^\alpha = \frac{\beta^\alpha(d^\alpha + \mu^\alpha + p\gamma^\alpha)}{(d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha)} f\left(S^0\right). \tag{4.1} $$

To find the endemic equilibrium, let

$$ \begin{cases} \Lambda^\alpha - \beta^\alpha f(S)(I + C) - d^\alpha S = 0, \\ \beta^\alpha f(S)(I + C) - (d^\alpha + \gamma^\alpha)I = 0, \\ p\gamma^\alpha I - (d^\alpha + \mu^\alpha)C = 0, \\ (1 - p)\gamma^\alpha I - d^\alpha R = 0. \end{cases} \tag{4.2} $$

Adding the first two equations of (4.2), one gets

$$S = \frac{\Lambda^\alpha - (d^\alpha + \gamma^\alpha)I}{d^\alpha}.$$

We can further get

$$C = \frac{p\gamma^\alpha}{d^\alpha + \mu^\alpha} I,\ \ R = \frac{(1-p)\gamma^\alpha}{d^\alpha} I.$$

Substituting the above equalities into the second equation of (4.2), one derives

$$\beta^\alpha f\left(\frac{\Lambda^\alpha - (d^\alpha + \gamma^\alpha)I}{d^\alpha}\right)\left(1 + \frac{p\gamma^\alpha}{d^\alpha + \mu^\alpha}\right) - (d^\alpha + \gamma^\alpha) = 0.$$

Define

$$H(I) = \beta^\alpha f\left(\frac{\Lambda^\alpha - (d^\alpha + \gamma^\alpha)I}{d^\alpha}\right)\left(1 + \frac{p\gamma^\alpha}{d^\alpha + \mu^\alpha}\right) - (d^\alpha + \gamma^\alpha).$$

Clearly, $H(0) = (d^\alpha + \gamma^\alpha)(\mathcal{R}_0^\alpha - 1)$, $H\left(\frac{\Lambda^\alpha}{d^\alpha + \gamma^\alpha}\right) = -(d^\alpha + \gamma^\alpha) < 0$ and

$$H'(I) = -\frac{d^\alpha + \gamma^\alpha}{d^\alpha}\beta^\alpha f'\left(\frac{\Lambda^\alpha - (d^\alpha + \gamma^\alpha)I}{d^\alpha}\right)\left(1 + \frac{p\gamma^\alpha}{d^\alpha + \mu^\alpha}\right) < 0.$$

So, system (1.2) has a unique endemic equilibrium $E^* = (S^*, I^*, C^*, R^*)$ when $\mathcal{R}_0^\alpha > 1$.

Theorem 4.1. *The disease-free equilibrium E^0 is locally asymptotically stable if $\mathcal{R}_0^\alpha < 1$ and unstable if $\mathcal{R}_0^\alpha > 1$.*

Proof. At E^0, the Jacobian matrix for system (1.2) is

$$J(E^0) = \begin{pmatrix} -d^\alpha & -\beta^\alpha f(S^0) & -\beta^\alpha f(S^0) & 0 \\ 0 & \beta^\alpha f(S^0) - (d^\alpha + \gamma^\alpha) & \beta^\alpha f(S^0) & 0 \\ 0 & p\gamma^\alpha & -(d^\alpha + \mu^\alpha) & 0 \\ 0 & (1-p)\gamma^\alpha & 0 & -d^\alpha \end{pmatrix}.$$

Obviously, two eigenvalues of $J(E^0)$ are $\lambda_1 = -d^\alpha < 0$ and $\lambda_2 = -d^\alpha < 0$. The other two eigenvalues λ_3 and λ_4 are determined by the following characteristic equation:

$$\lambda^2 + a_1\lambda + a_0 = 0,$$

where

$$\begin{aligned} a_1 &= d^\alpha + \gamma^\alpha - \beta^\alpha f(S^0) + d^\alpha + \mu^\alpha > (d^\alpha + \gamma^\alpha)(1 - \mathcal{R}_0^\alpha), \\ a_0 &= (d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha - \beta^\alpha f(S^0)) - p\gamma^\alpha\beta^\alpha f(S^0) \\ &= (d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha)(1 - \mathcal{R}_0^\alpha). \end{aligned}$$

Hence, one has

$$\lambda_3 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2},\ \ \lambda_4 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}.$$

Since $\mathcal{R}_0^\alpha < 1$, we can obtain that $a_1 > 0$ and $a_0 > 0$. This yields that $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$ $(i = 1, 2, 3, 4)$ is satisfied [5]. Hence, E^0 is locally asymptotically stable. If $\mathcal{R}_0^\alpha > 1$, then we have $a_0 < 0$ and $|\arg(\lambda_4)| < \frac{\alpha\pi}{2}$. Thus, E^0 is unstable [6]. □

