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LOCAL WELL-POSEDNESS FOR A 3D LIQUID-GAS TWO PHASE MODEL WITH VACUUM

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Abstract In this paper we prove the local well-posedness of strong solutions to a 3D liquid-gas two-phase flow model with vacuum in a bounded domain without the standard compatibility conditions.

Keywords Liquid-gas two-phase flow model, vacuum, local well-posedness.

MSC(2010) 76T10, 35Q35, 35D35, 76N10.

1. Introduction

In this paper we consider the following liquid-gas two-phase flow model in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho, n) = \mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u, \tag{1.2}$$

$$\partial_t n + \operatorname{div}(nu) = 0, \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.3}$$

$$u = 0, \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{1.4}$$

$$(\rho, \rho u, n)(\cdot, 0) = (\rho_0, \rho_0 u_0, n_0)(\cdot) \quad \text{in} \quad \Omega \subset \mathbb{R}^3. \tag{1.5}$$

Here ρ, n and u denote the liquid mass, gas mass, and velocity of the liquid and gas, respectively. μ and λ are viscosity constants satisfying

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0.$$

$p > 0$ is the common pressure for both phases, which satisfies

$$p(\rho, n) := C_0(-b(\rho, n) + \sqrt{b^2(\rho, n) + c(n)}), \tag{1.6}$$

with

$$b(\rho, n) := k_0 - \rho - a_0 n, \quad c(n) := 4k_0 a_0 n,$$

and C_0, k_0, and a_0 are positive constants.

Below we review some results to two-phase flow models. Evje and Karlsen [3] showed global existence of weak solutions to a two-phase model in 1D case, see also [4] for some improvements. Guo et al. [6] obtained the global strong solution for a 3D viscous liquid-gas two-phase flow model with vacuum when the energy of

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the initial data is small enough, see also [13]. Wen et al. [10] proved the local well-posedness and a blow-up criterion of strong solutions to the problem (1.1)–(1.5) following natural compatibility condition:

$$\nabla p(\rho_0, n_0) - \mu\Delta u_0 - (\lambda+\mu)\nabla \mathrm{div}\, u_0 = \sqrt{\rho_0} g \tag{1.7}$$

for some $g \in L^2(\Omega)$. Wu and Zhang [12] obtained the global existence and asymptotic behavior of strong solutions for the viscous liquid-gas two-phase flow model in a bounded domain when the initial data are near their equilibrium. Zhang [15] obtained the weak solutions of an inviscid two-phase flow model in physical vacuum in one-dimensional case. Recently, Li et al. [8] considered the large time behavior for a compressible two-fluid model with algebraic pressure closure and large initial data. Chen and Zhu [1] proved the existence of weak solutions to a steady two-phase flow. For more results on the two-phase flow models, see the review paper [11].

The aim of this paper is to prove the local well-posedness of strong solutions to the problem (1.1)–(1.5) without the assumption (1.7). We will prove

Theorem 1.1. *Let $0 \le \rho_0, n_0 \in W^{1,q}$ $(3 < q < 6)$ and $u_0 \in H_0^1$. Then the problem* (1.1)–(1.5) *has a unique local strong solution (ρ, n, u) satisfying*

$$\begin{cases} 0 \le \rho, n \in L^\infty(0,T;W^{1,q}), \partial_t\rho, \partial_t n \in L^\infty(0,T;L^2), \\ u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2), \sqrt{\rho}\partial_t u \in L^2(0,T;L^2), \\ \sqrt{t\rho}\partial_t u \in L^\infty(0,T;L^2), \sqrt{t}\partial_t u \in L^2(0,T;H_0^1), \end{cases} \tag{1.8}$$

for some $0 < T < \infty$.

Remark 1.1. Recently, Gong et al. [5] and Huang [7] obtained similar results to the isentropic compressible Navier-Stokes equations without the compatibility condition similar to (1.7). Their proofs are in the sprit of Choe and Kim [2] by taking more delicate estimates. Our arguments are different from those in [5, 7]. Here we construct a priori estimates by using some ideas developed in the study of low Mach number limit problem [9], see the details below.

