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A FREE BOUNDARY PROBLEM OF SOME MODIFIED LESLIE-GOWER PREDATOR-PREY MODEL WITH SHIFTING ENVIRONMENTS

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Abstract In this paper, we mainly study the long time dynamical behavior of the Leslie-Gower prey-predator model with two free boundaries in some shifting environments. We assume that the unfavourable region of the environment moves into the otherwise favourable homogeneous environment with a given speed $c > 0$ in the spreading direction of the prey and predator. We focus on the invasion of introduced predator in the new habitat. We show that such shifting environments could reverse the fates of the prey and the predator can be able to successfully invade. A complete discussion of the long time behavior of the model can be obtained for such cases.

Keywords Shifting environments, predator-prey model, free boundary, spreading and vanishing, spreading speed

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1. Introduction

Climate change has a profound impact on the survival and spreading of ecological species, resulting in changes in species abundance, diversity and habitat, and leading to the extinction of some vulnerable species around the world. In order to gain insight into the consequences of climate change, many mathematicians have proposed some models which can be deduced that climate change may threaten the survive of species by shifting environment from favorable to unfavorable conditions.

In [7], Du et al. considered the following free boundary problem of the diffusive logistic equation

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0, 0 < x < h(t), \\ u(t, h(t)) = 0, u_x(t, 0) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ h(0) = h_0, u(0, x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.1}$$

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where $u(t,x)$ stands for the population density of the species at time t and spatial position x, $x = h(t)$ is the moving boundary to be determined together with $u(t,x)$, a, b, d are given positive constants, $h_0 > 0$ denotes the size of initial habitat, $\mu > 0$ is the ratio of expanding speed of the free boundary and population gradient at expanding front, and u_0 is a given positive initial function. They have obtained the spreading-vanishing dichotomy results.

In [12], Du et al. investigated a similar situation by the following free boundary model

$$\begin{cases} u_t = d_1 u_{xx} + A_1(x-ct)u - b_1 u^2, \, t>0, 0<x<h(t), \\ u_x(t,0) = u(t,h(t)) = 0, & t>0, \\ h'(t) = -\mu_1 u_x(t,h(t)), & t>0, \\ h(0) = h_0, u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.2}$$

where μ_1, d_1, b_1 are given positive constants. The function $A_1(\xi)(\xi \in \mathbf{R})$ is assumed to be Lipschitz continuous, strictly increasing on $[-l_0, 0]$ and satisfied

$$A_1(\xi) = \begin{cases} a_{0,1}, \, \xi < -l_0, \\ a_1, \quad \xi \ge 0, \end{cases}$$

where $l_0 > 0$, $a_{0,1} < 0$ and $a_1 > 0$ are constants.

In [22], Z. Guo et al. considered the Leslie-Gower predator-prey model with a free boundary

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_{xx} + u(1-u) - \delta uv, & t>0, 0<x<h(t), \\ \dfrac{\partial v}{\partial t} = Dv_{xx} + \kappa v\left(1 - \dfrac{v}{u+\alpha}\right), & t>0, 0<x<h(t), \\ h'(t) = -\mu(u_x(t,h(t))) + \rho v_x(t,h(t)), & t>0, \\ h(0) = h_0, u_x(t,0) = v_x(t,0) = u(t,h(t)) = v(t,h(t)) = 0, & t>0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in [0,h_0], \end{cases} \tag{1.3}$$

with the positive parameters $\mu > 0$, $\rho > 0$. The initial data (u_0, v_0) satisfy

$$\begin{cases} u_0, v_0 \in C^2([0,h_0]), \\ u_0'(0) = v_0'(0) = u_0(h_0) = v_0(h_0) = 0, \\ h_0 > 0, u_0(x) > 0, v_0(x) > 0 \text{ for all } x \in [0,h_0). \end{cases}$$

The model (1.3) describes how two species $u(t,x)$ and $v(t,x)$ evolve if they initially occupy the bounded region $[0,h_0]$. They have established a spreading-vanishing dichotomy for the long-time dynamical behavior, which unique solution (u,v,h) is satisfied one of the following condition.

(i) Spreading: if $\lim\limits_{t\to\infty} h(t) = +\infty$, then $\lim\limits_{t\to\infty} u(t,x) = p_0$ and $\lim\limits_{t\to\infty} v(t,x) = q_0$, where (p_0, q_0) is the unique interior equilibrium of (1.3).