Let $D(\xi)$ be the discriminant of (4.3), where

$$D(\xi) = -\begin{vmatrix} 1 & b_2 & b_1 & b_0 & 0 \\ 0 & 1 & b_2 & b_1 & b_0 \\ 3 & 2b_2 & b_1 & 0 & 0 \\ 0 & 3 & 2b_2 & b_1 & 0 \\ 0 & 0 & 3 & 2b_2 & b_1 \end{vmatrix} = 18b_0b_1b_2 + (b_1b_2)^2 - 4b_0b_2^3 - 4b_1^3 - 27b_0^2.$$

Theorem 4.2. *The endemic equilibrium E^* is locally asymptotically stable if $\mathcal{R}_0^\alpha > 1$ and $D(\xi) > 0$.*

Proof. At E^*, the Jacobian matrix for system (1.2) is

$$J(E^*) = \begin{pmatrix} -d^\alpha - \beta^\alpha f'(S^*)(I^* + C^*) & -\beta^\alpha f(S^*) & -\beta^\alpha f(S^*) & 0 \\ \beta^\alpha f'(S^*)(I^* + C^*) & \beta^\alpha f(S^*) - (d^\alpha + \gamma^\alpha) & \beta^\alpha f(S^*) & 0 \\ 0 & p\gamma^\alpha & -(d^\alpha + \mu^\alpha) & 0 \\ 0 & (1-p)\gamma^\alpha & 0 & -d^\alpha \end{pmatrix}.$$

It is easy to see that one eigenvalue of $J(E^*)$ is $\lambda_1 = -d^\alpha < 0$. The other three eigenvalues λ_i $(i = 1, 2, 3)$ are determined by the following characteristic equation:

$$\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0, \tag{4.3}$$

where

$$\begin{aligned} b_2 &= \beta^\alpha f'(S^*)(I^* + C^*) + 2d^\alpha + \mu^\alpha + d^\alpha + \gamma^\alpha - \beta^\alpha f(S^*), \\ b_1 &= d^\alpha(d^\alpha + \gamma^\alpha - \beta^\alpha f(S^*)) + (d^\alpha + \gamma^\alpha)\beta^\alpha f'(S^*)(I^* + C^*) \\ &\quad + (d^\alpha + \mu^\alpha)(2d^\alpha + \gamma^\alpha - \beta^\alpha f(S^*) + \beta^\alpha f'(S^*)(I^* + C^*)) - p\gamma^\alpha\beta^\alpha f(S^*), \\ b_0 &= (d^\alpha + \mu^\alpha)[d^\alpha(d^\alpha + \gamma^\alpha - \beta^\alpha f(S^*)) + (d^\alpha + \gamma^\alpha)\beta^\alpha f'(S^*)(I^* + C^*)] - pd^\alpha\gamma^\alpha\beta^\alpha f(S^*). \end{aligned}$$

From (4.2), one has

$$\beta^\alpha f(S^*)(I^* + C^*) = (d^\alpha + \gamma^\alpha)I^*, \; C^* = \frac{p\gamma^\alpha}{d^\alpha + \mu^\alpha} I^*,$$

which implies that $\beta^\alpha f(S^*)(d^\alpha + \mu^\alpha + p\gamma^\alpha) = (d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha)$. Therefore, we obtain $d^\alpha + \gamma^\alpha > \beta^\alpha f(S^*)$. By some calculations, we derive that $b_2 > 0$,

$$b_1 = d^\alpha(d^\alpha + \mu^\alpha) + (2d^\alpha + \gamma^\alpha + \mu^\alpha)\beta^\alpha f'(S^*)(I^* + C^*) + d^\alpha(d^\alpha + \gamma^\alpha - \beta^\alpha f(S^*)) > 0$$

and

$$b_0 = (d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha)\beta^\alpha f'(S^*)(I^* + C^*) > 0.$$

It is clear that $b_1b_2 - b_0 > 0$. From [2, Proposition 1], we derive the theorem. □

Similarly proof as in [14, Lemma 4.5], we have the following result.

Lemma 4.1. *The solutions S, I, C and R of system (1.2) are uniformly continuous.*

We next study the global stability of equilibria for system (1.2).

Theorem 4.3. *If $\mathcal{R}_0^\alpha < 1$, then E^0 is globally asymptotically stable (GAS) in Ω.*

Proof. Define the following Lyapunov function:

$$\mathcal{U}(t) = I(t) + k_1 C(t),$$

where $k_1 > 0$ is to be chosen later. Clearly, $\mathcal{U}(t)$ is positive definite function in Ω. Since $X_0 \in \Omega$, one has $S(t) \le S^0$. By system (1.2), it follows that

$$\begin{aligned}
{}_0^C D_t^\alpha \mathcal{U}(t) &= {}_0^C D_t^\alpha I(t) + k_1 \, {}_0^C D_t^\alpha C(t) \\
&= \beta^\alpha f(S(t))(I(t)+C(t)) - (d^\alpha+\gamma^\alpha)I(t) + k_1 p\gamma^\alpha I(t) - k_1(d^\alpha+\mu^\alpha)C(t) \\
&\le \beta^\alpha f(S^0)(I(t)+C(t)) - (d^\alpha+\gamma^\alpha)I(t) + k_1 p\gamma^\alpha I(t) - k_1(d^\alpha+\mu^\alpha)C(t) \\
&= -(d^\alpha+\gamma^\alpha - k_1 p\gamma^\alpha - \beta^\alpha f(S^0))I(t) - (k_1(d^\alpha+\mu^\alpha) - \beta^\alpha f(S^0))C(t).
\end{aligned}$$

