BOUNDEDNESS CRITERION FOR SUBLINEAR OPERATORS AND COMMUTATORS ON GENERALIZED MIXED MORREY SPACES

Mingquan Wei[1,†]

Abstract In this paper, the author studies the boundedness for a large class of sublinear operators $T_\alpha, \alpha \in [0, n)$ generated by Calderón-Zygmund operators ($\alpha = 0$) and generated by fractional integral operator ($\alpha > 0$) on generalized mixed Morrey spaces $M_{\vec{q}}^{\varphi}(\mathbb{R}^n)$. Moreover, the boundeness for commutators of $T_\alpha, \alpha \in [0, n)$ on generalized mixed Morrey spaces $M_{\vec{q}}^{\varphi}(\mathbb{R}^n)$ is also studied. As applications, we obtain the boundedness for Hardy-Littlewood maximal operator, Calderón-Zygmund singular integral operators, fractional integral operator, fractional maximal operator and their commutators on generalzied mixed Morrey spaces.

Keywords Generalized mixed Morrey spaces, sublinear operator, commutators, Hardy-Littlewood maximal operator, fractional integral operator.

MSC(2010) 42B20, 42B25, 42B35.

1. Introduction

For $x \in \mathbb{R}^n$, and $r > 0$, let $B(x, r)$ be the open ball centered at x with the radius r, and $B^c(x, r)$ be its complement. The well-known fractional maximal operator M_α and fractional integral operator I_α are defined by

$$M_\alpha f(x) = \sup_{r > 0} \frac{1}{|B(x, r)|^{1 - \alpha/n}} \int_{B(x, r)} |f(y)| dy, \;\; 0 \leq \alpha < n,$$

and

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n - \alpha}} dy, \;\; 0 < \alpha < n$$

respectively, where f is locally integrable and $|B(x, r)|$ is the Lebesgue measure of $B(x, r)$. If $\alpha = 0$, then $M \equiv M_0$ is the classical Hardy-Littlewood maximal operator.

Other than fractional maximal operator M_α and fractional integral operator I_α, Calderón-Zygmund singular integral operators (see [8]) are also basic integral operators in harmonic analyis. A Calderón-Zygmund singular integral operator is a linear operator bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, which takes all infinitely

---
[†]The corresponding author. Email: weimingquan11@mails.ucas.ac.cn(M. Wei)
[1]School of Mathematics and Stastics, Xinyang Normal University, Henan, Xinyang 464000, China

continuously differentiable functions f with compact support to the functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and can be represented by

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy, \quad \text{a.e. off } \text{supp}f.$$

Here $k(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exists some $0 < \epsilon \leq 1$, such that

$$|k(x,y)| \lesssim \frac{1}{|x-y|^n}$$

for all $x, y \in \mathbb{R}^n, x \neq y$ and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \lesssim \frac{|x-x'|^\epsilon}{|x-y|^{n+\epsilon}}$$

whenever $2|x - x'| < |x - y|$.

In order to study the above operators and some related operators in harmonic analysis uniformly, many researchers introduced the following sublinear operators satisfying some size conditions.

The first one is $T \equiv T_0$, which is a sublinear opertor, and satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp}f$,

$$|Tf(x)| \lesssim \int \frac{|f(y)|}{|x-y|^n} dy. \tag{1.1}$$

Another one is T_α ($0 < \alpha < n$), the fractional version of T_0, which is also a sublinear opertor, and satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp}f$,

$$|T_\alpha f(x)| \lesssim \int \frac{|f(y)|}{|x-y|^{n-\alpha}} dy. \tag{1.2}$$

We point out that condition (1.1) was first introduced by Soria and Weiss [26] and condition (1.2) was introduced by Guliyev et al. [12]. Conditions (1.1) and (1.2) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund singular integral operators, the Carleson's maximal operators, the Hardy-Littlewood maximal operators, the Fefferman's singular multipliers, the Fefferman's singular integrals, the Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on (see [16, 26] for more details).

As is well known, commutators are also important operators and play a key role in harmonic analysis. Recall that for a locally integrable function b and an integral operator T, the commutator formed by b and T is defined by $[b, T] = bT - Tb$. Commutators of fractional maximal operator, fractional integral operator and Calderón-Zygmund singular integral operators have been intensively studied, see [8] for more details. It is worthy pointing out that there are two different commutators of the fractional maximal operator M_α. In this paper, the commutator of fractional maximal operator $[b, M_\alpha]$ under consideration is of the form

$$[b, M_\alpha]f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |b(x) - b(y)| \, |f(y)|dy, \quad 0 \leq \alpha < n,$$

for all locally integrable functions f on $\mathbb{R}^n$. For simplicity, we denote by $[b, M] = [b, M_0]$.

To study a class of commutators uniformly, one can also introduce some sublinear operators with additional size conditions as before. For a function b, suppose that the operator $T_b \equiv T_{b,0}$ represents a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$,

$$|T_b f(x)| \lesssim \int \frac{|b(x) - b(y)|}{|x - y|^n} |f(y)| dy. \tag{1.3}$$

Similarly, we assume that the operator $T_{b,\alpha}, \alpha \in (0, n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$,

$$|T_{b,\alpha} f(x)| \lesssim \int \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} |f(y)| dy. \tag{1.4}$$

The operator $T_{b,\alpha}, \alpha \in [0, n)$ has been studied in [12, 16].

As we know, the classical Morrey space is an important generalization of Lebesgue spaces. The classical Morrey space was introduced by Morrey in [18] to study the regularity of elliptic partial differential equations. Nowadays, the Morrey space has become one of the most important function spaces in the theory of function spaces. By a slight modification, Guliyev et al. [3, 12] introduced generalized Morrey spaces and studied the boundedness of many important operators in harmonic analysis on generalized Morrey spaces. In particular, the boundedness of sublinear operators T, T_α, T_b and $T_{b,\alpha}$ was considered in [12]. Another interesting extension of the classical Morrey space is the mixed Morrey space, which was introduced by Nogayama et al. [20, 21, 22]. The boundedness of many integral operators and their commutators on mixed Morrey spaces was studied in [20, 21].

Highly inspired by the work of Guliyev [3, 12, 13] and Nogayama et al. [20, 21, 22], we are going to study the boundedness of the operators T, T_α, T_b and $T_{b,\alpha}$ under some size conditions on generalized mixed Morrey spaces in this paper. Here, generalized mixed Morrey spaces are the combination of generalized Morrey spaces and mixed Morrey spaces (see Definition 2.1 in the following section), and theorefore are much more general. Our main results extend the boundedness of many operators on Morrey spaces [1, 24, 25, 28], mixed Lebesgue spaces [2, 4], generalized Morrey spaces [3, 12, 13], and mixed Morrey spaces [20, 21]. Moreover, one can obtain the boundedness of many integral operators in harmonic analysis on generalized mixed Morrey spaces from our main theorems.

This paper is organized as follows. The definitions and some preliminaries are presented in Sect. 2. The boundedness of T and T_α on generalized mixed Morrey spaces is studied in Sect. 3. The boundedness of T_b and $T_{b,\alpha}$ on generalized mixed Morrey spaces is obtained in Sect. 4. Some applications are given in Sect. 5 to show the power of our main theorems.

2. Definitions and preliminaries

Throughout the paper, we use the following notations.

For any $r > 0$ and $x \in \mathbb{R}^n$, let $B(x, r) = \{y : |y - x| < r\}$ be the ball centered at x with radius r. Let $\mathbb{B} = \{B(x, r) : x \in \mathbb{R}^n, r > 0\}$ be the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E.

