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EXISTENCE AND COMPUTATION OF INVARIANT ALGEBRAIC CURVES FOR PLANAR QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract Some necessary conditions are given for the existence of invariant algebraic curves for planar quadratic differential systems in a special canonical form. An efficient algorithm is then designed for computations of invariant algebraic curves. From the algorithm, a quadratic differential system is found with two Hopf bifurcations as the parameter varies, each leading to an invariant algebraic limit cycle of degree 5. A family of degree 6 invariant algebraic limit cycles is also produced. To further demonstrate the capability of the algorithm, we provide a quadratic system with a family of degree 7 invariant algebraic curves enclosing one or two centers, and a system possessing a degree 16 irreducible invariant algebraic curve with a singular point of multiplicity 8 on the curve.

Keywords Invariant algebraic curve, algebraic limit cycle, quadratic polynomial differential system, algorithm.

MSC(2010) 34C05, 34C07, 70K05.

1. Introduction

Invariant algebraic curves can play an important role in planar dynamical systems with polynomial vector fields, for example, in the theory of Darboux integrability [6, 7, 12, 16]. Despite extensive investigations in this field during the past century, there are still many open questions, especially for the existence of invariant algebraic limit cycles with high degree [8, 9, 11, 15]. One of the difficulties lies in the construction of polynomial systems that possess irreducible invariant algebraic curves with a given degree. From the computational point of view, it usually involves large scale symbolic computations which may not succeed even with the help of high performance computers.

Even for planar quadratic differential systems where the vector field is determined by polynomials of degree 2, there are many unanswered questions. For many decades, mathematicians have been searching for planar quadratic systems that possess invariant algebraic limit cycles, and trying to determine the maximum number of algebraic limit cycles a quadratic system may have [2–4, 8, 17–20]. It is well known (see, for example, [2, 19]) there is exactly one family possessing an invariant algebraic limit cycle of degree 2. No quadratic system has an algebraic limit cycle

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of degree 3. Exactly four families of quadratic systems have an invariant algebraic limit cycle of degree 4. But for higher degree, so far, only two families are found to have an invariant algebraic limit cycle of degree 5, and one family has an algebraic limit cycle of degree 6. It is still unknown weather there is any other quadratic system possessing an invariant algebraic limit cycle of degree 5 or 6. No discovery of any degree 7 or higher invariant algebraic limit cycles has been made for planar quadratic systems.

The main goal of this paper is to develop an algorithm that is promising to find invariant algebraic curves and algebraic limit cycles of high degree for quadratic systems in a certain canonical form. The idea is similar to that presented in [20]. We transform the problem of finding an invariant algebraic curve to a problem of solving large system of linear equations. In general, this process produces a large block-multi-diagonal coefficient matrix for the linear system of equations obtained from a planar quadratic system. Some invariant algebraic curves are reported in [20] for some special quadratic systems. However, no algebraic limit cycle is produced partially because of the large degrees of freedom in quadratic systems. The algorithm present in this paper is based on the observation of a special canonical form for planar quadratic systems. When the unknowns are ordered properly, the coefficient matrix of the linear system is lower triangular that makes the solving procedure much more efficient.

To further increase the efficiency of our algorithm, we derive some necessary conditions for the existence of invariant algebraic curves. In the canonical form (1.3) described below, there are five unknown parameters. To determine invariant algebraic curves, there are two more unknowns in the cofactor. From the necessary conditions, the number of unknowns can be reduced from a total of seven to five. Also, under each condition, there are special cases when some diagonal entry in the linear system becomes zero. This further reduces the number of unknowns to 4. The combination of the special canonical form of planar quadratic systems and the reduction of the number of unknowns makes our algorithm promising to find high degree invariant algebraic curves and algebraic limit cycles.

We briefly recall some definitions and introduce the canonical form used in this paper for planar quadratic systems. Let $p(x, y)$ and $q(x, y)$ be real coprime polynomials with degree 2. The following differential equations

$$\dot{x} = p(x, y), \qquad \dot{y} = q(x, y) \tag{1.1}$$

define a planar quadratic differential system in real space, where the dot denotes the derivative with respect to the independent time variable t. For a polynomial $\phi(x, y)$, the algebraic curve $\phi(x, y) = 0$ is called an invariant algebraic curve of the system if there exists a polynomial $K(x, y)$ with degree one such that

$$p\frac{\partial \phi}{\partial x} + q\frac{\partial \phi}{\partial y} = K\,\phi. \tag{1.2}$$

The polynomial K is called the cofactor of the curve $\phi = 0$. It is well-known that every invariant algebraic curve is formed by orbits of the differential system. An algebraic limit cycle of degree N is a limit cycle contained in an irreducible invariant algebraic curve of degree N for the system. Until now, the known maximum degree of invariant algebraic limit cycles is found to be six [2, 8, 19].

There have been some different classifications of planar quadratic systems (see e.g., [10,17,22]). In this paper, we adopt the following canonical form used in [13,14]:

$$\begin{aligned} \dot{x} &= x^2 + xy + y \\ \dot{y} &= \alpha_2 x + \beta_2 y + a_2 x^2 + b_2 xy + c_2 y^2. \end{aligned} \tag{1.3}$$

The authors in [13, 14] introduce this canonical form to investigate the number of limit cycles a planar quadratic system may have. A criterion on parameters $\alpha_2, \beta_2, a_2, b_2$, and c_2 for the existence of four limit cycles is successfully derived using perturbation theory. Although not every quadratic differential system can be transformed to this canonical form, we have found that it is well suited for symbolic computations of invariant algebraic curves. In this paper, we further assume $a_2 \neq 0$. We are especially interested in algebraic limit cycles enclosing a weak focus at the origin that makes it reasonable to assume $\phi(0,0) \neq 0$. Otherwise, the origin is an isolated point on the invariant algebraic curve. Therefore, we assume the cofactor takes the form $K(x,y) = mx + ny$ for some real numbers m and n.

The rest of the paper is organized as follows. Section 2 gives some necessary conditions for the existence of invariant algebraic curves for planar quadratic systems in the canonical form (1.3). A computational method is described in Section 3 to search for invariant algebraic curves and algebraic limit cycles. A quadratic system obtained from the algorithm is presented in Section 4 that undergoes two Hopf bifurcations leading to two degree five invariant algebraic limit cycles. Another planar quadratic system possessing a degree six invariant algebraic limit cycle is presented in Section 5. Since no degree seven invariant algebraic limit cycle is found after combing through all degree seven invariant algebraic curves obtained from the algorithm, we choose to present a quadratic system in Section 6 that has one or two centers depending on parameter values. All centers are enclosed in a family of degree seven irreducible invariant algebraic curves. Finally, a degree sixteen irreducible invariant algebraic curve with a multiplicity eight singular point is presented in Section 7 for a quadratic system. Section 8 concludes the paper.