Since $\mathcal{R}_0^\alpha < 1$, we can choose $k_1 > 0$ such that

$$d^\alpha + \gamma^\alpha - k_1 p\gamma^\alpha - \beta^\alpha f(S^0) > 0, \;\; k_1(d^\alpha+\mu^\alpha) - \beta^\alpha f(S^0) > 0.$$

Set

$$\mathcal{K}_1(t) := (d^\alpha + \gamma^\alpha - k_1 p\gamma^\alpha - \beta^\alpha f(S^0))I(t) + (k_1(d^\alpha+\mu^\alpha) - \beta^\alpha f(S^0))C(t).$$

Then, ${}_0^C D_t^\alpha \mathcal{U}(t) \le -\mathcal{K}_1(t)$, that is $\mathcal{U}(t) - \mathcal{U}(0) \le -J^\alpha \mathcal{K}_1(t)$. So, one derives that

$$\mathcal{U}(t) + J^\alpha \mathcal{K}_1(t) \le \mathcal{U}(0).$$

It gives that $J^\alpha I(t) \le \widetilde{C}_1$ and $J^\alpha C(t) \le \widetilde{C}_1$ with $\widetilde{C}_1 > 0$. Using Lemmas 2.4 and 4.1, we obtain

$$\lim_{t\to\infty} I(t) = 0 \text{ and } \lim_{t\to\infty} C(t) = 0.$$

According to the first and fourth equations of system (1.2), when $t \to \infty$, one gets

$$\begin{cases} {}_0^C D_t^\alpha S(t) = \Lambda^\alpha - d^\alpha S(t), \\ {}_0^C D_t^\alpha R(t) = -d^\alpha R(t). \end{cases}$$

We can find that $S(t) = S^0 + (S(0) - S^0)E_\alpha(-d^\alpha t^\alpha)$ and $R(t) = R(0)E_\alpha(-d^\alpha t^\alpha)$. By Lemma 2.1, we conclude that $\lim_{t\to\infty} S(t) = S^0$ and $\lim_{t\to\infty} R(t) = 0$. □

Theorem 4.4. *If $\mathcal{R}_0^\alpha > 1$, then E^* is globally asymptotically stable (GAS) in Ω.*

Proof. Define the following Lyapunov function:

$$\mathcal{V}(t) = \int_{S^*}^{S(t)} \frac{f(\tau) - f(S^*)}{f(\tau)} d\tau + \int_{I^*}^{I(t)} \frac{\tau - I^*}{\tau} d\tau + \frac{\beta^\alpha f(S^*)C^*}{p\gamma^\alpha I^*} \int_{C^*}^{C(t)} \frac{\tau - C^*}{\tau} d\tau.$$

Clearly, $\mathcal{V}(t)$ is positive definite in Ω. For $\alpha = 1$, the proof is similar as in [30, Theorem 1.4] and [31, Theorem 3.3], we omit here. For $0 < \alpha < 1$, from Lemma 2.6, one derives

$$\begin{aligned}
{}_0^C D_t^\alpha \mathcal{V}(t) \le& \left(1 - \frac{f(S^*)}{f(S(t))}\right) {}_0^C D_t^\alpha S(t) + \left(1 - \frac{I^*}{I(t)}\right) {}_0^C D_t^\alpha I(t) \\
&+ \frac{\beta^\alpha f(S^*)C^*}{p\gamma^\alpha I^*} \left(1 - \frac{C^*}{C(t)}\right) {}_0^C D_t^\alpha C(t)
\end{aligned}$$

Applying

$$\Lambda^\alpha = \beta^\alpha f(S^*)(I^* + C^*) + d^\alpha S^*,\ d^\alpha + \gamma^\alpha = \frac{\beta^\alpha f(S^*)(I^* + C^*)}{I^*},\ C^* = \frac{p\gamma^\alpha I^*}{d^\alpha + \mu^\alpha},$$