(ii) Vanishing: if $\lim_{t\to\infty} h(t) < +\infty$, then

$$\lim_{t\to\infty} \|u(t,x)\|_{C[0,h(t)]} = 0 \text{ and } \lim_{t\to\infty} \|u(t,x)\|_{C[0,h(t)]} = 0.$$

Free boundary problems with different free boundaries for two-species model have been studied by many authors. In [32], Wang et al. investigated a free boundary problem for the diffusive Leslie-Gower prey-predator model with double free boundaries in one space dimension. They also provided a spreading-vanishing dichotomy. Liu et al. in [23] discussed the diffusive competition model with two different free boundaries, and gave some sharper estimates of asymptotic spreading speeds of two free boundaries when both species spread successfully. Similar works but for the mutualist model can be found in [38]. In [16], Huang et al. studied a free boundary problem of the diffusive competition model with different habitats. At the same time, they investigated the existence, uniqueness, regularity, uniform estimates and long-time behavior of the global solution. Wang et al. in [34] studied a diffusive competition model with seasonal succession and different free boundaries. In [20], Li et al. investigated some free boundary models with nonlocal diffusions and different free boundaries. They proved that such kind of nonlocal diffusion problems has a unique global solution. They also studied the long-time behavior of global solution and criteria of spreading and vanishing for the classical Lotka-Volterra competition, prey-predator and mutualist models.

In this paper, we will consider the free boundary problem of Leslie-Gower predator-prey model with shifting environments, that is the following model

$$\begin{cases} u_t - d_1 u_{xx} = A_1(x-ct)u - u^2 - \beta uv, \, t>0, 0<x<h(t), \\ v_t - d_2 v_{xx} = A_2(x-ct)v - \dfrac{v^2}{u+\alpha}, & t>0, 0<x<g(t), \\ u_x(t,0) = 0, u(t,x) = 0, & t>0, h(t)\le x<+\infty, \\ v_x(t,0) = 0, v(t,x) = 0, & t>0, g(t)\le x<+\infty, \\ h'(t) = -\mu_1 u_x(t,h(t)), & t>0, \\ g'(t) = -\mu_2 v_x(t,g(t)), & t>0, \\ h(0) = h_0, u(0,x) = u_0(x), & 0\le x\le h_0, \\ g(0) = g_0, v(0,x) = v_0(x), & 0\le x\le g_0, \end{cases} \tag{1.4}$$

where d_i, $\mu_i (i=1,2)$, α, β, h_0 and g_0 are positive constants, $A_2(\xi)$ satisfies the same assumption as $A_1(\xi)$ except that $(a_{0,1}, a_1)$ is replaced by $(a_{0,2}, a_2)$ with $a_{0,2} < 0$, $a_2 > 0$, and the initial functions $u_0(x)$ and $v_0(x)$ satisfy

$$\begin{cases} u_0 \in C^2([0,h_0]), v_0 \in C^2([0,g_0]), u_0'(0) = v_0'(0) = 0, \\ u_0 > 0 \text{ in } [0,h_0), u_0(h_0) = 0, \\ v_0 > 0 \text{ in } [0,g_0), v_0(g_0) = 0. \end{cases} \tag{1.5}$$

By the mathematical analysis in [12], there exist positive constants $c_{0,1}$ and $c_{0,2}$, which are determined by the following problems

$$\begin{cases} d_1 p_1'' - c_{0,1} p_1' + a_1 p_1 - p_1^2 = 0, p_1(z) > 0,\ z > 0, \\ p_1(0) = 0, p_1(\infty) = a_1, c_{0,1} = \mu_1 p_1'(0), \end{cases} \tag{1.6}$$

and

$$\begin{cases} d_2 p_2'' - c_{0,2} p_2' + a_2 p_2 - \dfrac{p_2^2}{\alpha} = 0, p_2(z) > 0,\ z > 0, \\ p_2(0) = 0, p_2(\infty) = a_2\alpha, c_{0,2} = \mu_2 p_2'(0), \end{cases} \tag{1.7}$$

respectively. It is also known $c_{0,i} < 2\sqrt{a_i d_i}$ $(i = 1, 2)$.

Next, we will show the main theorems of this paper. Firstly, when $c > \max\{c_{0,1}, c_{0,2}\}$, we can prove that both u and v are extinct in their respective environmental conditions, that is the following theorem.