2. Necessary conditions for the existence of invariant algebraic curves

We write the invariant algebraic curve with degree N as

$$\phi(x,y) = \sum_{i=0}^{N} \phi_i(x,y) = 0, \tag{2.1}$$

where $\phi_i(x,y)$ is a homogeneous polynomial with degree i. In our algorithm presented in the section below, $\phi_N, \phi_{N-1}, \cdots, \phi_0$ will be solved sequenceially. We first provide some necessary conditions for the planar quadratic system in canonical form (1.3) to possess any invariant algebraic curves. The highest degree terms, ϕ_N, in the polynomial ϕ can also be constructed theoretically.

Lemma 2.1. *Suppose the planar quadratic system (1) has an invariant algebraic curve $\phi(x,y) = 0$, and $\phi_N(x,y)$ is the homogeneous part of ϕ with the highest degree. Then every irreducible factor of ϕ_N must be a factor of $xq_2 - yp_2$, where p_2 and q_2 are homogeneous parts of p and q with highest degree, respectively.*

The above lemma is proved in [5] for general polynomial differential system with any degree. Apply it to the canonical form (1.3) of quadratic systems, we have

$$xq_2 - yp_2 = a_2x(x - x_1y)(x - x_2y),$$

where,

$$x_{1,2} = \frac{1 - b_2 \pm q}{2a_2}, \quad q = \sqrt{(b_2 - 1)^2 - 4a_2(c_2 - 1)}. \tag{2.2}$$

We assume $a_2 \neq 0$ and allow complex numbers in the analysis for x_1 and x_2 appeared in the proof of the main theorem below.

Theorem 2.1. *If the canonical form (1.3) for planar quadratic differential systems has an invariant algebraic curve of degree N with cofactor $K(x, y) = mx + ny$, then one of the following conditions must be satisfied:*

(i) $m = n = N$,

(ii) $c_2 = \dfrac{n - r}{N - r}$ *for some integer* $r = 0, 1, \cdots, N - 1$, *and either*

$$b_2 = \frac{2m - N - r}{N - r}, \text{ and } N - r \text{ is even}, \tag{2.3}$$

or

$$n = N + \frac{(N - r)}{4a_2}\left((b_2 - 1)^2 - \frac{(b_2(N - r) + N + r - 2m)^2}{(N - r - 2k)^2}\right), \tag{2.4}$$

where k is an integer, $0 \leq k < \frac{1}{2}(N - r)$.

Proof. Let the highest degree homogeneous polynomial in the invariant algebraic curve be $\phi_N = \sum_{j=r}^{N} a_j x^j y^{N-j}$, where r is an integer, $0 \leq r \leq N$, and $a_r = 1$. The coefficient of $x^r y^{N-r+1}$ in the expression

$$p\frac{\partial \phi}{\partial x} + q\frac{\partial \phi}{\partial y} - (mx + ny)\phi \tag{2.5}$$

is given by $c_2(N - r) + r - n$. It must vanish. If $r = N$, then $n = N$. In this case, the coefficient of x^{N+1} in the above expression (2.5) becomes $N - m$, which also must vanish. Therefore, we get condition (i). If the integer $r < N$, then $c_2 = \frac{n-r}{N-r}$. This gives the first part of condition (ii). Now we prove that either (2.3) or (2.4) has to be valid if there is an invariant algebraic curve.

According to Lemma 2.1, the homogeneous part of the invariant algebraic curve with the highest degree N can be written as

$$\phi_N = x^r(x - x_1y)^{j-r}(x - x_2y)^{N-j}, \quad r \leq j \leq N, \tag{2.6}$$

where, x_1 and x_2 are given in (2.2). Then the following expression contains all terms with the highest degree $N + 1$ in (2.5) :

$$\begin{aligned} H &= \frac{\partial \phi_N}{\partial x}(x^2 + xy) + \frac{\partial \phi_N}{\partial y}(a_2x^2 + b_2xy + c_2y^2) - (mx + ny)\phi_N \\ &= x^{r+1}(x - x_1y)^{j-r-1}(x - x_2y)^{N-j-1}(\delta_1x^2 + \delta_2xy + \delta_3y^2), \end{aligned}$$

where,

$$
\begin{aligned}
\delta_1 &= N - m + a_2(rx_1 - Nx_2) + a_2 j(x_2 - x_1) \\
&= \frac{1}{2}\left((b_2+1)N - (b_2-1)r - 2m - q(N+r-2j)\right), \\
\delta_2 &= N - n + a_2(N-r)x_1x_2 + (b_2-1)j(x_2-x_1) \\
&\quad - N(b_2x_2 + x_1) + (b_2-1)rx_1 + m(x_1+x_2) \\
&= \frac{(b_2-1)}{2a_2}\left((b_2+1)N - (b_2-1)r - 2m - q(N+r-2j)\right), \\
\delta_3 &= \frac{1}{N-r}\left[\left(b_2N^2 - 2b_2Nr + (b_2-1)r^2 - m(N-r) + Nr\right)x_1x_2\right. \\
&\qquad \left. + j(N-n)(x_1-x_2) + (N-n)(rx_2 - Nx_1)\right] \\
&= -\frac{(N-n)}{2a_2(N-r)}\left((b_2+1)N - (b_2-1)r - 2m - q(N+r-2j)\right).
\end{aligned}
$$

Since $H = 0$ for all x and y, we must have $\delta_1 = \delta_2 = \delta_3 = 0$. So their common factor

$$(b_2+1)N - (b_2-1)r - 2m - q(N+r-2j) = 0.$$

If $N+r-2j = 0$, that is, $N-r$ is even, then we get the condition (2.3). Otherwise, we have

$$q = \frac{(b_2+1)N - (b_2-1)r - 2m}{N+r-2j}.$$

Then, together with (2.2), we can solve for n,

$$n = N + \frac{(N-r)}{4a_2}\left((b_2-1)^2 - \frac{(b_2(N-r)+N+r-2m)^2}{(N+r-2j)^2}\right),$$

where, $r \le j < \frac{1}{2}(N+r)$. Set $k = j - r$, we get the condition (2.4). □

Depending on N even or odd, there are $\frac{1}{4}(N+2)^2$ or $\frac{1}{4}(N+1)(N+3)$ conditions in the above theorem. For example, to find degree 6 invariant algebraic limit cycles, we need to comb through planar quadratic differential systems from 16 conditions. In the next section, we will develop an algorithm to find all invariant algebraic curves under each condition.