We will prove Theorem 1.1 in the following way: For $\delta > 0$, we choose $0 < \delta \le \rho_0^\delta, n_0^\delta \in H^2$ and $u_0^\delta \in H_0^1 \cap H^2$ satisfying

$$(\rho_0^\delta, n_0^\delta) \to (\rho_0, n_0) \ \text{ in } \ W^{1,q} \ \ u_0^\delta \to u_0 \ \text{ in } \ H_0^1 \ \text{ as } \ \delta \to 0. \tag{1.9}$$

Then it is easy to verify that the problem has a unique local strong solution $(\rho^\delta, n^\delta, u^\delta)$ in $[0, T_\delta)$. We point out that the condition $3 < q < 6$ is used at this point to guarantee that H^2 is compactly embedded in $W^{1,q}$ (since $q < 6$) which is compactly embedded in L^∞ (since $q > 3$).

Now we define, similar to [14],

$$\begin{aligned} M^\delta(t) := &1 + \sup_{0\le s\le t}\{\|(\rho^\delta, n^\delta)(\cdot,s)\|_{W^{1,q}} + \|(\partial_t\rho^\delta, \partial_t n^\delta)(\cdot,s)\|_{L^2} \\ &+ \|u^\delta(\cdot,s)\|_{H^1} + \sqrt{s}\|\sqrt{\rho^\delta}u_t^\delta(\cdot,s)\|_{L^2}\} + \|u^\delta\|_{L^2(0,t;H^2)} \\ &+ \|\sqrt{\rho^\delta}\partial_t u^\delta\|_{L^2(0,t;L^2)} + \|\sqrt{s}\nabla u_t^\delta\|_{L^2(0,t;L^2)}. \end{aligned} \tag{1.10}$$

We can prove

Theorem 1.2. *For any $t \in (0, T_\delta)$, we have that*

$$M^\delta(t) \le C_0(M_0^\delta) \exp(t^{\frac{6-q}{4q}} C(M^\delta(t))) \tag{1.11}$$

for some nondecreasing continuous function $C_0(\cdot)$ and $C(\cdot)$.

It follows from (1.11) that (see [9]):

$$M^\delta(t) \le C \tag{1.12}$$

and thus the proof of existence part is complete by taking $\delta \to 0$ and the standard compactness principle.

In the remainder of this paper we give the proof of Theorem 1.2 in section 2 and present the proof of uniqueness part of Theorem 1.1 with the regularity (1.8) in section 3.

2. Proof of Theorem 1.2

Below, for the sake of simplicity, we shall drop the superscript "δ" of ρ^δ, n^δ, u^δ, M_0^δ and $M^\delta(t)$, and denote $M \equiv M(t)$.

First, it is easy to see that

$$0 < \rho, n \quad \text{and} \quad \int \rho \mathrm{d}x = \int \rho_0 \mathrm{d}x, \int n \mathrm{d}x = \int n_0 \mathrm{d}x. \tag{2.1}$$

We note that

$$\begin{aligned}
\frac{\partial p}{\partial \rho} &= C_0 \left(1 - \frac{b}{\sqrt{b^2 + c(n)}}\right) > 0, \\
\frac{\partial p}{\partial n} &= C_0 a_0 \left(1 + \frac{-b + 2k_0}{\sqrt{b^2 + c(n)}}\right) > 0, \\
\frac{\partial^2 p}{\partial \rho^2} &= C_0 \frac{c(n)}{\sqrt{(b^2 + c(n))^3}} > 0, \\
\frac{\partial^2 p}{\partial \rho \partial n} &= C_0 a_0 \frac{2bk_0 + c(n)}{\sqrt{(b^2 + c(n))^3}} > 0, \\
\frac{\partial^2 p}{\partial n^2} &= -4C_0 a_0^2 k_0 \frac{\rho}{\sqrt{(b^2 + c(n))^3}} < 0,
\end{aligned}$$

and thus

$$\left|\frac{\partial p}{\partial \rho}\right| + \left|\frac{\partial p}{\partial n}\right| \le C(M). \tag{2.2}$$

Equation (1.2) can be written as

$$-\mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u = f := -\rho \partial_t u - \rho u \cdot \nabla u - \nabla p. \tag{2.3}$$