we obtain

$$\begin{aligned}
{}_0^C D_t^\alpha \mathcal{V}(t) \leq & \left(1 - \frac{f(S^*)}{f(S(t))}\right)(\Lambda^\alpha - d^\alpha S(t) - \beta^\alpha f(S(t))(I(t) + C(t))) \\
& + \left(1 - \frac{I^*}{I(t)}\right)(\beta^\alpha f(S(t))(I(t) + C(t)) - (d^\alpha + \gamma^\alpha)I(t)) \\
& + \frac{\beta^\alpha f(S^*)C^*}{p\gamma^\alpha I^*}\left(1 - \frac{C^*}{C(t)}\right)(p\gamma^\alpha I(t) - (d^\alpha + \mu^\alpha)C(t)) \\
= & \left(1 - \frac{f(S^*)}{f(S(t))}\right)(d^\alpha S^* - d^\alpha S(t) - \beta^\alpha f(S(t))(I(t) + C(t)) + \beta^\alpha f(S^*)(I^* + C^*)) \\
& + \left(1 - \frac{I^*}{I(t)}\right)\left(\beta^\alpha f(S(t))(I(t) + C(t)) - \frac{\beta^\alpha f(S^*)(I^* + C^*)}{I^*}I(t)\right) \\
& + \beta^\alpha f(S^*)C^*\left(1 - \frac{C^*}{C(t)}\right)\left(\frac{I(t)}{I^*} - \frac{C(t)}{C^*}\right) \\
= & -d^\alpha(S(t) - S^*)\left(1 - \frac{f(S^*)}{f(S(t))}\right) + \beta^\alpha f(S^*)I^*\left(2 - \frac{f(S^*)}{f(S(t))} - \frac{f(S(t))}{f(S^*)}\right) \\
& + \beta^\alpha f(S^*)C^*\left(3 - \frac{f(S^*)}{f(S(t))} - \frac{f(S(t))I^*C(t)}{f(S^*)I(t)C^*} - \frac{I(t)C^*}{I^*C(t)}\right).
\end{aligned} \tag{4.4}$$

By (A) and the arithmetic and geometric means, we get

$${}_0^C D_t^\alpha \mathcal{V}(t) \leq 0,\ \text{for } t \geq 0.$$

This means that

$$\mathcal{V}(t) \leq \mathcal{V}(0),\ \text{for } t \geq 0.$$

We now show that $\liminf_{t\to\infty} S(t) > 0$. If not, then one has $\liminf_{t\to\infty} S(t) = 0$. Since

$$\int_{S^*}^{S(t)} \frac{f(\tau) - f(S^*)}{f(\tau)} d\tau \leq \mathcal{V}(0),\ \text{for } t \geq 0,$$

we get

$$\limsup_{t\to\infty} \int_{S(t)}^{S^*} \frac{1}{f(\tau)} d\tau \leq \frac{1}{f(S^*)}(\mathcal{V}(0) + S^*),\ \text{for } t \geq 0.$$

By (A) and $\liminf_{t\to\infty} S(t) = 0$, we have

$$\int_0^{S^*} \frac{1}{f(\tau)} d\tau \leq \frac{1}{f(S^*)}(\mathcal{V}(0) + S^*),\ \text{for } t \geq 0.$$

Since $\lim_{\eta\to 0} \frac{f(\eta)}{\eta} = f'(0)$, there exists a constant $k_2 > 0$ such that

$$\frac{1}{f(\eta)} \geq \frac{1}{k_2\eta},\ \text{as } \eta \to 0.$$

By above discussions, we derive

$$\int_0^{S^*} \frac{1}{f(\tau)} d\tau = +\infty.$$

It is a contradiction. Similarly, we can show $\liminf_{t\to\infty} I(t) > 0$ and $\liminf_{t\to\infty} C(t) > 0$. Using the positivity and continuity of the solution $X(t)$ for system (1.2), we can define $\inf_{t\geq 0} S(t) := \zeta > 0$. From (4.4), we get

$$\begin{aligned}
{}_0^C D_t^\alpha \mathcal{V}(t) &\leq -d^\alpha (S(t) - S^*) \left(1 - \frac{f(S^*)}{f(S(t))}\right) \\
&= -\frac{d^\alpha}{f(S(t))}(S(t) - S^*)(f(S(t)) - f(S^*)) \\
&= -\frac{d^\alpha}{f(S(t))}(S(t) - S^*)^2 \frac{f(S(t)) - f(S^*)}{S(t) - S^*} \\
&= -\frac{d^\alpha}{f(S(t))}(S(t) - S^*)^2 f'(\nu(t)) \\
&\leq -\frac{d^\alpha M}{f(S^0)}(S(t) - S^*)^2,
\end{aligned} \tag{4.5}$$

where $\nu(t)$ is between S^* and $S(t)$ and $M = \max_{\zeta\leq\nu\leq S^0} |f'(\nu(t))|$. Define

$$\mathcal{K}_2(t) := \frac{d^\alpha M}{f(S^0)}(S(t) - S^*)^2.$$

By (4.5), we derive ${}_0^C D_t^\alpha \mathcal{V}(t) \leq -\mathcal{K}_2(t)$. This yields that $\mathcal{V}(t) - \mathcal{V}(0) \leq -J^\alpha \mathcal{K}_2(t)$, that is, $\mathcal{V}(t) + J^\alpha \mathcal{K}_2(t) \leq \mathcal{V}(0)$. Then, one has $J^\alpha (S(t) - S^*)^2 \leq \widetilde{C}_2$ with $\widetilde{C}_2 > 0$. By Lemma 4.1, the uniform continuity of $(S(t) - S^*)^2$ is obtained. Then, applying Lemma 2.4, we conclude that $\lim_{t\to\infty} S(t) = S^*$.