3. Symbolic computation of invariant algebraic curves

Unlike the algorithm proposed in [20] where a large linear system of equations is generated, using the canonical form (1.3), the algorithm presented in this section has the advantage to divide the whole process into two major stages in the computation of the invariant algebraic curve $\phi(x, y) = 0$. The first stage is to solve for coefficients of all monomials in ϕ as functions of parameters in the differential system. During this stage, other necessary conditions are produced for the existence of invariant algebraic curves. The second stage is to solve for unknown parameters to get sufficient conditions. Then invariant algebraic curves can be constructed accordingly.

3.1. Solving for invariant algebraic curves as functions of unknown parameters

Let $K(x,y)=mx+ny$ be the cofactor of an invariant algebraic curve $\phi=\sum_{i=0}^{N}\phi_i(x,y)$ for planar quadratic system (1.3). Theorem 2 provides necessary conditions for the existence of algebraic curves on the parameter variables, $\alpha_2, \beta_2, a_2, b_2, c_2, m$ and n. For any one condition, from the proof of the theorem, there is always a solution for the highest degree homogeneous polynomial ϕ_N. To determine the existence of ϕ, we try to solve for $\phi_{N-1}, \phi_{N-2}, \cdots, \phi_0$ as functions of parameters sequentially. The necessary conditions in the theorem essentially eliminates two unknown parameters. In each step of the algorithm, one or more equations will be produced for other unknown parameters to satisfy.

Suppose ϕ_k's have been solved for $i < k \leq N$, and are substituted into equation (1.2). Let $\phi_i = \sum_{j=0}^{i} a_{ij}x^{i-j}y^j$. Then, from the homogeneous part of (1.2) with degree i, the coefficients $a_{ij}, 0 \leq j \leq i$, in ϕ_i can be determined from a linear system of equations:

$$A\rho = \delta, \tag{3.1}$$

wheres, ρ is the unknown vector comprising $(a_{i0}, a_{i1}, \cdots, a_{ii})$, δ is the right hand side vector with each component a function of the parameters. For the canonical form (1.3), the coefficient matrix $A = (\sigma_{ij})$ is lower-triangular with 3 diagonals

$$\begin{aligned} &\sigma_{jj} = j - (i-j)c_2 - n, \quad j = 0, 1, \cdots, i, \\ &\sigma_{j+1,j} = j + (i-j)b_2 - m, \quad j = 0, 1, \cdots i, \\ &\sigma_{j+2,j} = (i-j)a_2, \quad j = 0, 1, \cdots i-1. \end{aligned} \tag{3.2}$$

Clearly, the matrix A has $i+2$ rows and $i+1$ columns. Forward substitution method can be performed on the first $i+1$ equations to solve for $(a_{i0}, a_{i1}, \cdots, a_{ii})$. Suppose no diagonal entry is zero, a unique solution is produced from the procedure. Substituting the solution into the last equation, we get a condition that has to be satisfied by all parameters.

We keep track of special parameter values that make one of the diagonal entries, say σ_{ll}, zero. In this case, we skip the l-th equation that usually solves for a_{il} during the process, and continue the forward substitution. After the procedure is finished, the solutions are substituted back into the skipped equation and the last equation. The result from the last equation may or may not contain a_{il}. If it does contain a_{il}, then a_{il} can be solved. Otherwise, a_{il} becomes a free parameter. In this scenario, including the l-th equation skipped earlier, two conditions are produced in the process that must be satisfied for all parameters, including the new one, a_{il}.

The above procedure is sequentially performed for $i = N-1, N-2, \cdots$ until the last step $i = 0$ where there are two equations for the unknown constant term in ϕ. One equation is used to solve the constant term and the other produces a condition for parameters to satisfy.

3.2. Determining solutions for parameters

After the above process is finished, N or more nonlinear algebraic equations are obtained for parameters to satisfy. The next step is to solve these equations to find sufficient conditions for the existence of invariant algebraic curves. In the case assuming no diagonal entry is zero, there are exactly N equations (some may be

0=0) and 5 unknown parameters. In other cases, there are more than N equations and 4 unknowns. Any solution produced from these equations generates an invariant algebraic curve with degree N for the planar quadratic system, although it may not be irreducible.

Mainly two simple techniques are used in solving nonlinear algebraic equations obtained from the above algorithm for parameters. One is pattern match. Because of the large number of monomials in the nonlinear system of equations, it is difficult to solve for all unknowns together. We select several values of one unknown, say, b_2. For each b_2 value, we solve the system of equations for other unknowns. Then we try to use rational functions of b_2 to fit the solution data. This approach works well and greatly reduces the time to find all families of planar quadratic systems possessing invariant algebraic curves with at least one free parameter b_2. The drawback is that it may lose some solutions, especially those isolated ones.

If solutions still cannot be found with fixed b_2 or some solutions cannot be fit with rational functions, we employ Groebner basis (see for example [1]) to generate an equivalent system of equations that can be solved directly. During this process, we are more interested in the nonlinear relation, say, between β_2 and b_2. In the case the relation between β_2 and b_2 can be expressed as a quadratic equation, a rational transform will be performed to avoid square roots during the solving process. In the cases that these two variables satisfy cubic or quartic equations, we discard the family of solutions due to their complicated expressions.

3.3. Post-processing

Substitute the solution for parameters to the solution for coefficients in the polynomial ϕ, the corresponding invariant algebraic curve can be explicitly constructed. Then we can investigate equilibrium points, their transitions, integrability, and other important issues of the quadratic system. We are more interested in algebraic limit cycles of planar quadratic differential systems [3,4,8,15,17–19]. We especially pay attention to Hopf bifurcations leading to limit cycles. When the equilibrium point at the origin has two purely imaginary eigenvalues in the Jacobian matrix, the first Lyapunov quantity is given by (see [13,14])

$$L_1 = -\frac{\pi(\alpha_2(b_2c_2 - 1) - a_2(b_2 + 2))}{4(-\alpha_2)^{5/2}}.$$

If L_1 is not zero, the equilibrium point is a weak focus of multiplicity 1. A Hopf bifurcation occurs at this point with limit cycles emerging when a suitable parameter changes. In this case, we further examine the invariant algebraic curve ϕ obtained in the above algorithm to figure out if it contains an oval. If it does, we can prove theoretically that the limit cycle is indeed algebraic.

We have implemented the algorithm using the software Mathematica [21], which provides some sophisticated methods to solve nonlinear algebraic equations, as well as a tool to generate Groebner basis. Some examples of invariant algebraic curves, including invariant algebraic limit cycles, obtained from our algorithm are presented in the following sections.

Remark 3.1. Unless otherwise stated, all invariant algebraic curves produced below are irreducible. The built-in function in Mathematica is used to check the absolute irreducibility of a polynomial in complex space.