The letter $\vec{p}$ denotes n-tuples of the numbers in $(0, \infty]$, $(n \geq 1)$, $\vec{p} = (p_1, \cdots, p_n)$. By definition, the inequality, for example, $0 < \vec{p} < \infty$ means $0 < p_i < \infty$ for all i. For $1 \leq \vec{p} \leq \infty$, we denote $\vec{p}' = (p_1', \cdots, p_n')$, where p_i' satisfies $\frac{1}{p_i} + \frac{1}{p_i'} = 1$. By $A \lesssim B$, we mean that $A \leq CB$ for some constant $C > 0$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

Let $\mathcal{M}(\mathbb{R}^n)$ be the class of Lebesgue measurable functions on $\mathbb{R}^n$. For $0 < \vec{p} < \infty$, a measurable function f on $\mathbb{R}^n$ belongs to $L_{\mathrm{loc}}^{\vec{p}}(\mathbb{R}^n)$ if $f\chi_E \in L^{\vec{p}}(\mathbb{R}^n)$ for any compact subset E of $\mathbb{R}^n$. We also use $\mathbb{C}$ to represent all the complex numbers, and $\mathbb{N}$ to represent the collection of all non-negative integers.

The classical Morrey space $M_q^p(\mathbb{R}^n)$ is a natural generalization of Lebesgue spaces, which consist of all functions $f \in L_{\mathrm{loc}}^p(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_q^p} = \sup_{x \in \mathbb{R}^n, r > 0} |B(x,r)|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(B(x,r))},$$

where $1 \leq q \leq p \leq \infty$. Note that $M_q^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $p = q$, and $M_q^p(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ when $p = \infty$. If $q > p$, then $M_q^p(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

The classical Morrey space is a proper substitution when we consider the boundedness of integral operators in harmonic analysis. For example, Chiarenza and Frasca [5] studied the boundedness of the maximal operator M in $M_q^p(\mathbb{R}^n)$. The well known Hardy-Littlewood-Sobolev inequality was also extented to Morrey spaces by Spanne (but published by Peetre [23]) and Adams [1]. Since $M_\alpha f(x) \lesssim I_\alpha |f|(x), 0 < \alpha < n$, the fractional maximal operator M_α is also bounded on $M_q^p(\mathbb{R}^n)$.

In recent years, classical Morrey spaces $M_q^p(\mathbb{R}^n)$ were extended to generalized Morrey spaces by Guliyev et al. [3, 10, 12, 13].

Let $1 \leq q < \infty$ and $\varphi(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. A function $f \in \mathcal{M}(\mathbb{R}^n)$ belongs to $M_q^\varphi(\mathbb{R}^n)$, the generalized Morrey spaces, if it satisfies

$$\|f\|_{M_q^\varphi} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{q}} \|f\|_{L^q(B(x,r))} < \infty.$$

Roughly speaking, generalized Morrey space $M_q^\varphi(\mathbb{R}^n)$ can be obtained by substituting $|B(x,r)|^{1/p - 1/q}$ with a more general function in the definition of $M_q^p(\mathbb{R}^n)$. From the definition, we recover the classical Morrey space $M_q^p(\mathbb{R}^n)$ by taking $\varphi(x,r) = |B(x,r)|^{-1/p}$. The boundedness of the sublinear operators T, T_α, T_b and $T_{b,\alpha}$ on spaces $M_q^\varphi(\mathbb{R}^n)$ under some size conditions was obtained by Guliyev et al. [12].

In 2019, Nogayama [21] considered a new Morrey type space, with the $L^p(\mathbb{R}^n)$ norm replaced by the mixed Lebesgue norm $L^{\vec{q}}(\mathbb{R}^n)$, which is called the mixed Morrey space.

We first recall the definition of mixed Lebesgue spaces defined in [2].

Let $\vec{p} = (p_1, \cdots, p_n) \in (0,\infty]^n$. Then the mixed Lebesgue norm $\|\cdot\|_{L^{\vec{p}}}$ is defined by

$$\|f\|_{L^{\vec{p}}} = \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \cdots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \cdots dx_n \right)^{\frac{1}{p_n}}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function. If $p_j = \infty$ for some $j = 1, \cdots, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ to be the set of all $f \in \mathcal{M}(\mathbb{R}^n)$ with $\|f\|_{L^{\vec{p}}} < \infty$.

Let $1 \leq \vec{q} < \infty, 1 \leq p < \infty$ and $n/p \leq \sum_{i=1}^n 1/q_i$. A function $f \in \mathcal{M}(\mathbb{R}^n)$ belongs to the mixed Morrey spaces $M_{\vec{q}}^p(\mathbb{R}^n)$ if

$$\|f\|_{M_{\vec{q}}^p} = \sup_{x \in \mathbb{R}^n, r > 0} |B(x,r)|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{i=1}^n \frac{1}{q_i}\right)} \|f\chi_{B(x,r)}\|_{L^{\vec{q}}} < \infty.$$

Obviously, we recover the classical Morrey space $M_q^p(\mathbb{R}^n)$ when $\vec{q} = q$. We point out that in [20, 21], the author used the cubes to define the mixed Morrey spaces. It is not hard to verify that the two definitions are equivalent.

In [21], the Hardy-Littlewood maximal operator M, the fractional interal operator I_α and the singular integral operators K were proved to be bounded in $M_{\vec{q}}^p(\mathbb{R}^n)$. The boundedness of the commutator of I_α on $M_{\vec{q}}^p(\mathbb{R}^n)$ was also obtained in [20].

Noting that generalized Morrey spaces and mixed Morrey spaces are different extensions of the classical Morrey spaces, it is natural for us to unify the two spaces.

Now we are in a position to give the difinition of generalized mixed Morrey spaces.

Definition 2.1. Let $1 \leq \vec{q} < \infty$, and $\varphi(x,r) : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. A function $f \in \mathcal{M}(\mathbb{R}^n)$ belongs to the mixed Morrey space $M_{\vec{q}}^\varphi(\mathbb{R}^n)$ if

$$\|f\|_{M_{\vec{q}}^\varphi} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1} \|\chi_{B(x,r)}\|_{L^{\vec{q}}}^{-1} \|f\chi_{B(x,r)}\|_{L^{\vec{q}}} < \infty.$$

Generalized mixed Morrey spaces contain generalized Morrey spaces and mixed Morrey spaces as special cases. In fact, $M_{\vec{q}}^\varphi(\mathbb{R}^n) = M_q^\varphi(\mathbb{R}^n)$ when $\vec{q} = q$, and $M_{\vec{q}}^\varphi(\mathbb{R}^n) = M_{\vec{q}}^p(\mathbb{R}^n)$ when $\varphi(x,r) = |B(x,r)|^{-1/p}$. We point out that generalized mixed Morrey spaces were also introduced recently by Zhang and Zhou [29] in different ways.

For a non-negative locally integrable function w on $\mathbb{R}^n$, the weighted Hardy operators H_w and H_w^* are defined by defined

$$H_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty$$

and

$$H_w^* g(t) := \int_t^\infty \left(1 + \ln\frac{s}{t}\right) g(s)w(s)ds, \quad 0 < t < \infty,$$

respectively, where $g \in L_{\text{loc}}^1(\mathbb{R}^n)$.

We have the following boundedness results for the weighted Hardy operators H_w and H_w^* proved in [9, 11].

Lemma 2.1. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w g(t) \leq C \operatorname*{ess\,sup}_{t>0} v_1(t)g(t)$$

holds for all non-negative and non-increasing g on $(0, \infty)$ if and only if

$$A = \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty,$$

and $C \sim A$.