By Lemma 2.5, we have $\lim_{\lambda\to 0^+} \lambda\mathcal{L}(S(t)) = S^*$. Adding the first two equations of system (1.2), we have

$${}_0^C D_t^\alpha (S(t) + I(t)) = \Lambda^\alpha - d^\alpha S(t) - (d^\alpha + \gamma^\alpha) I(t).$$

Using the Laplace transform to the above equality, we obtain

$$\lambda^\alpha \mathcal{L}(S(t) + I(t)) - \lambda^{\alpha-1}(S(0) + I(0)) = \frac{\Lambda^\alpha}{\lambda} - d^\alpha \mathcal{L}(S(t)) - (d^\alpha + \gamma^\alpha)\mathcal{L}(I(t)),$$

which gives that

$$\mathcal{L}(I(t)) = \frac{\frac{\Lambda^\alpha}{\lambda} + \lambda^{\alpha-1}(S(0) + I(0)) - (\lambda^\alpha + d^\alpha)\mathcal{L}(S(t))}{\lambda^\alpha + d^\alpha + \gamma^\alpha}.$$

Hence,

$$\begin{aligned}
\lim_{\lambda\to 0^+} \lambda\mathcal{L}(I(t)) &= \lim_{\lambda\to 0^+} \frac{\Lambda^\alpha}{\lambda^\alpha + d^\alpha + \gamma^\alpha} + \lim_{\lambda\to 0^+} \frac{\lambda^\alpha (S(0) + I(0))}{\lambda^\alpha + d^\alpha + \gamma^\alpha} \\
&\quad - \left(\lim_{\lambda\to 0^+} \frac{\lambda^\alpha + d^\alpha}{\lambda^\alpha + d^\alpha + \gamma^\alpha}\right) \cdot (\lim_{\lambda\to 0^+} \lambda\mathcal{L}(S(t))) \\
&= \frac{\Lambda^\alpha}{d^\alpha + \gamma^\alpha} - \frac{d^\alpha S^*}{d^\alpha + \gamma^\alpha} = I^*.
\end{aligned}$$

By Lemma 2.5, we derive $\lim_{t\to\infty} I(t) = I^*$. From the third and fourth equations of system (1.2), similarly as the above discussions, we can obtain that $\lim_{t\to\infty} C(t) = C^*$ and $\lim_{t\to\infty} R(t) = R^*$. Therefore, we get

$$\lim_{t\to\infty}(S(t), I(t), C(t), R(t)) = (S^*, I^*, C^*, R^*).$$

□

5. Numerical simulations and discussion

In this section, we choose $f(S) = \frac{S}{1+\delta S}$ for $\delta \geq 0$. System (1.2) can reduce to

$$\begin{cases} {}_0^C D_t^\alpha S(t) = \Lambda^\alpha - \frac{\beta^\alpha S(t)(I(t)+C(t))}{1+\delta S(t)} - d^\alpha S(t), \\ {}_0^C D_t^\alpha I(t) = \frac{\beta^\alpha S(t)(I(t)+C(t))}{1+\delta S(t)} - (d^\alpha + \gamma^\alpha) I(t), \\ {}_0^C D_t^\alpha C(t) = p\gamma^\alpha I(t) - (d^\alpha + \mu^\alpha) C(t), \\ {}_0^C D_t^\alpha R(t) = (1-p)\gamma^\alpha I(t) - d^\alpha R(t). \end{cases} \tag{5.1}$$

System (5.1) always has a disease-free equilibrium $E^0 = (\Lambda^\alpha/d^\alpha, 0, 0, 0)$. From (4.1), the basic reproduction number for system (5.1) is

$$\mathcal{R}_0^\alpha = \frac{\Lambda^\alpha \beta^\alpha (d^\alpha + \mu^\alpha + p\gamma^\alpha)}{(d^\alpha + \delta\Lambda^\alpha)(d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha)}. \tag{5.2}$$

From (5.2), we can see that $\mathcal{R}_0^\alpha$ is monotonically decreasing with respect to δ. By some calculations, when $\mathcal{R}_0^\alpha > 1$, the endemic equilibrium $E^* = (S^*, I^*, C^*, R^*)$ of system (5.1) is

$$I^* = \frac{(d^\alpha + \delta\Lambda^\alpha)(d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha) - \Lambda^\alpha \beta^\alpha (d^\alpha + \mu^\alpha + p\gamma^\alpha)}{(d^\alpha + \gamma^\alpha)(\delta(d^\alpha + \mu^\alpha)(d^\alpha + \gamma^\alpha) - \beta^\alpha (d^\alpha + \mu^\alpha + p\gamma^\alpha))},$$

and

$$S^* = \frac{\Lambda^\alpha - (d^\alpha + \gamma^\alpha) I^*}{d^\alpha}, \; C^* = \frac{p\gamma^\alpha I^*}{d^\alpha + \mu^\alpha}, \; R^* = \frac{(1-p)\gamma^\alpha I^*}{d^\alpha}.$$