Theorem 1.1. *Let* (u, v, h, g) *be the solution of* (1.4) *with initial functions satisfying* (1.5).
(i) If $c \geq c_{0,1}$*, then* u *vanishes, i.e.,*

$$\lim_{t\to\infty} h(t) = h_\infty < +\infty \text{ and } \lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0.$$

(ii) If $c \geq c_{0,2}$*, then* v *vanishes, i.e.,*

$$\lim_{t\to\infty} g(t) = g_\infty < +\infty \text{ and } \lim_{t\to\infty} \|v(t,\cdot)\|_{C([0,g(t)])} = 0.$$

Secondly, we discuss the case of $c \in [c_{0,1}, c_{0,2})$. In this case, the prey u becomes extinct and the long-time behavior of the predator v can be described by a trichotomy consisting of vanishing, spreading and borderline spreading.

Theorem 1.2. *Let* (u, v, h, g) *be the unique solution of* (1.4) *with initial functions satisfying* (1.5)*. Suppose that*

$$c_{0,2} > c \geq c_{0,1}, \tag{1.8}$$

then (u, h) *satisfies*

$$\lim_{t\to\infty} h(t) = h_\infty < +\infty, \lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0, \tag{1.9}$$

and
(i) vanishing of v*:* $\lim_{t\to\infty} g(t) = g_\infty < +\infty$ *and*

$$\lim_{t\to\infty} \left[\max_{x\in[0,g(t)]} v(t,x)\right] = 0;$$

(ii) spreading of v*:* $\lim_{t\to\infty} [g(t) - c_{0,2}t] = G_0$ *for some* $G_0 \in \mathbf{R}$*, and for any* $\tilde{c} \in (c, c_{0,2})$*,*

$$\lim_{t\to\infty} \left[\max_{[0,\tilde{c}t]} |v(t,x) - \phi(x - ct)|\right] = 0,$$

$$\lim_{t\to\infty} \left[\max_{[\tilde{c}t,g(t)]} |v(t,x) - p_2(g(t) - x)|\right] = 0,$$

where p_2 is the unique solution to (1.7) *and $\phi(x)$ is the unique solution of*

$$\begin{cases} d_2\phi'' + c\phi' + A_2(x)\phi - \dfrac{\phi^2}{\alpha} = 0, \, x \in (-\infty, +\infty), \\ \phi(-\infty) = 0, \phi(+\infty) = a_2\alpha; \end{cases} \tag{1.10}$$

(iii) borderline spreading of v: $\lim_{t\to\infty}[g(t) - ct] = L_*$ *and*

$$\lim_{t\to\infty}\left[\max_{[0,g(t)]}|v(t,x) - V_*(x - g(t) + L_*)|\right] = 0,$$

where L_*, V_* are uniquely determined by

$$\begin{cases} d_2V_*'' + cV_*' + A_2(x)V_* - \dfrac{V_*^2}{\alpha} = 0, V_* > 0, \, x \in (-\infty, L_*), \\ V_*(-\infty) = V_*(L_*) = 0, -\mu_2 V_*'(L_*) = c. \end{cases} \tag{1.11}$$

In order to better understand the trichotomy described in Theorem 1.2, we treat μ_2 as a parameter and fix all the other parameters in (1.4) as well as the initial functions u_0, v_0 satisfying (1.5).

Theorem 1.3. *Suppose that $c_{0,1} \le c < 2\sqrt{a_2 d_2}$, and (u, v, h, g) is the solution of* (1.4) *with initial functions u_0, v_0 satisfying* (1.5). *Then* (1.9) *always holds, and there exists $\tilde{\mu} \in (0, +\infty)$ such that,*
(i) vanishing of v happens if $\mu_2 \in (0, \tilde{\mu})$;
(ii) borderline spreading of v happens if $\mu_2 = \tilde{\mu}$;
(iii) spreading of v happens if $\mu_2 > \tilde{\mu}$.

The rest of this paper is organized as follows. In Section 2, we will show that some preliminaries which are either known or easily obtained from existing results. In Section 3, we will study the long-time dynamical behavior of u and v. In this section, we are devote to the proof of Theorems 1.1-1.2. In Section 4, we will prove Theorem 1.3.