Lemma 2.2. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w^* g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t)$$

holds for all non-negative and non-increasing g *on* $(0, \infty)$ *if and only if*

$$A = \sup_{t>0} v_2(t) \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{w(s) ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty,$$

and $C \sim A$.

3. Sublinear operators T and T_α in spaces $M_{\vec{q}}^{\varphi}(\mathbb{R}^n)$

In this section, we investigate the boundedness of T and T_α satisfying the size conditions (1.1) and (1.2) respectively, on generalized mixed Morrey spaces $M_{\vec{q}}^{\varphi}(\mathbb{R}^n)$.

We first prove two lemmas, which give us the explicit estimats for the $L^{\vec{q}}(\mathbb{R}^n)$ norm of T and T_α on a given ball $B(x_0, r)$.

Lemma 3.1. *Let* $1 < \vec{q} < \infty$, T *be a sublinear operator satisfying condition* (1.1), *and bounded on* $L^{\vec{q}}(\mathbb{R}^n)$.

Then for $1 < \vec{q} < \infty$, *the inequality*

$$\|Tf\|_{L^{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{q_i}} \int_{2r}^\infty t^{-1-\sum_{i=1}^n \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ *and all* $f \in L_{\text{loc}}^{\vec{q}}(\mathbb{R}^n)$.

Proof. For any ball $B = B(x_0, r)$, let $2B = B(x_0, 2r)$ be the ball centered at x_0, with the radius $2r$. We represent f as $f = f_1 + f_2$, where

$$f_1(y) = f\chi_{2B}(y), \quad f_2(y) = f\chi_{(2B)^c}(y), \quad r > 0.$$

Since T is a sublinear operator, we have

$$\|Tf\|_{L^{\vec{q}}(B)} \le \|Tf_1\|_{L^{\vec{q}}(B)} + \|Tf_2\|_{L^{\vec{q}}(B)}.$$

Noting that $f_1 \in L^{\vec{q}}(\mathbb{R}^n)$ and T is bounded in $L^{\vec{q}}(\mathbb{R}^n)$, we have

$$\|Tf_1\|_{L^{\vec{q}}(B)} \le \|Tf_1\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} = \|f\|_{L^{\vec{q}}(2B)}.$$

It is clear that $x \in B$, $y \in (2B)^c$ imply $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$, which further yields

$$|Tf_2(x)| \lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem, we have

$$\int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^n} dy \sim \int_{(2B)^c} |f(y)| \int_{|x_0-y|}^\infty \frac{dt}{t^{n+1}} dy$$

$$\sim \int_{2r}^\infty \int_{2r \le |x_0-y|<t} |f(y)| dy \frac{dt}{t^{n+1}}$$

$$\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality on mixed Lebesgue spaces (see [2]), we obtain

$$\int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}. \tag{3.1}$$

Moreover, for all $1 < \vec{q} < \infty$, we have

$$\|Tf_2\|_{L^{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}.$$

Therefore, we get

$$\|Tf\|_{L^{\vec{q}}(B(x_0,r))} \lesssim \|f\|_{L^{\vec{q}}(2B)} + r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}.$$

On the other hand,

$$\begin{aligned}
\|f\|_{L^{\vec{q}}(2B)} &\sim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}} \\
&\lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}. \tag{3.2}
\end{aligned}$$

Thus

$$\|Tf\|_{L^{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}.$$

$\square$

The second lemma is about the estimates of $\|T_\alpha f\|_{L^{\vec{p}}(B(x_0,r))}$.

Lemma 3.2. *Let $1 < \vec{q} < \infty$, $0 < \alpha < n/\vec{q}$, $\frac{1}{\vec{p}} = \frac{1}{\vec{q}} - \frac{\alpha}{n}$, T_α be a sublinear operator satisfying condition* (1.2)*, and bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$.*

Then for $1 < \vec{q} < \infty$, the inequality

$$\|T_\alpha f\|_{L^{\vec{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} t^{-1-\sum_{i=1}^{n} \frac{1}{p_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and all $f \in L^{\vec{q}}_{\text{loc}}(\mathbb{R}^n)$.

Proof. For any ball $B = B(x_0,r)$, we also let $2B = B(x_0, 2r)$ as before. Write $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$.

By the fact that T_α is a sublinear operator, we have

$$\|T_\alpha f\|_{L^{\vec{p}}(B)} \leq \|T_\alpha f_1\|_{L^{\vec{p}}(B)} + \|T_\alpha f_2\|_{L^{\vec{p}}(B)}.$$

Noting that $f_1 \in L^{\vec{q}}(\mathbb{R}^n)$ and T_α is bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$, we have

$$\|T_\alpha f_1\|_{L^{\vec{p}}(B)} \leq \|T_\alpha f_1\|_{L^{\vec{p}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} = \|f\|_{L^{\vec{q}}(2B)}.$$

Since $x \in B$, $y \in (2B)^c$ imply $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$, it yields that

$$|T_\alpha f_2(x)| \lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By Fubini's theorem, we have

$$
\begin{aligned}
\int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy &\sim \int_{(2B)^c} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy \\
&\sim \int_{2r}^{\infty} \int_{2r \leq |x_0-y| < t} |f(y)| dy \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1-\alpha}}.
\end{aligned}
$$

Applying Hölder's inequality on mixed Lebesgue spaces, we obtain

$$
\int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy \lesssim \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}. \tag{3.3}
$$

Moreover, for all $1 < \vec{q} < \infty$, we have

$$
\|T_\alpha f_2\|_{L^{\vec{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.
$$

Therefore,

$$
\|T_\alpha f\|_{L^{\vec{p}}(B(x_0,r))} \lesssim \|f\|_{L^{\vec{q}}(2B)} + r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.
$$

On the other hand,

$$
\begin{aligned}
\|f\|_{L^{\vec{q}}(2B)} &\sim r^{\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\vec{q}}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}} \\
&\lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}. \tag{3.4}
\end{aligned}
$$

Thus

$$
\|T_\alpha f\|_{L^{\vec{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}}.
$$

$\square$

Now we can present the first main result in this section.

Theorem 3.1. *Let* $1 < \vec{q} < \infty$*, and* (φ_1, φ_2) *satisfy the condition*

$$
\int_r^{\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{q_i}}}{t^{1+\sum_{i=1}^n \frac{1}{q_i}}} dt \lesssim \varphi_2(x,r).
$$

Suppose T *is a sublinear operator satisfying condition* (1.1) *which is bounded on* $L^{\vec{q}}(\mathbb{R}^n)$. *Then for* $1 < \vec{q} < \infty$, *the operator* T *is bounded form* $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ *to* $M_{\vec{q}}^{\varphi_2}(\mathbb{R}^n)$. *Moreover,*

$$
\|Tf\|_{M_{\vec{q}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.
$$

Proof. By Lemma 3.1 and Lemma 2.1 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1} r^{-\sum_{i=1}^n \frac{1}{q_i}}$, $g(r) = \|f\|_{L^{\vec{q}}(B(x,r))}$ and $w(r) = r^{-1-\sum_{i=1}^n \frac{1}{q_i}}$, we have

$$
\|Tf\|_{M_{\vec{q}}^{\varphi_2}} \lesssim \sup_{x \in \mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} t^{-1-\sum_{i=1}^n \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x,t))} dt
$$

$$\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x,r)^{-1} r^{-\sum_{i=1}^n \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x,t))} \sim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

$\square$

By taking $\vec{q} = q$ in Theorem 3.1, we recover the result of Guliyev et al. [12, Theorem 4.5], which gave the boundedness of T on generalized Morrey spaces.

Similarly, for the sublinear operator T_α with the size condition (1.2), we have the second theorem.