Because the system (2.3) is a strong elliptic system so that one can use the elliptic regularity to obtain

$$\|u\|_{W^{2,q}} \le C\|f\|_{L^q} \le C\|\rho \partial_t u\|_{L^q} + C\|\rho u \cdot \nabla u\|_{L^q} + C\|\nabla p\|_{L^q}$$

$$\begin{aligned}
&\leq C\|\rho\|_{L^\infty}^{\frac{5q-6}{4q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|u_t\|_{L^6}^{\frac{3q-6}{2q}}\\
&\quad + C(M)\|u\|_{L^\infty}\|\nabla u\|_{L^q} + C(M)(\|\nabla\rho\|_{L^q} + \|\nabla n\|_{L^q})\\
&\leq C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C(M)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{3}{2}} + C(M)\\
&\leq C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C(M)\|u\|_{H^2}^{\frac{3}{2}} + C(M),
\end{aligned} \tag{2.4}$$

which gives

$$\begin{aligned}
\int_0^t \|u\|_{W^{2,q}}\mathrm{d}s &\leq C(M)\int_0^t \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}}\mathrm{d}s + C(M)\int_0^t \|u\|_{H^2}^{\frac{3}{2}}\mathrm{d}s + C(M)t\\
&\leq C(M)\int_0^t s^{-\frac{3q-6}{4q}}(\sqrt{s}\|\nabla u_t\|_{L^2})^{\frac{3q-6}{2q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\mathrm{d}s\\
&\quad + C(M)\left(\int_0^t \mathrm{d}s\right)^{\frac{1}{4}}\left(\int_0^t \|u\|_{H^2}^2\mathrm{d}s\right)^{\frac{3}{4}} + C(M)t\\
&\leq C(M)\left(\int_0^t s^{-\frac{3q-6}{2q}}\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t s\|\nabla u_t\|_{L^2}^2\mathrm{d}s\right)^{\frac{3q-6}{4q}}\\
&\quad \times\left(\int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}s\right)^{\frac{6-q}{4q}} + C(M)t^{\frac{1}{4}}\\
&\leq C(M)t^{\frac{6-q}{4q}} + C(M)t^{\frac{1}{4}} \leq C(M)t^{\frac{6-q}{4q}}
\end{aligned} \tag{2.5}$$

for all $0 < t \leq 1$. Here, in (2.4), we have used the continuous embedding $W^{1,q} \hookrightarrow L^\infty$ (since $q > 3$) and Agmon's inequality

$$\|u\|_{L^\infty} \leq C\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}.$$

Using the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^\infty} \leq C\|\nabla u\|_{L^2}^{\frac{2q-6}{5q-6}}\|u\|_{W^{2,q}}^{\frac{3q}{5q-6}}, \tag{2.6}$$

we observe that

$$\begin{aligned}
\int_0^t \|\nabla u\|_{L^\infty}\mathrm{d}s &\leq C(M)\int_0^t \|u\|_{W^{2,q}}^{\frac{3q}{5q-6}}\mathrm{d}s\\
&\leq C(M)\left(\int_0^t \mathrm{d}s\right)^{\frac{2q-6}{5q-6}}\left(\int_0^t \|u\|_{W^{2,q}}\mathrm{d}s\right)^{\frac{3q}{5q-6}}\\
&\leq C(M)t^{\frac{2q-6}{5q-6}}\cdot t^{\frac{6-q}{4q}\cdot\frac{3q}{5q-6}} = C(M)t^{\frac{1}{4}} \leq C(M)t^{\frac{6-q}{4q}}.
\end{aligned} \tag{2.7}$$

Testing (1.1) by ρ^{m-1}, we see that

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^m\mathrm{d}x = -\int \mathrm{div}\,(\rho u)\rho^{m-1}\mathrm{d}x = \int \rho u\nabla\rho^{m-1}\mathrm{d}x = -\frac{m-1}{m}\int \rho^m\mathrm{div}\,u\mathrm{d}x,$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^m} \leq \|\mathrm{div}\,u\|_{L^\infty}\|\rho\|_{L^m},$$

and thus (using Gronwall's inequality and the Sobolev embedding $W^{1,q} \hookrightarrow L^\infty$ (since $q > 3$))

$$
\begin{aligned}
\|\rho\|_{L^m} &\le \|\rho_0\|_{L^m} \exp\left(\int_0^t \|\mathrm{div}\, u\|_{L^\infty} \mathrm{d}s\right)\\
&\le \|\rho_0\|_{L^m} \exp(t^{\frac{6-q}{4q}} C(M)) \quad (2 \le m \le \infty).
\end{aligned} \tag{2.8}
$$