2. Some preliminaries

The existence and uniqueness conclusion for (1.4) can be obtained by simple modification of the proof of Proposition 2 in [15](with some corrections as given in [13]), or follow the argument in [33], which are rather different from those in [15].

Theorem 2.1. *For any given (u_0, v_0) satisfied* (1.5) *and any $\varpi \in (0, 1)$, there exists $T > 0$ such that the problem* (1.4) *admits a unique bounded solution*

$$(u, v, h, g) \in C^{\frac{1+\varpi}{2}, 1+\varpi}(I_T) \times C^{\frac{1+\varpi}{2}, 1+\varpi}(J_T) \times C^{1+\frac{\varpi}{2}}([0, T]).$$

Moreover,

$$\|u\|_{C^{\frac{1+\varpi}{2},1+\varpi}(I_T)} + \|v\|_{C^{\frac{1+\varpi}{2},1+\kappa}(J_T)} + \|h\|_{C^{1+\frac{\varpi}{2}}([0,T])} + \|g\|_{C^{1+\frac{\varpi}{2}}([0,T])} \le c,$$

where

$$I_T = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], x \in [0, h(t)]\},$$
$$J_T = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], x \in [0, g(t)]\},$$

and c, T only depend on h_0, g_0, ϖ, $\|u_0\|_{C^2([0,h_0])}$, $\|v_0\|_{C^2([0,g_0])}$.

By similar arguments as in the proof of Theorem 1 of [15], we have the following global existence result.

Theorem 2.2. *The unique solution (u, v, h, g) in Theorem 2.1 can be extended to all $t > 0$, and there exist positive constants M_1, M_2, M_3, M_4 such that*

$$
\begin{aligned}
&0 < u(t,x) \le M_1, \quad \textit{for} \quad t \in (0,+\infty),\ 0 \le x < h(t),\\
&0 < v(t,x) \le M_2, \quad \textit{for} \quad t \in (0,+\infty),\ 0 \le x < g(t),\\
&0 < h'(t) \le M_3,\ 0 < g'(t) \le M_4, \quad \textit{for} \quad t \in (0,+\infty).
\end{aligned}
$$

Next, we state a comparison principle, which is extracted from Lemma 4.1 and Lemma 4.2 of [28] with minor modifications.

Lemma 2.1. *Suppose that $T \in (0,\infty)$, $\bar{h}$, $\underline{h}$, $\bar{g}$, $\underline{g} \in C^1([0,T])$ and $\bar{h}$, $\underline{h}$, $\bar{g}$, $\underline{g} > 0$ in $[0,T]$. Denote by*

$$
\begin{aligned}
\Omega_1 &= (t,x) : t > 0, x \in [0,\bar{h}(t)],\\
\Omega_2 &= (t,x) : t > 0, x \in [0,\underline{h}(t)],\\
\Omega_3 &= (t,x) : t > 0, x \in [0,\bar{g}(t)],\\
\Omega_4 &= (t,x) : t > 0, x \in [0,\underline{g}(t)].
\end{aligned}
$$

Let $\bar{u} \in C(\bar{\Omega}_1) \cap C^{1,2}(\Omega_1)$, $\bar{v} \in C(\bar{\Omega}_3) \cap C^{1,2}(\Omega_3)$, $\underline{u} \in C(\bar{\Omega}_2) \cap C^{1,2}(\Omega_2)$, $\underline{v} \in C(\bar{\Omega}_4) \cap C^{1,2}(\Omega_4)$. Assume that

$$0 < \bar{u}, \underline{u} \le M_1 \ \textit{and}\ 0 < \bar{v}, \underline{v} \le M_2,$$

$(\bar{u}, \bar{v}, \bar{h}, \bar{g})$ satisfies

$$
\begin{cases}
\bar{u}_t - d_1 \bar{u}_{xx} \ge A_1(x-ct)\bar{u} - \bar{u}^2, & t > 0, 0 < x < \bar{h}(t),\\
\bar{v}_t - d_2 \bar{v}_{xx} \ge A_2(x-ct)\bar{v} - \dfrac{\bar{v}^2}{M_1+\alpha}, & t > 0, 0 < x < \bar{g}(t),\\
\bar{u}_x(t,0) \le 0, \bar{u}(t,x) = 0, & t > 0, \bar{h}(t) \le x < +\infty,\\
\bar{v}_x(t,0) \le 0, \bar{v}(t,x) = 0, & t > 0, \bar{g}(t) \le x < +\infty,\\
\bar{h}'(t) \ge -\mu_1 \bar{u}_x(t,\bar{h}(t)), & t > 0,\\
\bar{g}'(t) \ge -\mu_2 \bar{v}_x(t,\bar{g}(t)), & t > 0,
\end{cases}
\tag{2.1}
$$