In the following, we fix $\delta = 1$. Firstly, we set the parameter values [34]: $\Lambda = 1$, $\beta = 0.2483$, $d = 0.007$, $\gamma = 0.5$, $p = 0.75$ and $\mu = 0.001$. By Figure 1, it is clear that $\mathcal{R}_0^\alpha$ is monotonically increasing with respect to α. From Table 1, we can observe that the acute infection and chronic infection increase as α increases (see Figures 2 and 3). The endemic equilibrium E^* is GAS by using Theorem 4.5 (see Figure 4). Then, we choose the parameter values [26]: $\Lambda = 1$, $\beta = 0.65$, $d = 0.5$, $\gamma = 0.5$, $p = 0.4$ and $\mu = 0.5$. The disease-free equilibrium E^0 is GAS by applying Theorem 4.4 (see Table 2 and Figure 5).

Due to capturememory and hereditary nature of real-world problems, fractional order has aroused great interests in different fields. By the above simulations, we observe that the incidence rate and fractional order α can effect the dynamical behavior of system (5.1). The acute infection and chronic infection of system (5.1)

Table 1. Endemic equilibrium and stability for different α

α	$\mathcal{R}_0^\alpha$	Equilibrium point	Stability
0.96	19.9847	$E^* = (0.0522, 1.9127, 74.8267, 28.7938)$	E^* is GAS
0.86	13.5983	$E^* = (0.0782, 1.7681, 43.8752, 17.3686)$	E^* is GAS
0.76	9.2084	$E^* = (0.1188, 1.6255, 25.4579, 10.4198)$	E^* is GAS
0.66	6.2029	$E^* = (0.1839, 1.4806, 14.5518, 6.1935)$	E^* is GAS

Table 2. Disease-free equilibrium and stability for different α

α	$\mathcal{R}_0^\alpha$	Equilibrium point	Stability
0.96	0.5098	$E^0 = (1.9453, 0, 0, 0)$	E^0 is GAS
0.86	0.4848	$E^0 = (1.8150, 0, 0, 0)$	E^0 is GAS
0.76	0.4605	$E^0 = (1.6935, 0, 0, 0)$	E^0 is GAS
0.66	0.4369	$E^0 = (1.5801, 0, 0, 0)$	E^0 is GAS

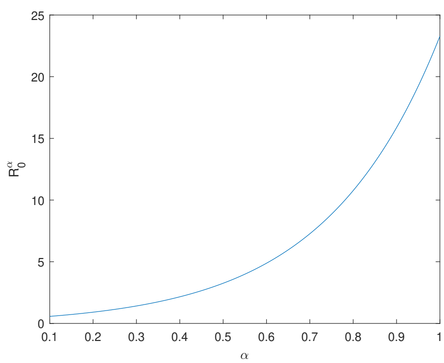

Figure 1. Plot of $\mathcal{R}_0^\alpha$ in terms of α.

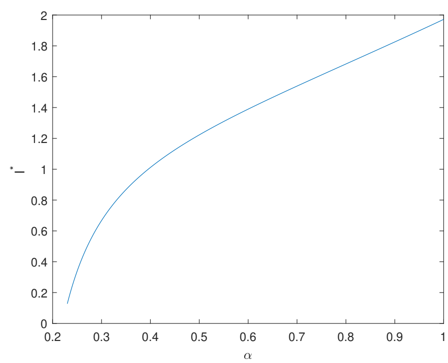

Figure 2. Plot of I^* in terms of α.

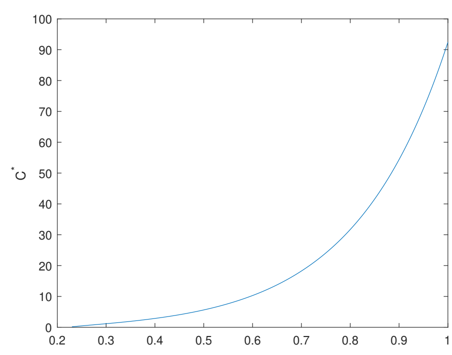

Figure 3. Plot of C^* in terms of α.

decrease as α decreases. This yields that memory can reduce the spread of HCV. Thus, this work is a novel analysis to study the transmission dynamics of HCV.