Theorem 3.2. *Let $1 < \vec{q} < \infty$, $0 < \alpha < n/\vec{q}$, $\frac{1}{\vec{p}} = \frac{1}{\vec{q}} - \frac{\alpha}{n}$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{q_i}}}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}} dt \lesssim \varphi_2(x,r).$$

Suppose T_α is a sublinear operator satisfying condition (1.2) *which is bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$. Then for $1 < \vec{q} < \infty$, the operator T_α is bounded form $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ to $M_{\vec{p}}^{\varphi_2}(\mathbb{R}^n)$. Moreover,*

$$\|T_\alpha f\|_{M_{\vec{p}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

Proof. By Lemma 3.2 and Lemma 2.1 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1} r^{-\sum_{i=1}^n \frac{1}{q_i}}$, $g(r) = \|f\|_{L^{\vec{q}}(B(x,r))}$ and $w(r) = r^{-1-\sum_{i=1}^n \frac{1}{p_i}}$, we have

$$\|T_\alpha f\|_{M_{\vec{p}}^{\varphi_2}} \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x,r)^{-1} \int_{2r}^\infty t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\vec{q}}(B(x,t))} dt$$

$$\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(x,r)^{-1} r^{-\sum_{i=1}^n \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x,t))} \sim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

$\square$

By taking $\vec{q} = q$ in Theorem 3.2, we recover the result of [12, Theorem 5.4].

4. Sublinear operators T_b and $T_{b,\alpha}$ in spaces $M_{\vec{q}}^{\varphi}(\mathbb{R}^n)$

In this section, we investigate the boundedness of T_α and $T_{b,\alpha}$ satisfying the size conditions (1.3) and (1.4) respectively, on generalized mixed Morrey space $M_{\vec{q}}^{\varphi}(\mathbb{R}^n)$.

First, we review the definition of $\mathrm{BMO}(\mathbb{R}^n)$, the bounded mean oscillation space. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty.$$

If one regards two functions whose difference is a constant as one, then the space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_{\mathrm{BMO}}$. The John-Nirenberg ineuqalitiy for $\mathrm{BMO}(\mathbb{R}^n)$ yields that for any $1 < q < \infty$ and $f \in \mathrm{BMO}(\mathbb{R}^n)$, the $\mathrm{BMO}(\mathbb{R}^n)$ norm of f is equivalent to

$$\|f\|_{\mathrm{BMO}^q} = \sup_{x \in \mathbb{R}^n, r > 0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^q dy \right)^{\frac{1}{q}}.$$

Recall that for any $\vec{q} = (q_1, \cdots, q_n) \in (1, \infty)^n$, the John-Nirenberg inequality for mixed norm spaces [15] shows that the $\mathrm{BMO}(\mathbb{R}^n)$ norm of all $f \in \mathrm{BMO}(\mathbb{R}^n)$ is also

equivalent to

$$\|f\|_{\mathrm{BMO}^{\vec{q}}} = \sup_{x \in \mathbb{R}^n, r>0} \frac{\|(f - f_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}}}{\|\chi_{B(x,r)}\|_{L^{\vec{q}}}}. \tag{4.1}$$

The following property for $\mathrm{BMO}(\mathbb{R}^n)$ functions is valid.

Lemma 4.1. *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then for all $0 < 2r < t$, we have*

$$|f_{B(x,r)} - f_{B(x,t)}| \lesssim \|f\|_{\mathrm{BMO}} \ln \frac{t}{r}. \tag{4.2}$$

We first prove two lemmas, which give us the explicit estimats for the $L^{\vec{q}}(\mathbb{R}^n)$ norm of T_b and $T_{b,\alpha}$ on a given ball $B(x_0, r)$.

Lemma 4.2. *Let $1 < \vec{q} < \infty$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, T_b be a sublinear operator satisfying condition (1.3), and bounded on $L^{\vec{q}}(\mathbb{R}^n)$.*
Then for $1 < \vec{q} < \infty$, the inequality

$$\|T_b f\|_{L^{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{q_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-1-\sum_{i=1}^n \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and all $f \in L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. For any ball $B = B(x_0, r)$, let $2B = B(x_0, 2r)$. Write f as $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$.
Since T_b is a sublinear operator, we have

$$\|T_b f\|_{L^{\vec{q}}(B)} \le \|T_b f_1\|_{L^{\vec{q}}(B)} + \|T_b f_2\|_{L^{\vec{q}}(B)}.$$

Noting that $f_1 \in L^{\vec{q}}(\mathbb{R}^n)$ and T_b is bounded in $L^{\vec{q}}(\mathbb{R}^n)$, we have

$$\|T_b f_1\|_{L^{\vec{q}}(B)} \le \|T_b f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} = \|f\|_{L^{\vec{q}}(2B)}.$$

Since $x \in B$, $y \in (2B)^c$ imply $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$, we get

$$\begin{aligned}
|T_b f_2(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^n} |f_2(y)| dy \\
&\sim \int_{(2B)^c} \frac{|b(x) - b(y)|}{|x_0 - y|^n} |f(y)| dy.
\end{aligned}$$

By using generalized Minkowski's inequality on mixed Lebesgue spaces, we have

$$\begin{aligned}
\|T_b f_2\|_{L^{\vec{q}}(B))} &\lesssim \left\| \int_{(2B)^c} \frac{|b(\cdot) - b(y)|}{|x_0 - y|^n} |f(y)| dy \right\|_{L^{\vec{q}}(B(x_0,r))} \\
&\lesssim \left\| \int_{(2B)^c} \frac{|b_B - b(y)|}{|x_0 - y|^n} |f(y)| dy \right\|_{L^{\vec{q}}(B(x_0,r))} \\
&\quad + \left\| \int_{(2B)^c} \frac{|b(\cdot) - b_B|}{|x_0 - y|^n} |f(y)| dy \right\|_{L^{\vec{q}}(B(x_0,r))} \\
&= I_1 + I_2.
\end{aligned}$$

For the term I_1, we have

$$I_1 \sim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{(2B)^c} \frac{|b_B - b(y)|}{|x_0 - y|^n} |f(y)| dy$$

$$\sim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{(2B)^c} |b(y) - b_B||f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy$$

$$\sim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \int_{2r \leq |x_0-y| < t} |b(y) - b_B||f(y)| dy \frac{dt}{t^{n+1}}$$

$$\lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_B||f(y)| dy \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality and (4.1), (4.2), we get

$$I_1 \lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)| dy \frac{dt}{t^{n+1}}$$

$$+ r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b_{B(x_0,r)} - b_{B(x_0,t)}||f(y)| dy \frac{dt}{t^{n+1}}$$

$$\lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} \|(b(\cdot) - b_{B(x_0,t)})\chi_{B(x_0,t)}\|_{L^{\vec{q}'}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dy \frac{dt}{t^{n+1}}$$

$$+ r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}$$

$$\lesssim \|b\|_{\mathrm{BMO}} r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}.$$

In order to estimate I_2, note that

$$I_2 = \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^n} dy \times \|b(\cdot) - b_B\|_{L^{\vec{q}}(B(x_0,r))}.$$

It follows from (4.1) that

$$I_2 \lesssim \|b\|_{\mathrm{BMO}} r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Thus by (3.1), we get

$$I_2 \lesssim \|b\|_{\mathrm{BMO}} r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}}.$$

Summing up I_1 and I_2, we get

$$\|T_b f_2\|_{L^{\vec{q}}(B)} \lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-1-\sum_{i=1}^{n} \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt.$$

Therefore, by (3.2), there holds

$$\|T_b f_2\|_{L^{\vec{q}}(B)} \lesssim \|f\|_{L^{\vec{q}}(2B)} + r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-1-\sum_{i=1}^{n} \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt$$

$$\lesssim r^{\sum_{i=1}^{n} \frac{1}{q_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-1-\sum_{i=1}^{n} \frac{1}{q_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt.$$

We are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

The second lemma in this section is about the estimates of $\|T_{b,\alpha} f\|_{L^{\vec{p}}(B(x_0,r))}$.