the couple $(\underline{u}, \underline{h})$ satisfies

$$
\begin{cases}
\underline{u}_t - d_1 \underline{u}_{xx} \le \underline{u}[A_1(x-ct) - \underline{u} - \beta M_2], & t > 0, 0 < x < \underline{h}(t),\\
\underline{u}_x(t,0) \ge 0, & t > 0, \underline{h}(t) \le x < +\infty,\\
\underline{u}(t,x) = 0, & t > 0, \underline{h}(t) \le x < +\infty,\\
\underline{h}'(t) \le -\mu_1 \underline{u}_x(t,\underline{h}(t)), & t > 0,
\end{cases}
\tag{2.2}
$$

and the couple $(\underline{v}, \underline{g})$ *satisfies*

$$\begin{cases} \underline{v}_t - d_2\underline{v}_{xx} \le \underline{v}\left[A_2(x-ct) - \dfrac{\underline{v}}{\alpha}\right], & t>0, 0<x<\underline{g}(t), \\ \underline{v}_x(t,0) \ge 0, & t>0, \underline{g}(t) \le x < +\infty, \\ \underline{v}(t,x) = 0, & t>0, \underline{g}(t) \le x < +\infty, \\ \underline{g}'(t) \le -\mu_2 \underline{v}_x(t,\underline{g}(t)), & t>0. \end{cases} \tag{2.3}$$

Assume that the initial data of (2.1) *satisfy*

$$\bar{h}(0) \ge h_0, \bar{g}(0) \ge g_0, \bar{u}(0,x) \ge 0, \bar{v}(0,x) \ge 0 \ \textit{on}\ [0,\bar{h}(0)],$$
$$\bar{u}(0,x) \ge u_0(x), \bar{v}(0,x) \ge v_0(x) \ \textit{on}\ [0,h(0)],$$

and the initial data of (2.2) *and* (2.3) *satisfy*

$$\underline{h}(0) \le h_0, \underline{g}(0) \le g_0, 0 < \underline{u}(0,x) \le u_0(x), 0 < \underline{v}(0,x) \le v_0(x) \ \textit{on}\ [0,\underline{h}(0)].$$

Then, the solution (u,v,h,g) *of* (1.4) *satisfies*

$$\underline{h}(t) \le h(t) \le \bar{h}(t),\ \underline{g}(t) \le g(t) \le \bar{g}(t) \ \textit{on}\ [0,+\infty),$$
$$u \le \bar{u} \ \textit{for all}\ t \ge 0, 0 \le x \le h(t),$$
$$v \le \bar{v} \ \textit{for all}\ t \ge 0, 0 \le x \le g(t),$$
$$u \ge \underline{u} \ \textit{for all}\ t \ge 0, 0 \le x \le \underline{h}(t),$$
$$v \ge \underline{v} \ \textit{for all}\ t \ge 0, 0 \le x \le \underline{g}(t).$$

The proof of Lemma 2.1 is very similar to the proofs of Lemma 5.1 of [14], Lemma 4.1 and Lemma 4.2 of [28]. Hence, we omit the details here.

For any constants $L > 0 \ge -l$, we consider the following problem

$$\begin{cases} d_2V'' + cV' + A_2(x)V - \dfrac{V^2}{\alpha} = 0, V(x) > 0 \text{ for } -l < x < L, \\ V(-l) = V(L) = 0. \end{cases} \tag{2.4}$$

Lemma 2.2. *Suppose* $0 < c < c_{0,2}$. *Then the following conclusions hold.*
(i) For each $l \ge 0$, *there is a unique* $L(l)$ *such that* (2.4) *with* $L = L(l)$ *has a unique positive solution* $V_l(x)$ *satisfying* $-\mu_2 V_l'(L(l)) = c$.
(ii) The function $l \to L(l)$ *is decreasing, denote* $L_* := \lim\limits_{l\to\infty} L(l) > -l_0$. *Moreover,* $V_*(x) := \lim\limits_{l\to\infty} V_l(x)$ *exists and* (L_*, V_*) *satisfies*

$$\begin{cases} d_2V_*'' + cV_*' + A_2(x)V_* - \dfrac{V_*^2}{\alpha} = 0, V_* > 0, x \in (-\infty, L_*), \\ V_*(-\infty) = V_*(L_*) = 0, -\mu_2 V_*'(L_*) = c. \end{cases} \tag{2.5}$$