Applying ∇ to (1.1), testing by $|\nabla\rho|^{q-2}\nabla\rho$, we find that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q} \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q} + C\|\rho\|_{L^\infty}\|\nabla\mathrm{div}\, u\|_{L^q},
$$

which implies

$$
\begin{aligned}
\|\nabla\rho\|_{L^q} &\le C\left(\|\nabla\rho_0\|_{L^q} + \int_0^t \|\rho\|_{L^\infty}\|\nabla\mathrm{div}\, u\|_{L^q}\mathrm{d}s\right)\exp\left(\int_0^t \|\nabla u\|_{L^\infty}\mathrm{d}s\right)\\
&\le C(1 + C(M)t^{\frac{6-q}{4q}})\exp(t^{\frac{6-q}{4q}} C(M))\\
&\le C_0(M_0)\exp(t^{\frac{6-q}{4q}} C(M)).
\end{aligned} \tag{2.9}
$$

Similarly, we have

$$
\|n\|_{L^m} \le \|n_0\|_{L^m} \exp(t^{\frac{6-q}{4q}} C(M)) \quad (2 \le m \le \infty), \tag{2.10}
$$

$$
\|\nabla n\|_{L^q} \le C_0(M_0)\exp(t^{\frac{6-q}{4q}} C(M)). \tag{2.11}
$$

Testing (1.2) by u_t, and using the equation (1.1), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}\, u)^2)\mathrm{d}x + \int \rho|u_t|^2\mathrm{d}x\\
&= -\int \rho u\cdot\nabla u\cdot u_t \mathrm{d}x + \int p\,\mathrm{div}\, u_t\mathrm{d}x\\
&=: I_1 + I_2.
\end{aligned} \tag{2.12}
$$

We bound I_1 and I_2 as follows.

$$
\begin{aligned}
|I_1| &\le \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\\
&\le C(M)\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}} \le C(M)\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{H^2}^{\frac{1}{2}}\\
&\le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C(M)\|u\|_{H^2};\\
I_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int p\,\mathrm{div}\, u\mathrm{d}x - \int p_t\mathrm{div}\, u\mathrm{d}x\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int p\,\mathrm{div}\, u\mathrm{d}x - \int\left(\frac{\partial p}{\partial\rho}\rho_t + \frac{\partial p}{\partial n}n_t\right)\mathrm{div}\, u\mathrm{d}x\\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int p\,\mathrm{div}\, u\mathrm{d}x + C(M)(\|\rho_t\|_{L^2} + \|n_t\|_{L^2})\|\nabla u\|_{L^2}\\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int p\,\mathrm{div}\, u\mathrm{d}x + C(M).
\end{aligned}
$$

Here we have used the using the continuous embedding $H^1 \hookrightarrow L^6$ and the Gagliardo-Nirenberg interpolation inequality

$$\|\nabla u\|_{L^3} \le C\|u\|_{H^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}. \tag{2.13}$$

Inserting the above estimates into (2.12) and integrating over $(0,t)$, we have

$$\|\nabla u\|_{L^2}^2 + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2 \mathrm{d}s \le C_0(M_0)\exp(t^{\frac{6-q}{4q}}C(M)). \tag{2.14}$$

Applying ∂_t to (1.2) and using (1.1), we infer that

$$\begin{aligned}&\rho\partial_t^2 u + \rho u\cdot\nabla u_t - \mu\Delta u_t - (\lambda+\mu)\nabla \mathrm{div}\, u_t\\ =&-\nabla p_t + \mathrm{div}\,(\rho u)(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u.\end{aligned} \tag{2.15}$$

Testing (2.15) by u_t and using (1.1), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \int(\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}\,u_t)^2)\mathrm{d}x\\ =&\int p_t \mathrm{div}\,u_t\mathrm{d}x - \int \rho u\nabla|u_t|^2\mathrm{d}x - \int\rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\mathrm{d}x - \int \rho u_t\cdot\nabla u\cdot u_t\mathrm{d}x\\ =:&\sum_{i=3}^{6} I_i.\end{aligned} \tag{2.16}$$

We bound I_i $(i=3,\dots,6)$ as follows.