4. Invariant algebraic limit cycles of degree 5

From the algorithm described in the last section, a quadratic system with the following parameters

$$\begin{aligned}
a_2 &= \frac{(2s+1)(9s-7)\left(27s^3-63s^2+89s-77\right)}{8(s-7)^2},\\
b_2 &= \frac{3\left(9s^3+27s^2+19s-63\right)}{8(s-7)},\\
c_2 &= 2,\\
\alpha_2 &= \frac{(s-1)(9s-7)^2\left(3s^2-7\right)}{4(s-7)^2},\\
\beta_2 &= \frac{(9s-7)\left(3s^2+2s-9\right)}{8(s-7)},
\end{aligned} \tag{4.1}$$

is generated to have an invariant algebraic curve of degree 5 that contains a limit cycle:

$$\begin{aligned}
\phi =\, & 3d_1(3s-1)^2\left(3s^2-14s+7\right)u^4\left(\left(27s^3-63s^2+89s-77\right)u+8v\right)\\
& + 4d_1u^2\left(d_2(9s-7)u^2+16\left(45s^3-81s^2+135s-91\right)uv+64v^2\right)\\
& + 3072d_3(s-2)(2s+1)(9s-7)^2\left(15s^2-10s+7\right)u^3\\
& + 9216(s+1)(9s-7)^2\left(9s^4+183s^3-417s^2+509s-308\right)u^2v\\
& - 18432(9s-7)\left(9s^4-48s^3+10s^2-184s+133\right)uv^2\\
& - 98304\left(3s^2-14s+7\right)v^3+144(s-3)^2(9s-7)^2\left(d_3u+16v\right)^2,
\end{aligned}$$

where, we have shifted the origin to a singular point on the curve with a partial scaling

$$u = x+\frac{9s-7}{3(3s-1)},\ v=(s-7)\left(y+\frac{(9s-7)^2}{12(3s-1)}\right),$$

and

$$\begin{aligned}
d_1 &= 27(s-3)(s+1)(3s-1)^2(3s+7)(9s-7),\\
d_2 &= 243s^5-585s^4+1062s^3-2042s^2+2039s-1197,\\
d_3 &= (9s-7)\left(9s^2+2s+25\right).
\end{aligned}$$

The cofactor is given by

$$K(x,y) = 6\left(\frac{3s^2-7}{s-7}x+y\right).$$

For the existence of invariant algebraic curves with degree $N=5$, one of the 12 conditions in Theorem 2 must be satisfied. This quadratic system is produced when $r=4, k=0$ in condition (2.4), equivalently,

$$c_2 = n-4, a_2 = \frac{(m-5)(b_2-m+4)}{n-5}.$$

For the special case $n=6$, one of the diagonal entries σ_{jj} in the coefficient matrix (3.2) is zero. Three complicated algebraic equations with 4 unknown parameters

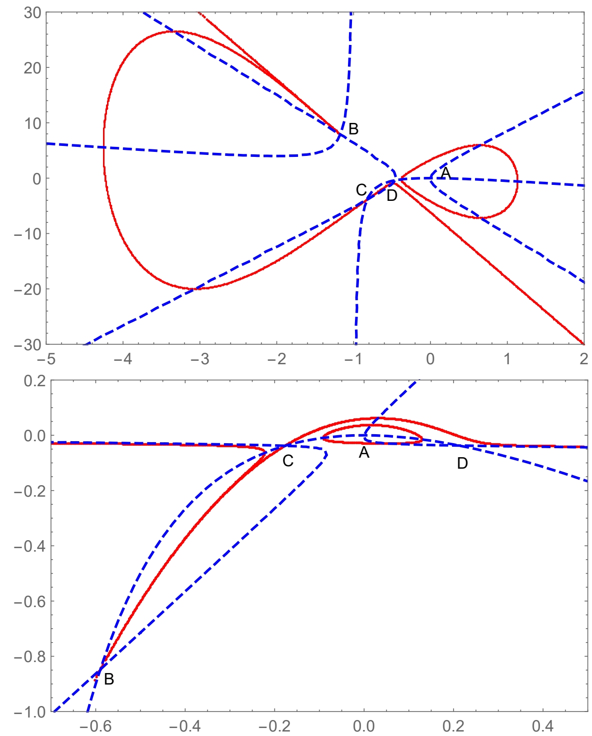

Figure 1. Top: Invariant algebraic limit cycle of degree 5 for the quadratic system (4.1) at $s = -\frac{20}{9}$. Bottom: Another invariant algebraic limit cycle at $s = \frac{17}{12}$. The dashed blue curves in this and other figures below are $\dot{x} = 0$ and $\dot{y} = 0$, respectively. Their intersections are equilibrium points of the dynamical system. Solid red curves are components of the invariant algebraic curve of the quadratic system.

α_2, β_2, b_2, and m are obtained from the algorithm (other two equations 0=0 are deleted). Using Groebner Basis, we found one of the invariant algebraic curves satisfying the equation

$$96b_2^2 - \left(m^2 + 120m - 144\right) b_2 + 2m^3 - 3m^2 + 288m - 1080 = 0.$$

With the help of the rational transform

$$m = \frac{6(3s^2 - 7)}{s - 7},$$

we produced parameter values presented in (4.1).

There are two Hopf bifurcations in this system each generating algebraic limit cycles as parameter s varies. One Hopf bifurcation occurs at $s = -\frac{1}{3}(1 + 2\sqrt{7})$ with Lyapunov number $L \approx -0.025$. Stable limit cycles emerge as a result for smaller s value in the interval $(-\frac{7}{3}, -\frac{1+2\sqrt{7}}{3})$. The invariant algebraic curve for $s = -\frac{20}{9}$ is shown in Figure 1 (top). Clearly, there is a periodic orbit in the algebraic curve.