Lemma 2.3. *Let* L_* *be given as in Lemma 2.2, and* M_2 *be given as in Theorem 2.2. For any given* $L < L_*$ *and* $-l < L$, *the following problem*

$$\begin{cases} d_2W'' + cW' + A_2(x)W - \dfrac{W^2}{\alpha} = 0, W(x) > 0 \textit{ for } -l < x < L, \\ W(-l) = M_2, W(L) = 0, \end{cases} \tag{2.6}$$

has a unique positive solution $W_{l,L}$. Moreover, for all sufficiently large l, we have

$$-\mu_2 W'_{l,L}(L) < c. \tag{2.7}$$

Lemmas 2.2 and 2.3 are minor modifications of Lemma 2.5 and Lemma 2.8 in [12].

3. The long-time dynamical behavior

3.1. The dynamical behavior for $c > \max\{c_{0,1}, c_{0,2}\}$

In this section, we will prove Theorem 1.1. **Proof.** We first prove part (i). Since $v \geq 0$ from (1.4), we can obtain

$$\begin{cases} u_t - d_1 u_{xx} \leq u A_1(x-ct) - u^2, \, t>0, 0<x<h(t), \\ u_x(t,0)=0, u(t,h(t))=0, & t>0, \\ h'(t) = -\mu_1 u_x(t,h(t)), & t>0, \\ u(0,x) = u_0(x), & 0 \leq x \leq h_0. \end{cases}$$

By the comparison principle, we have

$$h(t) \leq \bar{h}(t) \text{ and } u(t,x) \leq \bar{u}(t,x) \text{ for } t \geq 0, 0<x<h(t), \tag{3.1}$$

where $(\bar{u}, \bar{h})$ is the unique solution of

$$\begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} = \bar{u} A_1(x-ct) - \bar{u}^2, \, t>0, 0<x<\bar{h}(t), \\ \bar{u}_x(t,0)=0, \bar{u}(t,\bar{h}(t))=0, & t>0, \\ \bar{h}'(t) = -\mu_1 \bar{u}_x(t,\bar{h}(t)), & t>0, \\ \bar{u}(0,x) = u_0(x), \bar{h}(0)=h_0, & 0 \leq x \leq h_0. \end{cases}$$

Since $c \geq c_{0,1}$, by Theorem 1.2 in [12], we can conclude that

$$\lim_{t\to\infty} \bar{h}(t) = \bar{h}_\infty < +\infty \text{ and } \lim_{t\to\infty} \|\bar{u}(t,\cdot)\|_{C([0,\bar{h}(t)])} = 0.$$

Combining this with (3.1), we can obtain that

$$\lim_{t\to\infty} h(t) = h_\infty < +\infty \text{ and } \lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0.$$

Next we establish part (ii). Since $u \leq M_1$ from (1.4), we will obtain

$$\begin{cases} v_t - d_2 v_{xx} \leq v A_2(x-ct) - \dfrac{v^2}{M_1+\alpha}, \, t>0, 0<x<g(t), \\ v_x(t,0)=0, v(t,g(t))=0, & t>0, \\ g'(t) = -\mu_2 v_x(t,g(t)), & t>0, \\ v(0,x) = v_0(x), & 0 \leq x \leq g_0. \end{cases}$$

Due to the comparison principle, we can deduce that

$$g(t) \le \bar{g}(t) \text{ and } v(t,x) \le \bar{v}(t,x) \text{ for } t \ge 0, 0 < x < g(t), \tag{3.2}$$

where $(\bar{v}, \bar{g})$ is the unique solution of

$$\begin{cases} \bar{v}_t - d_2\bar{v}_{xx} = \bar{v}A_2(x-ct) - \dfrac{\bar{v}^2}{M_1+\alpha}, & t>0, 0<x<\bar{g}(t), \\ \bar{v}_x(t,0) = 0, \bar{v}(t,\bar{g}(t)) = 0, & t>0, \\ \bar{g}'(t) = -\mu_2\bar{v}_x(t,\bar{g}(t)), & t>0, \\ \bar{v}(0,x) = v_0(x), \bar{g}(0) = g_0, & 0 \le x \le g_0. \end{cases}$$