Lemma 4.3. *Let $1 < \vec{q} < \infty$, $0 < \alpha < n/\vec{q}$, $\frac{1}{\vec{p}} = \frac{1}{\vec{q}} - \frac{\alpha}{n}$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $T_{b,\alpha}$ be a sublinear operator satisfying condition* (1.4), *and bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$.*
Then for $1 < \vec{q} < \infty$, the inequality

$$\|T_{b,\alpha}f\|_{L^{\vec{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{-1-\sum_{i=1}^{n} \frac{1}{p_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and all $f \in L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof.   For any ball $B = B(x_0,r)$, let $2B = B(x_0,2r)$. Write f as $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$ as before.
    Since $T_{b,\alpha}$ is a sublinear operator, we have

$$\|T_{b,\alpha}f\|_{L^{\vec{q}}(B)} \leq \|T_{b,\alpha}f_1\|_{L^{\vec{q}}(B)} + \|T_{b,\alpha}f_2\|_{L^{\vec{q}}(B)}.$$

Noting that $f_1 \in L^{\vec{q}}(\mathbb{R}^n)$ and $T_{b,\alpha}$ is bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$, we have

$$\|T_{b,\alpha}f_1\|_{L^{\vec{q}}(B)} \leq \|T_{b,\alpha}f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} = \|f\|_{L^{\vec{q}}(2B)}.$$

Since $x \in B$, $y \in (2B)^c$ imply $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$, we get

$$|T_{b,\alpha}f_2(x)| \lesssim \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} |f_2(y)| dy$$
$$\sim \int_{(2B)^c} \frac{|b(x) - b(y)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy.$$

By using generalized Minkowski's inequality, we have

$$\|T_{b,\alpha}f_2\|_{L^{\vec{p}}(B))} \lesssim \left\| \int_{(2B)^c} \frac{|b(\cdot) - b(y)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy \right\|_{L^{\vec{p}}(B(x_0,r))}$$
$$\lesssim \left\| \int_{(2B)^c} \frac{|b_B - b(y)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy \right\|_{L^{\vec{p}}(B(x_0,r))}$$
$$+ \left\| \int_{(2B)^c} \frac{|b(\cdot) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)| dy \right\|_{L^{\vec{p}}(B(x_0,r))}$$
$$= I_1 + I_2.$$

For the term I_1, we have

$$I_1 \sim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{(2B)^c} \frac{|b_B - b(y)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy$$
$$\sim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{(2B)^c} |b(y) - b_B| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy$$
$$\sim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \int_{2r \leq |x_0-y| < t} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1-\alpha}}$$
$$\lesssim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1-\alpha}}.$$

Applying Hölder's inequality and (4.1), (4.2), we get

$$
\begin{aligned}
I_1 &\lesssim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| dy \frac{dt}{t^{n+1-\alpha}} \\
&\quad + r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b_{B(x_0,r)} - b_{B(x_0,t)}| |f(y)| dy \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \int_{B(x_0,t)} \|(b(\cdot) - b_{B(x_0,t)}) \chi_{B(x_0,t)}\|_{L^{\vec{q}'}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dy \frac{dt}{t^{n+1-\alpha}} \\
&\quad + r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \, \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1-\alpha+\sum_{i=1}^{n} \frac{1}{q_i}}} \\
&\lesssim \|b\|_{\mathrm{BMO}} r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{p_i}}}.
\end{aligned}
$$

For the term I_2, note that

$$
I_2 = \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy \times \|b(\cdot) - b_B\|_{L^{\vec{p}}(B(x_0,r))}.
$$

It follows from (4.1) that

$$
I_2 \lesssim \|b\|_{\mathrm{BMO}} r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.
$$

Thus by (3.3), we get

$$
I_2 \lesssim \|b\|_{\mathrm{BMO}} r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \|f\|_{L^{\vec{q}}(B(x_0,t))} \frac{dt}{t^{1+\sum_{i=1}^{n} \frac{1}{p_i}}}.
$$

Summing up I_1 and I_2, we get

$$
\|T_{b,\alpha} f_2\|_{L^{\vec{p}}(B)} \lesssim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) t^{-1-\sum_{i=1}^{n} \frac{1}{p_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt.
$$

Therefore, by (3.4), there holds

$$
\begin{aligned}
\|T_{b,\alpha} f\|_{L^{\vec{p}}(B)} &\lesssim \|f\|_{L^{\vec{q}}(2B)} + r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) t^{-1-\sum_{i=1}^{n} \frac{1}{p_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt \\
&\lesssim r^{\sum_{i=1}^{n} \frac{1}{p_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) t^{-1-\sum_{i=1}^{n} \frac{1}{p_i}} \|f\|_{L^{\vec{q}}(B(x_0,t))} dt.
\end{aligned}
$$

We are done. □

Now we give the boundedness of T_b and $T_{b,\alpha}$ on generalized mixed Morrey spaces.

Theorem 4.1. *Let $1 < \vec{q} < \infty$, and (φ_1, φ_2) satisfy the condition*

$$
\int_{r}^{\infty} \left(1 + \ln \frac{t}{r} \right) \frac{\mathrm{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^{n} \frac{1}{q_i}}}{t^{1+\sum_{i=1}^{n} \frac{1}{q_i}}} dt \lesssim \varphi_2(x,r).
$$

Suppose $b \in \mathrm{BMO}(\mathbb{R}^n)$, and T_b is a sublinear operator satisfying condition (1.3), and bounded on $L^{\vec{q}}(\mathbb{R}^n)$. Then for $1 < \vec{q} < \infty$, the operator T_b is bounded form $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ to $M_{\vec{q}}^{\varphi_2}(\mathbb{R}^n)$. Moreover,

$$
\|T_b f\|_{M_{\vec{q}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.
$$

Proof. The proof of Theorem 4.1 follows by Lemma 4.2 and Lemma 2.2 in the same manner as in the proof of Theorem 3.1. $\square$

From Theorem 4.1, on can recover the result of Guliyev et al. [12, Theorem 6.6] by taking $\vec{q} = q$.

Similarly, for the sublinear operator $T_{b,\alpha}$ with size condition (1.4), we have the following theorem.

Theorem 4.2. *Let* $1 < \vec{q} < \infty$, $0 < \alpha < n/\vec{q}$, $\frac{1}{\vec{p}} = \frac{1}{\vec{q}} - \frac{\alpha}{n}$, *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{q_i}}}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}} dt \lesssim \varphi_2(x,r).$$

Suppose $b \in \mathrm{BMO}(\mathbb{R}^n)$, *and* $T_{b,\alpha}$ *is a sublinear operator satisfying condition* (1.4), *and bounded from* $L^{\vec{q}}(\mathbb{R}^n)$ *to* $L^{\vec{p}}(\mathbb{R}^n)$. *Then for* $1 < \vec{q} < \infty$, *the operator* $T_{b,\alpha}$ *is bounded from* $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ *to* $M_{\vec{p}}^{\varphi_2}(\mathbb{R}^n)$. *Moreover,*

$$\|T_{b,\alpha} f\|_{M_{\vec{p}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

Proof. The statement of Theorem 4.2 follows by Lemma 4.3 and Lemma 2.2 in the same manner as in the proof of Theorem 3.2. $\square$

By taking $\vec{q} = q$ in Theorem 4.2, we recover the result of Guliyev et al. [12, Theorem 7.4], which proved the boundedness of $T_{b,\alpha}$ on the generalized Morrey spaces.