Theorem 4.1. *Let $f(x, y) = 0$ be a real invariant algebraic curve of degree larger than 1 for the quadratic system (1.1). Let $k(x, y)$ be the cofactor of f. We define*

$$P(X, Y, Z) = Z^2 p(\frac{X}{Z}, \frac{Y}{Z}),\ Q(X, Y, Z) = Z^2 q(\frac{X}{Z}, \frac{Y}{Z}),\ K(X, Y, Z) = Zk(\frac{X}{Z}, \frac{Y}{Z}).$$

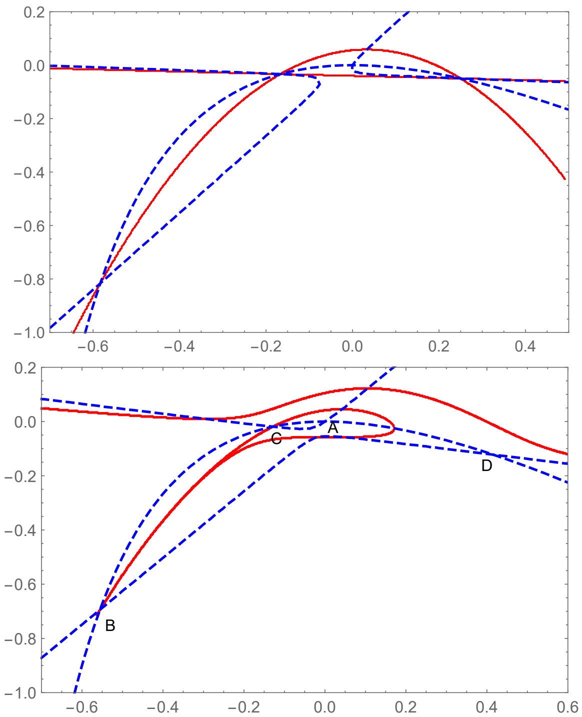

Figure 2. Top: The degree 5 invariant algebraic curve at $s = \frac{7}{5}$ for the quadratic system (4.1). There is a heteroclinic loop enclosing the equilibrium point at the origin. For $s > \frac{7}{5}$, algebraic limit cycle emerges. Bottom: The degree 5 invariant algebraic curve of (4.1) at $s = \frac{4}{3} < \frac{7}{5}$. It contains a heteroclinic loop connecting two equilibrium points B and C.

Suppose that there are two points A_1 and A_2 in the complex projective plane such that $P(A_i) = Q(A_i) = K(A_i) = 0$, $i = 1, 2$. Then all the limit cycles of the quadratic system are contained in $f = 0$, so in particular they are algebraic.

The above theorem is proved in [3]. For the quadratic system with parameters in (4.1), one easily checks that, the following two points

$$A_1(0,0,1), \quad A_2\left(\frac{7-3s^2}{s(3s-1)}, \frac{(7-3s^2)^2}{s(s-7)(3s-1)}, 1\right)$$

in the complex projective plane satisfy the assumption in Theorem 4.1. Therefore, there is an unique degree 5 invariant algebraic limit cycle for the system that is on $\phi = 0$. As s decreases, the size of the limit cycle increases. When s approaches $s = -\frac{7}{3}$, the size increases to infinity.

From Figure 1 (top), besides the algebraic limit cycle enclosing the unstable focus at the origin A, the invariant algebraic curve contains another branch homeomorphic to a straight line passing through two equilibrium points B and C, where B is an unstable node while C is stable. The other equilibrium point D not on the algebraic curve is a hyperbolic saddle. We remark that, at the transition point $s = -\frac{7}{3}$, the invariant algebraic curve degenerates to two straight lines. For $s < -\frac{7}{3}$, it contains three disjoint branches with each homeomorphic to a straight line.

There is another Hopf bifurcation point of the quadratic system at $s = \frac{1}{3}(-1 + 2\sqrt{7})$ with Lyapunov number $L \approx -179.8$. Stable algebraic limit cycles emerge and persist in the interval $\frac{7}{5} < s < \frac{1}{3}(-1+2\sqrt{7})$. An example is shown in Figure 1 (bottom) for $s = \frac{17}{12}$. Similar to the case discussed earlier, the invariant algebraic curve contains one limit cycle enclosing the unstable focus A and another branch passing through two equilibrium points B and C. Unlike the limit cycle presented above, the size of the limit cycle is always bounded in this case. At the transition point $s = \frac{7}{5}$, there is a heteroclinic loop enclosing the unstable focus at the origin. This loop is formed by a straight line and a parabola with two hyperbolic saddles on it (Figure 2 top). For $s < \frac{7}{5}$, There are two disjoint branches in the invariant algebraic curve. One of them forms a heteroclinic loop with a cusp on it (Figure 2 bottom).

5. Invariant algebraic limit cycles of degree 6

The quadratic system with the following parameters

$$\begin{aligned}
a_2 &= -s\left(2s^2+16s+59\right)/70,\\
b_2 &= -(s^2+23s-168)/70,\\
c_2 &= 2,\\
\alpha_2 &= -s^2(2s-3)/70,\\
\beta_2 &= -s(s-19)/70,
\end{aligned} \tag{5.1}$$

has an irreducible invariant algebraic limit cycle with degree 6. Define

$$u = x + \frac{s}{s+2},\ v = y + \frac{s^2}{2s+4},$$

the invariant algebraic curve can be expressed as

$$\begin{aligned}
\phi =&(-14+s)^2(2+s)^6(21+s)(7+2s)u^6\\
&+30(-14+s)^2(2+s)^4(21+s)u^4(s(9+s)u-14v)\\
&+1176000v^2\left(3s\left(3s^2+62s+14\right)u-196(s+1)v\right)\\
&+3000s^3(s+16)\left(2s^3-111s^2-741s-1708\right)u^3\\
&-126000s^2(-1512-160s-41s^2+2s^3)u^2v\\
&+300(-14+s)^2(2+s)^2u^2\left(s^2(179+18s+s^2)u^2-28s(23+s)uv+588v^2\right)\\
&+202500s^2(s(s+16)u-28v)^2.
\end{aligned}$$

The cofactor is $K(x,y) = 6x+6y$. This planar quadratic system is generated when the condition $m = n = 6$ is satisfied in Theorem 2. For the special case $c_2 = 2$, one of the diagonal entries σ_{jj} in the coefficient matrix (3.2) is zero. Three algebraic equations with 4 unknowns α_2, β_2, a_2 and b_2 are obtained from the algorithm (other three equations 0=0 are deleted). Using Groebner basis, for one family of invariant algebraic curves, the parameters b_2 and β_2 in the system satisfy the relation

$$100b_2^2 - 200b_2\beta_2 + 660b_2 + 100\beta_2^2 + 1860\beta_2 - 2160 = 0.$$

The transform $b_2 = -(s^2+23s-168)/70$ is then used to produce all parameter values of the system given in (5.1).

For $14 < s < 19$, the invariant algebraic curve of the system contains an oval enclosing the unstable focus at the origin A as shown in Figure 3 (top) when $s = 18$.