$$\begin{aligned}|I_3| =& \left|\int\left(\frac{\partial p}{\partial\rho}\rho_t + \frac{\partial p}{\partial n}n_t\right)\mathrm{div}\,u_t\mathrm{d}x\right|\\ \le& C(M)\|\nabla u_t\|_{L^2}\\ \le&\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M);\\ |I_4| \le& 2\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2}\\ \le& C(M)\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\\ \le& C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac12}\|\nabla u_t\|_{L^2}\\ \le& C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\\ \le& \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M)\|\sqrt{\rho}u_t\|_{L^2}^2;\\ |I_5| \le&\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6} + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6}\\ &+\|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\ \le& C(M)(\|\nabla u\|_{L^3}^2 + \|u\|_{H^2})\|\nabla u_t\|_{L^2}\\ \le& C(M)(\|\nabla u\|_{L^2}\|u\|_{H^2} + \|u\|_{H^2})\|\nabla u_t\|_{L^2}\\ \le& C(M)\|u\|_{H^2}\|\nabla u_t\|_{L^2}\\ \le& \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M)\|u\|_{H^2}^2;\\ |I_6| \le& \|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^4}^2 \le \|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac32}\\ \le& C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\end{aligned}$$

$$\leq \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M)\|\sqrt{\rho}u_t\|_{L^2}^2.$$

Inserting the above estimates into (2.16), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \frac{3}{4}\mu\int |\nabla u_t|^2\mathrm{d}x \leq C(M) + C(M)\|\sqrt{\rho}u_t\|_{L^2}^2 + C(M)\|u\|_{H^2}^2. \quad (2.17)$$

Multiplying the above inequality by t and integrating over $(0,t)$, we have

$$t\int \rho|u_t|^2\mathrm{d}x + \int_0^t s\|\nabla u_t\|_{L^2}^2\mathrm{d}s \leq C_0(M_0)\exp(t^{\frac{6-q}{4q}}C(M)). \quad (2.18)$$

It follows from (2.3) that, using the Gagliardo-Nirenberg interpolation inequality (2.13) and Young's inequality,

$$\begin{aligned}
\|u\|_{H^2} &\leq C\|f\|_{L^2} \leq C\|\rho u_t + \rho u\cdot\nabla u + \nabla p\|_{L^2} \\
&\leq C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\nabla p\|_{L^2} \\
&\leq C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M)\|u\|_{H^2}^{\frac{1}{2}} + C(M),
\end{aligned}$$

which yields

$$\|u\|_{H^2} \leq C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M),$$

and therefore

$$\|u\|_{L^2(0,t;H^2)} \leq C_0(M_0)\exp(t^{\frac{6-q}{4q}}C(M)). \quad (2.19)$$

It follows from (1.1), (2.8), (2.9) and (2.14) that

$$\begin{aligned}
\|\rho_t\|_{L^2} &= \|u\cdot\nabla\rho + \rho\operatorname{div} u\|_{L^2} \\
&\leq \|u\|_{L^{\frac{2q}{q-2}}}\|\nabla\rho\|_{L^q} + \|\rho\|_{L^\infty}\|\operatorname{div} u\|_{L^2} \\
&\leq C\|\rho\|_{W^{1,q}}\|\nabla u\|_{L^2} \\
&\leq C_0(M_0)\exp(t^{\frac{6-q}{4q}}C(M)).
\end{aligned} \quad (2.20)$$

Similarly, we have

$$\|n_t\|_{L^2} \leq C_0(M_0)\exp(t^{\frac{6-q}{4q}}C(M)). \quad (2.21)$$

Combining (2.8)–(2.11), (2.14), and (2.18)–(2.21), we conclude that (1.11) holds true. This completes the proof of Theorem 1.2. □

In the remainder of this paper we give the proof of Theorem 1.2 in section 2 and present the proof of uniqueness part of Theorem 1.1 with the regularity (1.8) in section 3.

3. Proof of the uniqueness part of Theorem 1.1

This section is devoted to the proof of the uniqueness part of Theorem 1.1. Let (ρ_i, u_i, n_i) $(i = 1, 2)$ be the two strong solutions satisfying (1.8) with the same initial data.