5. Some applications

This section gives some applications of our main theorems. We will show that many important integral operators and commutators appearing in harmonic analysis satisfy the assumptions mentioned above. Therefore, we can obtain the boundedness of various operators on generalized mixed Morrey spaces by using our main results.

As stated in the introduction, the Hardy-Littlewood maximal operator M and the Calderón-Zygmund singular integral operator K are all sublinear operators satisfying the condition (1.1). One can also see that $[b, M]$ and $[b, K]$ are also sublinear operators and satisfiy the condition (1.3). Note that the fractional integral operator I_α is a linear operator satisfying the condition (1.2), and the commutator $[b, I_\alpha]$ is also a linear operator satisfying (1.4). So in order to apply our main theorems to the mentioned operators, we need to show the boundedness of $M, K, [b, M], [b, K]$ on $L^{\vec{q}}(\mathbb{R}^n)$, and the boundedness of $I_\alpha, [b, I_\alpha]$ from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$.

As we know, the Hardy-Littlewood maximal operator M is bounded on $L^{\vec{q}}(\mathbb{R}^n)$, $1 < \vec{q} < \infty$ (see [21]), but there is no complete boundedness results for some other operators on mixed Lebesgue spaces. To prove the boundedness of some important operators on mixed Lebesgue spaces in a uniform way, we will give the extrapolation theorems on mixed Lebesgue spaces, which have their own interest.

The extrapolaton theory on mixed Lebesgue spaces relies on the classical A_p weight (see [8]).

Definition 5.1. For $1 < p < \infty$, a positive function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be an A_p weight if

$$[w]_{A_p} = \sup_{B \in \mathbb{B}} \left(\frac{1}{|B|} \int_B w(x) dx\right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{p'}{p}} dx\right)^{\frac{p}{p'}} < \infty.$$

A positive locally integrable function w is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B w(y) dy \le C w(x), \quad \text{a.e.} \ \ x \in B$$

for some constant $C > 0$. The infimum of all such C is denoted by $[w]_{A_1}$. We denote A_∞ by the union of all A_p ($1 \le p < \infty$) functions.

We also need the boundedness of M on mixed norm spaces $L^{\vec{q}}(\mathbb{R}^n)$, see [21].

Lemma 5.1. *For $1 < \vec{q} < \infty$, there holds*

$$\|Mf\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{q}}}.$$

By $\mathfrak{F}$, we mean a family of pair (f,g) of non-negative measurable functions that are not identical to zero. For such a family $\mathfrak{F}$, $p > 0$ and a weight $w \in A_q$, the expression

$$\int_{\mathbb{R}^n} f(x)^p w(x) dx \lesssim \int_{\mathbb{R}^n} g(x)^p w(x) dx, \quad (f,g) \in \mathfrak{F}$$

means that this inequality holds for all pair $(f,g) \in \mathfrak{F}$ if the left hand side is finite, and the implicated constant depends only on p and $[w]_{A_q}$.

Now we give the extrapolation theorems on mixed Lebesgue spaces. The first one is the diagonal extrapolation theorem.

Theorem 5.1. *Let $0 < q_0 < \infty$ and $\vec{q} = (q_1, \cdots, q_n) \in (0, \infty)^n$. Let $f, g \in \mathcal{M}(\mathbb{R}^n)$. Suppose for every $w \in A_1$, we have*

$$\int_{\mathbb{R}^n} f(x)^{q_0} w(x) dx \lesssim \int_{\mathbb{R}^n} g(x)^{q_0} w(x) dx, \quad (f,g) \in \mathfrak{F}.$$

Then if $\vec{q} > q_0$, we have

$$\|f\|_{L^{\vec{q}}} \lesssim \|g\|_{L^{\vec{q}}}, \quad (f,g) \in \mathfrak{F}.$$

Our second main result in this section is the off-diagonal extrapolation on mixed Lebesgue spaces.

Theorem 5.2. *Let $f, g \in \mathcal{M}$. Suppose that for some p_0 and q_0 with $0 < q_0 < p_0 < \infty$ and every $w \in A_1$,*

$$\left(\int_{\mathbb{R}^n} f(x)^{p_0} w(x) dx \right)^{\frac{1}{p_0}} \lesssim \left(\int_{\mathbb{R}^n} g(x)^{q_0} w(x)^{\frac{q_0}{p_0}} dx \right)^{\frac{1}{q_0}}, \quad (f,g) \in \mathfrak{F}. \tag{5.1}$$

Then for all $q_0 < \vec{q} < \frac{p_0 q_0}{p_0 - q_0}$, and $\vec{p}$ satisfies $1/\vec{q} - 1/\vec{p} = 1/q_0 - 1/p_0$, we have

$$\|f\|_{L^{\vec{p}}} \lesssim \|g\|_{L^{\vec{q}}}, \quad (f,g) \in \mathfrak{F}. \tag{5.2}$$

Proof. By the similarity, we only prove Theorem 5.2.

We use the Rubio de Francia iteration algorithm presented in [7].

Let $\bar{\vec{q}} = \vec{q}/q_0$ and $\bar{\vec{p}} = \vec{p}/p_0$. By the assumptions and Lemma 5.1, the maximal operator is bounded on $L^{\bar{\vec{p}}'}(\mathbb{R}^n)$, so there exists a positive constant B such that

$$\|Mf\|_{L^{\bar{\vec{p}}'}} \le B \|f\|_{L^{\bar{\vec{p}}'}}.$$

For any non-negative function h, define a new operator $\mathfrak{R}h$ by

$$\mathfrak{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k B^k},$$

where for $k \geq 1$, M^k denotes k iterations of the maximal operator, and M^0 is the identity operator.

The operator $\mathfrak{R}$ satisfies

$$h(x) \leq \mathfrak{R}h(x), \tag{5.3}$$

$$\|\mathfrak{R}h\|_{L^{\bar{p}'}} \leq 2\|h\|_{L^{\bar{p}'}}, \tag{5.4}$$

$$\|\mathfrak{R}h\|_{A_1} \leq 2B. \tag{5.5}$$

The inequality (5.3) is straight-forward.

Since

$$M(\mathfrak{R}h) \leq \sum_{k=0}^{\infty} \frac{M^{k+1}h}{2^k B^k} \leq 2B \sum_{k=1}^{\infty} \frac{M^k h}{2^k B^k} \leq 2B\mathfrak{R}h,$$

the properties (5.4) and (5.5) are consequences of Lemma 5.1 and the definition of A_1.