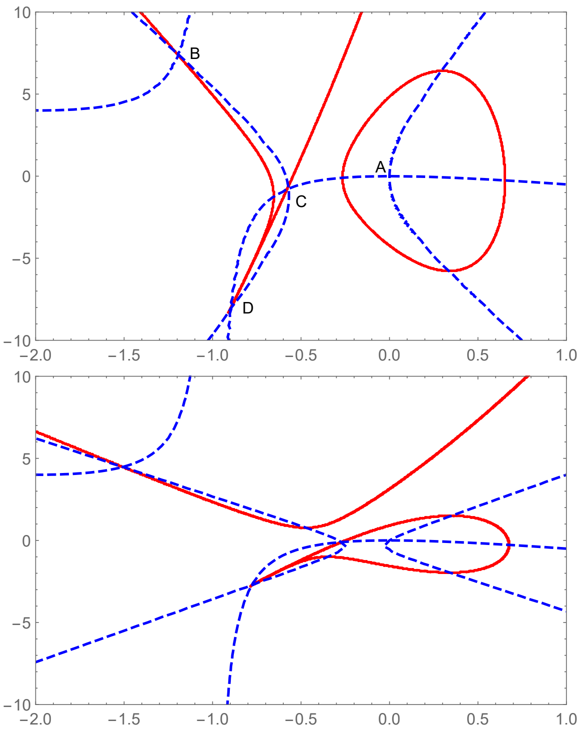

Figure 3. Top: The invariant algebraic curve for the quadratic system (5.1) when $s = 18$. One of the branches is an algebraic limit cycle of degree 6 enclosing the focus at the origin. Bottom: The invariant algebraic curve at $s = 7$. One of the branches is a heteroclinic loop enclosing the focus at the origin with a cusp.

Theorem 5.1. *The quadratic system (1.3) with parameters in (5.1) has a unique limit cycle, the algebraic one of degree 6 on $\phi = 0$.*

Proof. Define the function

$$H = (s^2(2s-3) + 2s(2s^2-4s+19)x + (2+s)(2s^2-9s+14)x^2 - 140y)\phi^{-1/3}.$$

Since $\phi = 0$ is an invariant algebraic curve with cofactor $K = 6x + 6y$, we have

$$\begin{aligned}\dot{H} =& [2s(2s^2-4s+19)\dot{x} + 2(2+s)(2s^2-9s+14)x\dot{x} - 140\dot{y}]\phi^{-1/3} \\ & - \frac{1}{3}[s^2(2s-3) + 2s(2s^2-4s+19)x + (2+s)(2s^2-9s+14)x^2 \\ & - 140y]K(x,y)\phi^{-1/3} \\ =& 40s(s+2)x^2\phi^{-1/3}.\end{aligned}$$

Suppose there is another limit cycle or periodic orbit γ for the system. Because the curve $\phi = 0$ is invariant, so γ does not have intersection with it. Therefore, ϕ does not change sign on γ. Then we would have

$$\int_\gamma \dot{H} = \int_\gamma 40s(s+2)x^2\phi^{-1/3} \neq 0, \text{ for } s \neq 0, -2,$$

a contradiction. Therefore, the algebraic curve $\phi = 0$ contains a unique degree 6 invariant algebraic limit cycle for the system. □

From Figure 3 (top), besides the oval, there is another branch in the algebraic curve passing through other three equilibrium points B, C, and D, where B is an unstable node, C a saddle, and D a stable node. For $s > 19$, there is only one branch in the invariant algebraic curve that is homeomorphic to a straight line. A Hopf bifurcation occurs at $s = 19$ with Lyapunov number $L \approx -0.0025$. Stable algebraic limit cycle emerges in the interval $14 < s < 19$ with the size of the cycle increasing to infinity when s approaches 14. At $s = 14$, the invariant algebraic curve degenerates to two crossing straight lines. In the interval $0 < s < 14$, there is a heteroclinic loop, with a cusp on it, enclosing the unstable focus at the origin as shown in Figure 3 (bottom) when $s = 7$.

6. Invariant algebraic curves of degree 7 enclosing one or two centers

No degree 7 invariant algebraic limit cycle for the quadratic system in canonical form (1.3) has been found from our algorithm. In this section, we report a quadratic system having a family of irreducible invariant algebraic curves of degree 7 that contain periodic orbits enclosing one or two centers for appropriate parameter values.

The planar quadratic differential system (1.3) with the following parameters

$$\begin{aligned}
a_2 &= -5(2b_2-5)(7b_2-4)/243,\\
b_2 &= b_2,\\
c_2 &= 2/5,\\
\alpha_2 &= -25(b_2+2)(7b_2-4)/243,\\
\beta_2 &= 0,
\end{aligned} \tag{6.1}$$

has a family of irreducible invariant algebraic curves with degree 7. Define $p = (35b_2-20)x-27y$, and $s = 7b_2-4$, the polynomial $\phi(x,y)$ of the algebraic curve can be expressed as

$$\begin{aligned}
&896p^7 + 78400sp^6 + 2940000s^2p^5 + 61250000\left(25b_2^2+64b_2-134\right)sp^4\\
&+30625000p^3s\left(1944\left(2b_2^2-b_2-10\right)y-5(13b_2-100)s^2\right)\\
&+6562500p^2\left[40824\left(11b_2{}^2-82b_2-46\right)sy^2-25(307b_2-484)s^4\right.\\
&\left.+45360(b_2+2)s^3y\right]+1093750p\left[972000(b_2+2)s^4y\right.\\
&\left.+2449440(5b_2+1)s^3y^2-39680928(b_2+2)(5b_2+1)sy^3-38125s^6\right]\\
&+78125\left(122472000(5b_2+1)s^4y^2-2499898464(5b_2+1)^2sy^4-953125s^7\right)\\
&+\lambda[-35(2b_2-5)x^2-5(7b_2-4)(7x+5)+189(x+1)y]^2,
\end{aligned} \tag{6.2}$$

where, λ is a free parameter for the system with fixed b_2. The cofactor is given by

$$K(x,y) = \frac{14}{9}(b_2+2)x + \frac{14}{5}y.$$

This system is obtained when $r = k = 0$ in condition (2.4), equivalently,

$$c_2 = \frac{n}{7}, a_2 = \frac{(m-7)(7b_2-m)}{7(n-7)}.$$

For the special case $n = \frac{14}{5}$, one of the diagonal entries σ_{jj} in the coefficient matrix (3.2) is zero. And, when $m = \frac{14}{9}(b_2 + 2)$, another diagonal entry becomes zero. During the first stage of the algorithm, the coefficient of x^2y^2 in the invariant algebraic curve, a_{22}, becomes a free parameter. Eight algebraic equations with 4 unknowns α_2, β_2, b_2 and a_{22} are obtained from the algorithm. Surprisingly, they are easy to solve without using Groebner basis in the software Mathematica. There are three families of solutions with either one or two free parameters, as well as one isolated solution. One family of the solutions produces the quadratic system (6.1) with a free parameter b_2.