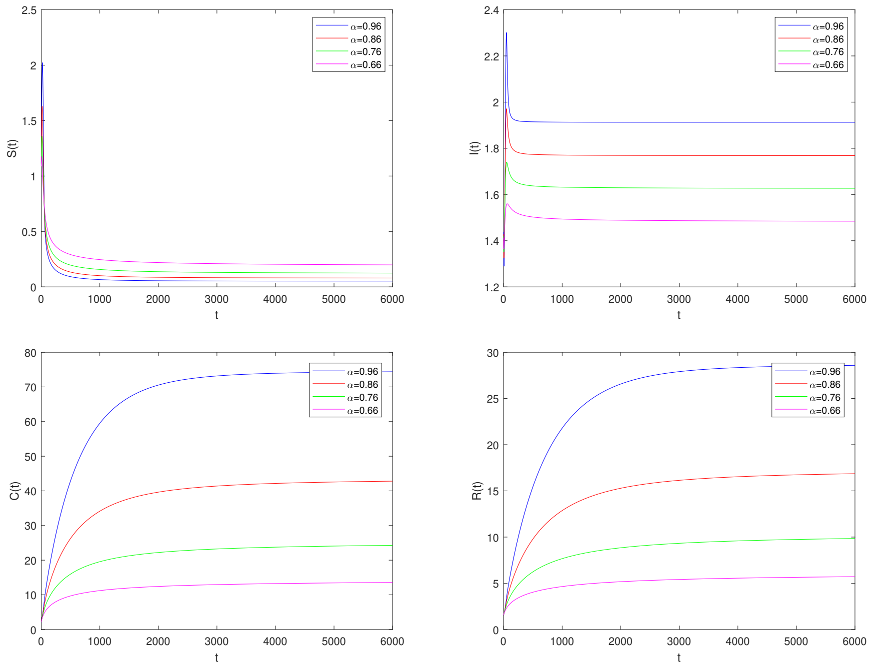

Figure 4. Dynamics of system (5.1) for different values α when $\mathcal{R}_0^{\alpha} > 1$.

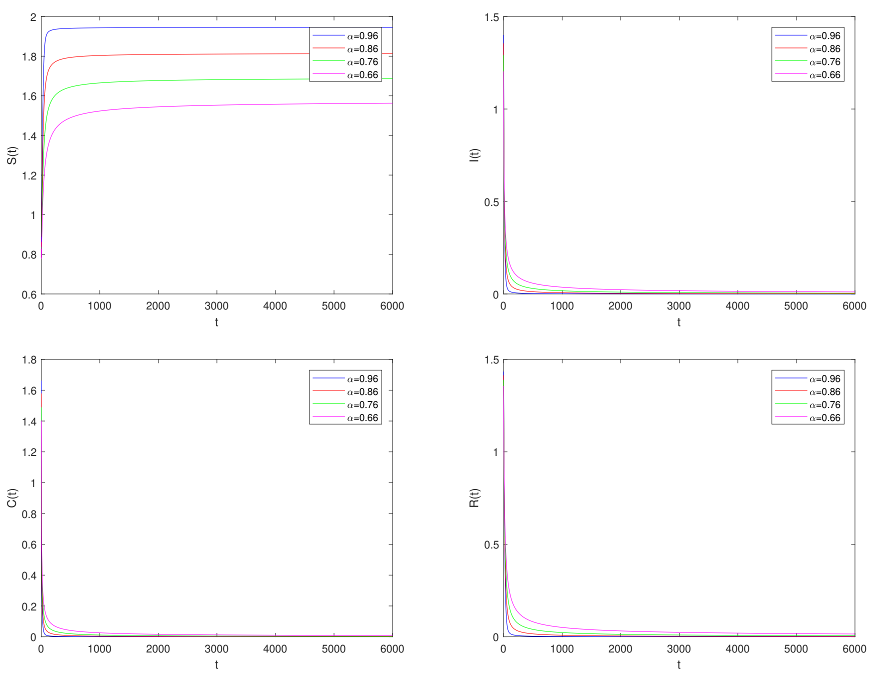

Figure 5. Dynamics of system (5.1) for different values α when $\mathcal{R}_0^{\alpha} < 1$.

This can help us to draft more scientific and reasonable public health policies. It is our future work to take time delay into this model.

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References

[1] T. A. R. Aggarwal and Y. A. Raj, *A fractional order HIV-TB co-infection model in the presence of exogenous reinfection and recurrent TB*, Nonlinear Dyn., 2021, 104, 4701–4725.

[2] E. Ahmed, A. M. A. El-Sayed and H. A. A. El-Saka, *On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems*, Phys. Lett. A, 2006, 358, 1–4.

[3] L. Cai and X. Li, *A note on global stability of an SEI epidemic model with acute and chronic stages*, Appl. Math. Comput., 2008, 196, 23–930.

[4] Y. Chen, F. Liu, Q. Yu and T. Li, *Review of fractional epidemic models*, Appl. Math. Model., 2021, 97, 281–307.

[5] N. D. Cong, T. S. Doan, S. Siegmund and H. T. Tuan, *Linearized asymptotic stability for fractional differential equations*, Electron. J. Qual. Theo. Diff. Equ., 2016, 39, 1–13.

[6] N. D. Cong, T. S. Doan, S. Siegmund and H. T. Tuan, *An instability theorem for nonlinear fractional differential systems*, Discrete Contin. Dyn. Syst. Ser. B, 2017, 22, 3079–3090.