Since the dual of $L^{\bar{p}}(\mathbb{R}^n)$ is $L^{\bar{p}'}(\mathbb{R}^n)$, we get

$$\|f\|_{L^{\bar{p}}}^{p_0} = \|f^{p_0}\|_{L^{\bar{p}}} \tag{5.6}$$

$$\lesssim \sup \left\{ \int_{\mathbb{R}^n} |f(x)|^{p_0} h(x) dx : \ \|h\|_{L^{\bar{p}'}} \leq 1, h \geq 0 \right\}.$$

By using Hölder's inequality on mixed Lebesgue spaces and (5.3), we have

$$\int_{\mathbb{R}^n} f(x)^{p_0} h(x) dx \lesssim \int_{\mathbb{R}^n} f(x)^{p_0} \mathfrak{R}h(x) dx$$

$$\lesssim \|f^{p_0}\|_{L^{\bar{p}}} \|h\|_{L^{\bar{p}'}} < \infty. \tag{5.7}$$

In view of (5.3) and $\mathfrak{R}h \in A_1$, we use (5.1) with $w = \mathfrak{R}h(x)$ to obtain

$$\int_{\mathbb{R}^n} f(x)^{p_0} h(x) dx \lesssim \int_{\mathbb{R}^n} f(x)^{p_0} \mathfrak{R}h(x) dx \lesssim \left(\int_{\mathbb{R}^n} g(x)^{q_0} [\mathfrak{R}h(x)]^{q_0/p_0} dx \right)^{p_0/q_0}.$$

Combining (5.4) with (5.7) and using Hölder's inequality on mixed Lebesgue spaces again, we arrive at

$$\int_{\mathbb{R}^n} f(x)^{p_0} h(x) dx \lesssim \|g^{q_0}\|_{L^{\bar{q}}}^{p_0/q_0} \|(\mathfrak{R}h)^{q_0/p_0}\|_{L^{\bar{q}'}}^{p_0/q_0} \tag{5.8}$$

$$\sim \|g\|_{L^{\bar{q}}}^{p_0} \|(\mathfrak{R}h)^{q_0/p_0}\|_{L^{\bar{q}'}}^{p_0/q_0}.$$

A direct calculation yields $q_0 \bar{q}' = p_0 \bar{p}'$. Therefore

$$\|(\mathfrak{R}h)^{q_0/p_0}\|_{L^{\bar{q}'}}^{p_0/q_0} = \|\mathfrak{R}h\|_{L^{\bar{p}'}} \lesssim \|h\|_{L^{\bar{p}'}}. \tag{5.9}$$

By taking the supremum over all $h \in L^{\bar{p}'}(\mathbb{R}^n)$ with $\|h\|_{L^{\bar{p}'}} \leq 1$ and $h \geq 0$, (5.6), (5.8) and (5.9) give us the desired conclusion (5.2). $\qquad\square$

We point out that when $n = 2$, there are other versions of the diagonal extrapolation theorem [14] and the off-diagonal extrapolation theorem [27] on mixed Lebesgue spaces, which are different from Theorem 5.1 and Theorem 5.2.

By the density of smooth functions with compact support $C_c^\infty(\mathbb{R}^n)$ in the mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$, $1 < \vec{q} < \infty$ (see [2]), one can apply Theorem 5.1 and Theorem 5.2 to the mapping property of some sublinear operators.

Theorem 5.3. *Suppose $0 < q_0 < \vec{q} < \infty$ and T is a sublinear operator such that for every $w \in A_1$,*

$$\int_{\mathbb{R}^n} |Tf(z)|^{q_0} w(z) dz \lesssim \int_{\mathbb{R}^n} |f(z)|^{q_0} w(z) dz, \quad f \in C_c^\infty(\mathbb{R}^n).$$

Then T can be extended to a bounded operator on $L^{\vec{q}}(\mathbb{R}^n)$.

Proof. By Theorem 5.1, for any $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$\|Tf\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{q}}}.$$

Since T is a sublinear operator, we have $|T(f) - T(g)| \leq |T(f - g)|$, and hence, for any $f, g \in C_c^\infty(\mathbb{R}^n)$, we have

$$\|T(f) - T(g)\|_{L^{\vec{q}}} \leq \|T(f - g)\|_{L^{\vec{q}}} \lesssim \|f - g\|_{L^{\vec{q}}}.$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{\vec{q}}(\mathbb{R}^n)$, the above inequalities guarantee that T can be extended to be a bounded operator on $L^{\vec{q}}(\mathbb{R}^n)$. $\square$

The following corollary is a consequence of Theorem 5.3 and the weighted boundedness of the corresponding operators.

Corollary 5.1. *Let $1 < \vec{q} < \infty$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $M, K, [b, M], [b, K]$ are all bounded on $L^{\vec{q}}(\mathbb{R}^n)$.*

Proof. It is well known that $M, K, [b, M], [b, K]$ are all sublinear operators, and bounded on $L_w^{q_0}(\mathbb{R}^n)$ for arbitrary $1 < q_0 < \infty$ and $w \in A_{q_0}$ (see [8] for example). Since $A_1 \subset A_{q_0}$, Theorem 5.3 implies that $M, K, [b, M], [b, K]$ are all bounded on $L^{\vec{q}}(\mathbb{R}^n)$ for all $q_0 < \vec{q} < \infty$. In view of the arbitrariness of $1 < q_0 < \infty$, M, $K, [b, M], [b, K]$ are also bounded on $L^{\vec{q}}(\mathbb{R}^n)$ for all $1 < \vec{q} < \infty$. $\square$

Similarly, we have the following theorem.

Theorem 5.4. *Suppose that $0 < q_0 < p_0 < \infty$, and T is a sublinear operator such that every $w \in A_1$,*

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x) dx \right)^{\frac{1}{p_0}} \lesssim \left(\int_{\mathbb{R}^n} |f(x)|^{q_0} w(x)^{\frac{q_0}{p_0}} dx \right)^{\frac{1}{q_0}}, \quad f \in C_c^\infty(\mathbb{R}^n).$$

Then for all $q_0 < \vec{q} < \frac{p_0 q_0}{p_0 - q_0}$, and $\vec{p}$ satisfies $1/\vec{q} - 1/\vec{p} = 1/q_0 - 1/p_0$, T can be extended to a bounded operator from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$.

Since the proof of Theorem 5.4 is similar to that of Theorem 5.3, we leave it to readers.

To apply Theorem 5.4 to fractional integral operator and its commutator, we need a different class of weights: Suppose $0 < \alpha < n$, $1 < q < n/\alpha$ and $1/q - 1/p = \alpha/n$, we say $w \in A_{q,p}(\mathbb{R}^n)$ if for any ball $B \in \mathbb{B}$,

$$\frac{1}{|B|} \int_B w(x) dx \left(\frac{1}{|B|} \int_B w(x)^{-q'/p} dx \right)^{p/q'} < \infty.$$

Note that this is equivalent to $w \in A_r$, where $r = 1 + p/q'$, so, in particular if $w \in A_1$, then $w \in A_{q,p}$. The boundedness of fractional integral operator I_α and its commutator $[b, I_\alpha]$ on weighted Lebesgue spaces was obtained in [17, 19].

Lemma 5.2. *Let $0 < \alpha < n$ and $1 < q < p < \infty$ with $1/q - 1/p = \alpha/n$. If $w \in A_{q,p}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then we have*

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^p w(x) dx \right)^{1/p} \lesssim \left(\int_{\mathbb{R}^n} |f(x)|^q w(x)^{q/p} dx \right)^{1/q},$$

$$\left(\int_{\mathbb{R}^n} |[b, I_\alpha] f(x)|^p w(x) dx \right)^{1/p} \lesssim \left(\int_{\mathbb{R}^n} |f(x)|^q w(x)^{q/p} dx \right)^{1/q}.$$

These results are usually stated with the class $A_{q,p}$ defined slightly differently, with w replaced by w^p (see [6, 17, 19] for example). Our formulation, though non-standard, is better for our purposes.

By Theorem 5.4 and Lemma 5.2, we can get the following result.

Corollary 5.2. *Let $0 < \alpha < n$, $1 < \vec{q} < n/\alpha$, $\frac{1}{\vec{q}} - \frac{1}{\vec{p}} = \frac{\alpha}{n}$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then both I_α and $[b, I_\alpha]$ are bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$.*

Proof.   The proof is just a repetition of the proof of Corollary 5.1, so we omit the details.   □

We point out that the boundedness of I_α on mixed Lebesgue spaces has already proved in [2] in a more general setting. However, our proof, relying on the extrapolation theory on mixed norm spaces, has its own interest.