For $b_2 < -2$, the quadratic system has 4 equilibrium points. The one at the origin is a center inside a homoclinic orbit. Figure 4 (top) shows 3 invariant algebraic curves of the system corresponding to 3 different λ values in the expression (6.2) when $b_2 = -3$. One contains a periodic orbit enclosing the center. One forms a homoclinic orbit bounding all algebraic periodic orbits. And the third is homeomorphic to a straight line with two cusps on it. For $-2 < b_2 < \frac{4}{7}$, the origin becomes a saddle point. For $b_2 > \frac{4}{7}$, the quadratic system has only two equilibrium points. Both of them are centers. Figure 4 (bottom) shows 6 invariant algebraic curves of the system corresponding to 6 different λ values in the expression (6.2) when $b_2 = 1$, each containing a line and a periodic orbit enclosing one of the two centers.

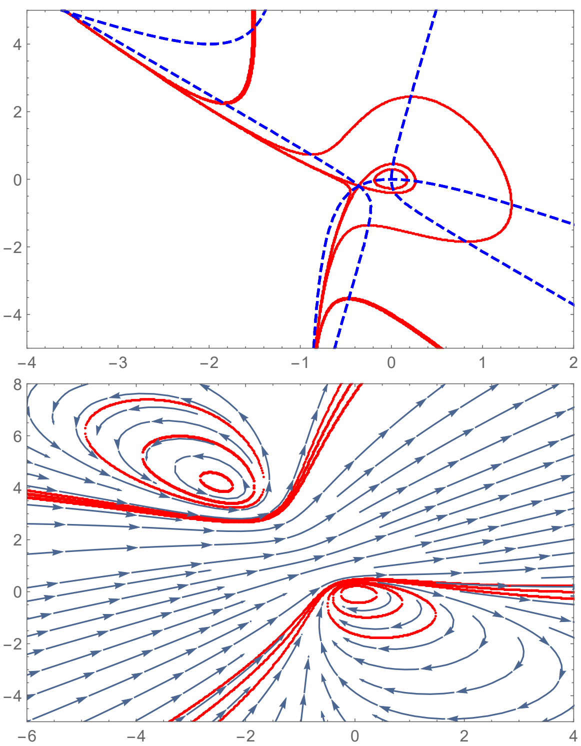

Figure 4. Top: Three invariant algebraic curves of the planar quadratic system (6.1) at $b_2 = -3$. One contains a homoclinic orbit enclosing the center at the origin. Bottom: Six invariant algebraic curves of the planar quadratic system (6.1) at $b_2 = 1$. Each contains a periodic orbit enclosing one of the two centers and a branch homeomorphic to a straight line. Flow field of the dynamical system is shown in the figure with arrowed lines.

7. Irreducible invariant algebraic curves of degree 16

A test run of our algorithm searching for irreducible invariant algebraic curves of degree 16 is performed when the condition $m = n = 16$ is satisfied in Theorem 2.1. Two quadratic systems are found with invariant algebraic limit cycles. However, both polynomials of the curves are reducible. What we have found turn out to be degree 4 irreducible algebraic limit cycles. So we do not report them in this paper. Instead, we show a quadratic differential system possessing a degree 16 irreducible invariant algebraic curve with a singular point of multiplicity 8 on it. The parameters in the system are given in (7.1) below,

$$\begin{aligned} a_2 &= -\frac{4}{225}(s-3)(s+4)(4s-29), \\ b_2 &= \frac{4}{15}\left(s^2-1\right), \\ c_2 &= 2, \\ \alpha_2 &= -\frac{16}{225}(s-11)(s-3)^2, \\ \beta_2 &= \frac{4}{15}(s-10)(s-3). \end{aligned} \tag{7.1}$$

For the condition $m = n = 16$, during the forward substitution procedure in the algorithm, one of the diagonal entries becomes 0 when $c_2 = 2$. Eight nontrivial nonlinear algebraic equations are obtained for 4 parameters from the first stage of the algorithm. For one family of solutions, using pattern match in the second stage of the algorithm, the following polynomial is produced in the Groebner basis

$$4224 - 2140b_2 + 75b_2^2 - 1240\beta_2 - 150b_2\beta_2 + 75\beta_2^2.$$

Then, the solution for all parameters in (7.1) are generated using the transform $b_2 = \frac{4}{15}\left(s^2-1\right)$. Due to the large number of monomials in the polynomial of the invariant algebraic curve, we omit the expression of ϕ in this paper. The cofactor is $K(x,y) = 16(x+y)$.

Take $s = -\frac{13}{3}$, for example, we get the following planar quadratic differential system

$$\begin{aligned} \dot{x} &= x^2 + xy + y, \\ \dot{y} &= \frac{475}{12}x + \frac{265}{12}y - \frac{7}{2}x^2 + \frac{51}{20}xy + 2y^2. \end{aligned} \tag{7.2}$$

The degree 16 polynomial ϕ of the invariant algebraic curve of the system is provided in Appendix A.

System (7.2) has four equilibrium points

$$P_1(0,0),\ P_2(-\frac{25}{27}, -\frac{625}{54}),\ P_3(-\frac{19}{9}, \frac{361}{90}),\ P_4(5, -\frac{25}{6}).$$

The invariant algebraic curve $\phi(x,y) = 0$ and the vector field of the differential system are shown in Figure 5. Two heteroclinic loops are clearly on the algebraic curve, each containing an unstable node (P_3 and P_4, respectively), and meeting at

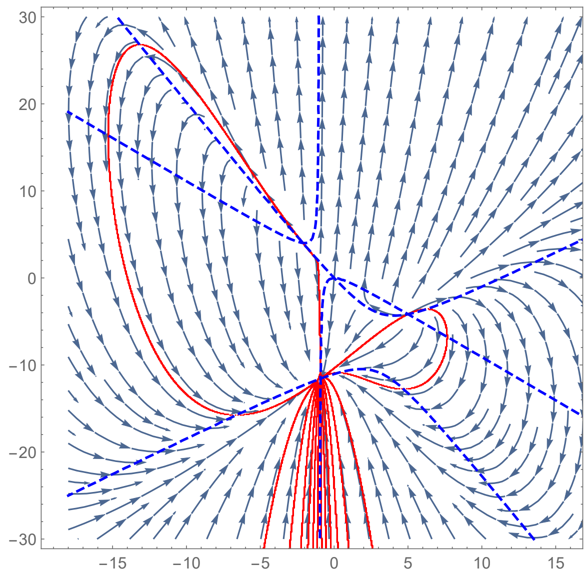

Figure 5. The algebraic curve (solid red) and the vector field (black arrows) of the differential system (7.2). Blue dashed lines are determined by $\dot{x} = 0$ and $\dot{y} = 0$.

the stable node P_2. The other equilibrium point P_1 is a saddle. From the figure, there are 16 orbits on the algebraic curve, each leading to the stable node P_2. From the expression of the polynomial ϕ in (A.1), one easily finds that

$$\frac{\partial^{i+j}\phi}{\partial^i x \partial^j y}(P_2) = 0$$

for $i + j = 0, 1, \cdots, 7$. Therefore, P_2 is a singular point of the polynomial ϕ with multiplicity 8.