[7] J. Cui, S. Zhao, S. Guo, Y. Bai, X. Wang and T. Chen, *Global dynamics of an epidemiological model with acute and chronic HCV infections*, Appl. Math. Lett., 2020, 103, 106203.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[9] K. Diethelm, *A fractional calculus based model for the simulation of an outbreak of dengue fever*, Nonlinear Dyn., 2013, 71, 613–619.

[10] K. M. Furati, I. O. Sarumi and A. Q. M. Khaliq, *Fractional model for the spread of COVID-19 subject to government intervention and public perception*, Appl. Math. Model., 2021, 95, 89–105.

[11] J. R. Graef, L. Kong, A. Ledoan and M. Wang, *Stability analysis of a fractional online social network model*, Math. Comput. Simulat., 2020, 178, 625–645.

[12] G. Huang, Y. Takeuchi, W. Ma and D. Wei, *Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate*, Bull. Math. Biol., 2010, 72, 1192–1207.

[13] H. Kheiri and M. Jafari, *Stability analysis of a fractional order model for the HIV/AIDS epidemic in a patchy environment*, J. Comput. Appl. Math., 2019, 346, 323–339.

[14] Q. Kong, *A Short Course in Ordinary Differential Equations*, Springer, New York, 2015.

[15] A. Lahrouz, R. Hajjami, M. E. Jarroudi and A. Settati, *Mittag-Leffler stability and bifurcation of a nonlinear fractional model with replase,* J. Comput. Appl. Math., 2021, 386, 113247.

[16] Y. Li, Y. Chen and I. Podlubny, *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability*, Comput. Math. Appl., 2010, 59, 1810–1823.

[17] Z. Lu, Y. Yu, Y. Chen, G. Ren, C. Xu, S. Wang and Z. Yin, *A fractional-order SEIHDR model for COVID-19 with inter-city networked coupling effects*, Nonlinear Dyn., 2020, 101, 1717–1730.

[18] M. Martcheva and C. Castillo-Chavez, *Diseases with chronic stage in a population with varying size,* Math. Biosci., 2003, 182, 1–25.

[19] C. C. McCluskey and Y. Yang, *Global stability of a diffusive virus dynamics model with general incidence function and time delay*, Nonlinear Anal. RWA., 2015, 25, 64–78.

[20] I. Podlubny, *Fractional Differential Equations,* Academic Press, New York, 1999.

[21] B. Reade, R. G. Bowers, M. Begon and R. Gaskell, *A model of disease and vaccination for infections with acute and chronic phases*, J. Theor. Biol., 1998, 190, 55–367.

[22] T. Sardar, S. Rana and J. Chattopadhyay, *A mathematical model of dengue transmission with memory*, Commun. Nonlinear Sci. Numer. Simulat., 2015, 22, 511–525.

[23] J. L. Schiff, *The Laplace Transform: Theory and Applications*, Springer, New York, 1999.

[24] R. Shi, T. Lu and C. Wang, *Dynamic analysis of a fractional-order model for HIV with drug-resistance and CTL immune response*, Math. Comput. Simulat., 2021, 188, 509–536.

[25] B. Sounvoravong and S. Guo, *Dynamics of a diffusive SIR epidemic model with time delay*, J. Nonlinear Model. Anal., 2019, 1, 319–334.

[26] R. Su and W. Yang, *Global stability of a diffusive HCV infections epidemic model with nonlinear incidence*, J. Appl. Math. Comput., 2021. https://doi.org/10.1007/s12190-021-01637-3.

[27] Y. Wang, Z. Zhao and M. Wang, *The transmissibility of hepatitis C virus: a modelling study in Xiamen City, China*, Epidemiol. Infect., 2020, 148, e291.

[28] World Health Organization, Hepatitis C, 2021. https://www.who.int/newsroom/fact-sheets/detail/hepatitis-c.

[29] Y. Yang and L. Xu, *Stability of a fractional order SEIR model with general incidence*, Appl. Math. Lett., 2020, 105, 106303.

[30] Y. Yang, J. Zhou and C. H. Hsu, *Threshold dynamics of a diffusive SIRI model with nonlinear incidence rate*, J. Math. Anal. Appl., 2019, 478, 874–896.

[31] Y. Yang, L. Zou, J. Zhou and C. H. Hsu, *Dynamics of a waterborne pathogen model with spatial heterogeneity and general incidence rate*, Nonlinear Anal. RWA., 2020, 53, 103065.

[32] J. Yuan and Z. Yang, *Global dynamics of an SEI model with acute and chronic stages*, J. Comput. Appl. Math., 2008, 213, 465–476.

[33] R. Zhang and Y. Liu, *A new Barbalat's lemma and Lyapunov stability theorem for fractional order systems*, 29th Chinese control and decision conference (CCDC), IEEE, 2017, 3676–3681.

[34] S. Zhang and Y. Zhou, *Dynamics and application of an epidemiological model for hepatitis C*, Math. Comput. Modell., 2012, 56, 36–42.

[35] Y. Zhou, J. Wang and L. Zhang, *Basic Theory of Fractional Differential Equations (Second Edition)*, World Scientific, Singapore, 2016.