From the statement at the beginning of this section, we can obtain the boundedness of M, K on generalized mixed Morrey spaces, whose proof is just a combination of Theorem 3.1 and Corollary 5.1.

Theorem 5.5. *Let $1 < \vec{q} < \infty$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\mathrm{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{q_i}}}{t^{1+\sum_{i=1}^n \frac{1}{q_i}}} dt \lesssim \varphi_2(x,r).$$

Then the Hardy-Littlewood maximal operator M and the Calderón-Zygmund singular integral operator K are both bounded from $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ to $M_{\vec{q}}^{\varphi_2}(\mathbb{R}^n)$. Moreover,

$$\|Mf\|_{M_{\vec{q}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}},$$

$$\|Kf\|_{M_{\vec{q}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

Similarly, we have

Theorem 5.6. *Let $1 < \vec{q} < \infty$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\mathrm{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{q_i}}}{t^{1+\sum_{i=1}^n \frac{1}{q_i}}} dt \lesssim \varphi_2(x,r).$$

If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator of the Hardy-Littlewood maximal operator $[b, M]$ and the commutator of the Calderón-Zygmund singular integral operator $[b, K]$ are both bounded from $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ to $M_{\vec{q}}^{\varphi_2}(\mathbb{R}^n)$. Moreover,

$$\|[b, M]f\|_{M_{\vec{q}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}},$$

$$\|[b, K]f\|_{M_{\vec{q}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

For the boundedness of $I_\alpha, [b, I_\alpha]$ on generalized mixed Morrey spaces, we have the following theorems.

Theorem 5.7. *Let $1 < \vec{q} < \infty$, $0 < \alpha < n/\vec{q}$, $\frac{1}{\vec{p}} = \frac{1}{\vec{q}} - \frac{\alpha}{n}$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\sum_{i=1}^n \frac{1}{q_i}}}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}} dt \lesssim \varphi_2(x,r).$$

Then the fractional integral operator I_α is bounded from $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ to $M_{\vec{p}}^{\varphi_2}(\mathbb{R}^n)$. Moreover,

$$\|I_\alpha f\|_{M_{\vec{p}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

Theorem 5.8. *Let $1 < \vec{q} < \infty$, $0 < \alpha < n/\vec{q}$, $\frac{1}{\vec{p}} = \frac{1}{\vec{q}} - \frac{\alpha}{n}$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\sum_{i=1}^n \frac{1}{q_i}}}{t^{1+\sum_{i=1}^n \frac{1}{p_i}}} dt \lesssim \varphi_2(x,r).$$

If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator of the fractional integral operator $[b, I_\alpha]$ is bounded from $M_{\vec{q}}^{\varphi_1}(\mathbb{R}^n)$ to $M_{\vec{p}}^{\varphi_2}(\mathbb{R}^n)$. Moreover,

$$\|[b, I_\alpha]f\|_{M_{\vec{p}}^{\varphi_2}} \lesssim \|f\|_{M_{\vec{q}}^{\varphi_1}}.$$

We point out that the results in Theorem 5.7 and Theorem 5.8 remain true for fractional maximal operator and its commutator.

Acknowledgements

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J., 1975, 42(4), 765–778.

[2] A. Benedek and R. Panzone, *The space L^p, with mixed norm*, Duke Math. J., 1961, 28(3), 301–324.

[3] V. I. Burenkov and V. S. Guliyev, *Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal., 2009, 30(3), 211–249.

[4] T. Chen and W. Sun, *Iterated weak and weak mixed-norm spaces with applications to geometric inequalities*, J. Geom. Anal., 2020, 30(4), 4268–4323.

[5] F. Chiarenza, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat., 1987, 7, 273–279.

[6] D. V. Cruz-Uribe, A. Fiorenza, J. Martell and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math., 2006, 31(1), 239–264.

[7] D. V. Cruz-Uribe, J. M. Martell and C. Pérez, *Weights, extrapolation and the theory of Rubio de Francia, In: Operator Theory: Advance and Applications, vol. 215*, Birkhäuser, Basel, 2011.

[8] L. Grafakos, *Classical Fourier analysis, 2en edn*, Springer, New York, 2008.

[9] V. S. Guliev, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J., 2012, 3(3), 33–61.

[10] V. Guliyev, M. Omarova, M. Ragusa and A. Scapellato, *Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces*, Anal. Math. Phys., 2021, 11(1), 1–20.

[11] V. S. Guliyev, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci., 2013, 193(2), 211–227.

[12] V. S. Guliyev, S. S. Aliyev, T. Karaman and P. S. Shukurov, *Boundedness of sublinear operators and commutators on generalized Morrey spaces*, Integral Equations and Operator Theory, 2011, 71(3), 327–355.

[13] V. S. Guliyev and L. G. Softova, *Global regularity in generalized Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients*, Potential Anal., 2013, 38(3), 843–862.

[14] K. P. Ho, *Strong maximal operator on mixed-norm spaces*, Ann. Univ. Ferrara Sez. VII Sci. Mat., 2016, 62(2), 275–291.

[15] K. P. Ho, *Mixed norm lebesgue spaces with variable exponents and applications*, Riv. Mat. Univ. Parma, 2018, 9(1), 21–44.

[16] G. Lu, S. Lu and D. Yang, *Singular integrals and commutators on homogeneous groups*, Anal. Math., 2002, 28(2), 103–134.

[17] S. Lu, Y. Ding and D. Yan, *Singular integrals and related topics*, World Scientific, Singapore, 2007.

[18] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., 1938, 43(1), 126–166.

[19] B. Muckenhoupt and R. L. Wheeden, *Norm inequalities for the littlewood-paley function g_λ^**, Trans. Amer. Math. Soc., 1974, 191, 95–111.

[20] T. Nogayama, *Boundedness of commutators of fractional integral operators on mixed Morrey spaces*, Integral Transforms and Spec. Funct., 2019, 30(10), 790–816.

[21] T. Nogayama, *Mixed Morrey spaces*, Positivity, 2019, 23(4), 961–1000.

[22] T. Nogayama, T. Ono, D. Salim and Y. Sawano, *Atomic decomposition for mixed Morrey spaces*, J. Geom. Anal., 2021, 31(9), 9338–9365.

[23] J. Peetre, *On the theory of $L_{p,\lambda}$ spaces*, J. Funct. Anal., 1969, 4(1), 71–87.

[24] M. Rosenthal and H. J. Schmeisser, *On the boundedness of singular integrals in Morrey spaces and its preduals*, J. Fourier Anal. Appl., 2016, 22(2), 462–490.

[25] L. G. Softova, *Parabolic oblique derivative problem with discontinuous coefficients in generalized Morrey spaces*, Ric. Mat., 2013, 62(2), 265–278.

[26] F. Soria and G. Weiss, *A remark on singular integrals and power weights*, Indiana Univ. Math. J., 1994, 43(1), 187–204.

[27] J. Tan, *Off-diagonal extrapolation on mixed variable Lebesgue spaces and its applications to strong fractional maximal operators*, Georgian Math. J., 2020, 27(4), 637–647.

[28] M. Yang, Z. Fu and J. Sun, *Existence and large time behavior to coupled chemotaxis-fluid equations in Besov-Morrey spaces*, J. Differential Equations, 2019, 266(9), 5867–5894.

[29] H. Zhang and J. Zhou, *The boundedness of fractional integral operators in local and global mixed Morrey-type spaces*, Positivity, 2022, 26(1), 26.