We denote

$$(\rho, u, n) := (\rho_1 - \rho_2, u_1 - u_2, n_1 - n_2).$$

Then it is easy to verify that

$$\partial_t\rho+u_2\cdot\nabla\rho+\rho\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u+u\cdot\nabla\rho_1=0, \tag{3.1}$$

$$\begin{aligned}\rho_1\partial_t u+\rho_1u_1\cdot\nabla u-\mu\Delta u-(\lambda+v\mu)\nabla\mathrm{div}\,u=&-\rho_1u\cdot\nabla u_2-\rho(\partial_tu_2+u_2\cdot\nabla u_2)\\&-\nabla(p(\rho_1,n_1)-p(\rho_2,n_2)),\end{aligned} \tag{3.2}$$

$$\partial_t n+u_2\cdot\nabla n+n\mathrm{div}\,u_2+n_1\mathrm{div}\,u+u\cdot\nabla n_1=0. \tag{3.3}$$

Testing (3.1) by ρ and using (1.8), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^2\mathrm{d}x=&-\int(u_2\nabla\rho+\rho\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u+u\cdot\nabla\rho_1)\rho\mathrm{d}x\\=&-\int\left(\frac{1}{2}\rho^2\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u\rho+u\nabla\rho_1\rho\right)\mathrm{d}x\\\le& C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}^2+C\|\rho_1\|_{L^\infty}\|\nabla u\|_{L^2}\|\rho\|_{L^2}\\&+C\|u\|_{L^6}\|\nabla\rho_1\|_{L^3}\|\rho\|_{L^2}\\\le& C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\rho\|_{L^2},\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^2}\le C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}+C\|\nabla u\|_{L^2}.$$

By the Gronwall inequality, we get

$$\|\rho\|_{L^2}\le C\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s. \tag{3.4}$$

Similarly, testing (3.3) by n and using (1.8), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int n^2\mathrm{d}x=&-\int(u_2\nabla n+n\mathrm{div}\,u_2+n_1\mathrm{div}\,u+u\cdot\nabla n_1)n\mathrm{d}x\\\le& C\|\nabla u_2\|_{L^\infty}\|n\|_{L^2}^2+C\|\nabla u\|_{L^2}\|n\|_{L^2},\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}\le C\|\nabla u_2\|_{L^\infty}\|n\|_{L^2}+C\|\nabla u\|_{L^2}.$$

Hence

$$\|n\|_{L^2}\le C\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s. \tag{3.5}$$

Testing (3.2) by u, using (1.1), (1.8), (2.2), (3.4), and (3.5) we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\rho_1|u|^2\mathrm{d}x+\int_0^t\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2)\mathrm{d}x\mathrm{d}s\right)\\&+\frac{1}{2}\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2)\mathrm{d}x\\\le& C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C\|\partial_tu_2\|_{L^6}\|u\|_{L^6}\|\rho\|_{L^{\frac{3}{2}}}\\&+C\|u_2\|_{L^6}\|\nabla u_2\|_{L^6}\|u\|_{L^6}\|\rho\|_{L^2}+C\|p(\rho_1,n_1)-p(\rho_2,n_2)\|_{L^2}\|\nabla u\|_{L^2}\\\le& C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C\|\nabla u_{2t}\|_{L^2}\|\nabla u\|_{L^2}\|\rho\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+ C\|u_2\|_{H^2}\|\nabla u\|_{L^2}\|\rho\|_{L^2} + C(\|\rho\|_{L^2} + \|n\|_{L^2})\|\nabla u\|_{L^2} \\
\le\, & C\|\nabla u_2\|_{L^\infty}\int \rho_1|u|^2\mathrm{d}x \\
&+ C(\|\nabla u_{2t}\|_{L^2} + \|u\|_{H^2} + 1)\|\nabla u\|_{L^2}\int_0^t \|\nabla u\|_{L^2}\mathrm{d}s \\
\le\, & \frac{\mu}{16}\|\nabla u\|_{L^2}^2 + C\|\nabla u_2\|_{L^\infty}\int \rho_1|u|^2\mathrm{d}x \\
&+ C(\|\nabla u_{2t}\|_{L^2}^2 + \|u\|_{H^2}^2 + 1)\left(\int_0^t \|\nabla u\|_{L^2}\mathrm{d}s\right)^2 \\
\le\, & \frac{\mu}{16}\|\nabla u\|_{L^2}^2 + C\|\nabla u_2\|_{L^\infty}\int \rho_1|u|^2\mathrm{d}x \\
&+ C(t\|\nabla u_{2t}\|_{L^2}^2 + \|u\|_{H^2}^2 + 1)\int_0^t \|\nabla u\|_{L^2}^2\mathrm{d}s. \qquad (3.6)
\end{aligned}
$$

Applying the Gronwall inequality to (3.6) gives $u = 0$, and then using this fact, we get $\rho = 0$ and $n = 0$ from (3.4) and (3.5). Thus, we complete the proof of the uniqueness part of Theorem 1.1 and finally get Theorem 1.1. □

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