This is one example to show the effectiveness of our algorithm finding high degree invariant algebraic curves. We are in the process generating other quadratic systems with complex dynamics.

8. Conclusion

In this paper, we have derived some necessary conditions for the existence of invariant algebraic curves for planar quadratic systems in the canonical form (1.3). An algorithm is then carefully designed to search for invariant algebraic curves with an arbitrary degree. From the algorithm, we have successfully produced one example of degree 5 and one example of degree 6 invariant algebraic limit cycles for two quadratic differential systems, respectively. A family of degree 7 invariant algebraic curves enclosing one or two centers is also produced from the algorithm. A test run for degree 16 irreducible invariant algebraic curves further shows that our algorithm is capable of producing high degree algebraic curves. Theoretically, based on the algorithm presented in this paper, going over all conditions in Theorem 2.1, it is possible to find all invariant algebraic limit cycles for quadratic systems in the canonical form (1.3).

A. Coefficients in the irreducible invariant algebraic curve of degree 16

The invariant algebraic curve of the quadratic system (7.2) can be written as

$$\phi(x,y) = \sum_{8\le i+j\le 16} a_{ij}\left(x+\frac{25}{27}\right)^i\left(y+\frac{625}{54}\right)^j = 0, \tag{A.1}$$

where the nonzero terms are given below in Figure 6 with the format (i, j, a_{ij}).

$$\begin{pmatrix}
8 & 0 & -2^{8}5^{16}11^{1}13^{2}\\
9 & 0 & 2^{10}3^{5}5^{14}13^{1}23^{1}41^{1}\\
10 & 0 & -2^{8}3^{9}5^{11}7^{1}13^{1}19^{1}23^{1}103^{1}\\
11 & 0 & 2^{8}3^{15}5^{10}13^{1}19^{1}23^{1}83^{1}\\
12 & 0 & -2^{5}3^{20}5^{9}7^{1}11^{1}13^{1}19^{1}23^{1}\\
13 & 0 & 2^{6}3^{23}5^{6}7^{1}13^{1}19^{1}23^{1}43^{1}\\
14 & 0 & -2^{4}3^{29}5^{4}7^{1}19^{1}23^{2}\\
15 & 0 & -2^{4}3^{34}5^{1}19^{1}23^{1}\\
16 & 0 & 3^{38}19^{1}23^{1}\\
7 & 1 & 2^{13}5^{18}13^{1}\\
8 & 1 & -2^{12}3^{5}5^{14}7^{2}13^{1}23^{1}\\
9 & 1 & 2^{11}3^{10}5^{11}7^{1}13^{1}17^{1}19^{1}23^{1}\\
10 & 1 & -2^{10}3^{15}5^{11}13^{2}19^{1}23^{1}\\
11 & 1 & 2^{9}3^{20}5^{9}11^{1}13^{1}19^{1}23^{1}\\
12 & 1 & -2^{8}3^{24}5^{7}7^{1}13^{1}19^{1}23^{1}\\
13 & 1 & 2^{7}3^{29}5^{4}7^{1}19^{1}23^{1}\\
14 & 1 & -2^{6}3^{34}5^{1}19^{1}23^{1}\\
6 & 2 & -2^{14}5^{14}7^{1}13^{1}107^{1}\\
7 & 2 & 2^{14}3^{6}5^{12}7^{1}13^{1}23^{1}29^{1}\\
8 & 2 & -2^{12}3^{10}5^{10}7^{1}13^{1}19^{1}23^{1}67^{1}\\
9 & 2 & 2^{13}3^{15}5^{9}13^{1}19^{1}23^{1}47^{1}\\
10 & 2 & -2^{10}3^{22}5^{7}11^{1}13^{1}19^{1}23^{1}\\
11 & 2 & 2^{10}3^{24}5^{4}7^{2}13^{1}19^{1}23^{1}
\end{pmatrix}
\begin{pmatrix}
12 & 2 & 2^{8}3^{29}5^{2}7^{1}13^{1}19^{1}23^{1}\\
5 & 3 & 2^{17}5^{13}7^{1}13^{1}89^{1}\\
6 & 3 & -2^{16}3^{5}5^{12}7^{1}13^{1}23^{2}\\
7 & 3 & 2^{16}3^{9}5^{9}7^{3}13^{1}19^{1}23^{1}\\
8 & 3 & -2^{15}3^{15}5^{8}13^{1}19^{1}23^{1}29^{1}\\
9 & 3 & 2^{13}3^{20}5^{6}11^{1}13^{1}19^{1}23^{1}\\
10 & 3 & 2^{12}3^{23}5^{3}7^{1}11^{1}13^{1}19^{1}23^{1}\\
4 & 4 & -2^{17}5^{13}7^{1}13^{1}71^{1}\\
5 & 4 & 2^{18}3^{5}5^{11}7^{1}13^{1}17^{1}23^{1}\\
6 & 4 & -2^{16}3^{10}5^{8}7^{1}13^{1}19^{1}23^{1}31^{1}\\
7 & 4 & 2^{16}3^{15}5^{7}11^{1}13^{1}19^{1}23^{1}\\
8 & 4 & 2^{13}3^{20}5^{5}11^{1}13^{1}19^{1}23^{1}\\
3 & 5 & 2^{21}5^{11}7^{1}13^{1}53^{1}\\
4 & 5 & -2^{20}3^{6}5^{9}7^{1}11^{1}13^{1}23^{1}\\
5 & 5 & 2^{19}3^{10}5^{6}7^{1}13^{2}19^{1}23^{1}\\
6 & 5 & 2^{18}3^{15}5^{5}7^{1}13^{1}19^{1}23^{1}\\
2 & 6 & -2^{22}5^{11}7^{2}13^{1}\\
3 & 6 & 2^{22}3^{5}5^{9}7^{1}13^{1}23^{1}\\
4 & 6 & 2^{20}3^{9}5^{6}7^{1}13^{1}19^{1}23^{1}\\
1 & 7 & 2^{25}5^{9}13^{1}17^{1}\\
2 & 7 & 2^{24}3^{5}5^{7}13^{1}23^{1}\\
0 & 8 & 2^{24}5^{8}13^{1}
\end{pmatrix}$$

Figure 6. Coefficients in the algebraic curve (A.1).

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