Due to $c \ge c_{0,2}$ and Theorem 1.2 in [12], then we have

$$\lim_{t\to\infty} \bar{g}(t) = \bar{g}_\infty < +\infty \text{ and } \lim_{t\to\infty} \|\bar{v}(t,\cdot)\|_{C([0,\bar{g}(t)])} = 0.$$

Thus by (3.2), we can obtain

$$\lim_{t\to\infty} g(t) = g_\infty < +\infty \text{ and } \lim_{t\to\infty} \|v(t,\cdot)\|_{C([0,g(t)])} = 0.$$

□

3.2. The dynamical behavior for $c_{0,2} > c \ge c_{0,1}$

Let (u, v, h, g) be the unique solution of (1.4) with initial functions satisfied (1.5). In this section, we assume $c \in [c_{0,1}, c_{0,2})$. Denote

$$G^* := \limsup_{t\to\infty}[g(t) - ct] \tag{3.3}$$

and L_* be given by Lemma 2.2. We break the proof of Theorem 1.2 into three main steps:
Step 1. $G^* < L_*$ implies vanishing of v;
Step 2. $G^* = L_*$ implies borderline spreading of v;
Step 3. $G^* > L_*$ implies spreading of v.
Proof of Step 1. $G^* < L_*$ implies vanishing of v.

Lemma 3.1. *If $G^* < L_*$, then $G^* = -\infty$ and hence* $\lim_{t\to\infty}[g(t) - ct] = -\infty$.

Proof. Fix $L_1 \in (G^*, L_*)$. By Theorem 1.1 (i), we know $h_\infty < +\infty$. Hence due to $G^* < L_1$, we can choose $T_0 > 0$ such that $ct - l > h_\infty$ and $g(t) - ct < L_1$ for all $t \ge T_0$. Since $ct - l > h_\infty$ for $t \ge T_0$, we see that $u(t,x) = 0$ for $t \ge T_0$, $x > ct-l$. From Lemma 2.3, there exists $l > 0$ large enough such that $-\mu_2 W'_{l,L_1} < c$, where $W_{l,L_1}(x)$ is the unique positive solution of (2.6) with L replaced by L_1. For $M_0 > M_2$, $L_0 > L_1$, denote ϕ_c as the unique positive solution of

$$\begin{cases} d_2\phi'' + c\phi' + A_2(x)\phi - \dfrac{\phi^2}{\alpha} = 0, -l < x < L_0, \\ \phi(-l) = M_0, \phi(L_0) = 0. \end{cases} \tag{3.4}$$

By the continuous dependence of ϕ_c on M_0 and L_0, if (M_0, L_0) is close to (M_2, L_1), then ϕ_c is close to W_{l,L_1}. Hence, $-\mu_2\phi'_c(L_0) < c$ if (M_0, L_0) is close enough to

(M_2, L_1). Fix (M_0, L_0), then the above inequality holds. According to the comparison principle, we can obtain that $\phi_c(x) > W_{l,L_1}(x)$ for $x \in [-l, L_1]$. Then we can find $\varepsilon_0 > 0$ such that

$$\phi_c(x) > W_{l,L_1}(x) + 2\varepsilon_0 \text{ for } x \in [-l, L_1].$$

Applying the continuous dependence of ϕ_c on c, there exists $\sigma > 0$ such that

$$\phi_{c-\sigma}(x) > W_{l,L_1}(x) + \varepsilon_0 \text{ for } x \in [-l, L_1] \text{ and } -\mu_2 \phi'_{c-\sigma}(L_0) < c - \sigma,$$

where $\phi_{c-\sigma}$ is the unique positive solution of (3.4) with c replaced by $c - \sigma$.

Define $\Phi(t, x)$ as the unique positive solution of

$$\begin{cases} \Phi_t = d_2\Phi_{xx} + c\Phi_x + A_2(x)\Phi - \dfrac{\Phi^2}{\alpha}, & t > T_0, -l < x < L_1, \\ \Phi(t, -l) = M_2, \Phi(t, L_1) = 0, & t > T_0, \\ \Phi(L_0, x) = M_2, & -l < x < L_1. \end{cases}$$

By the properties of logistic type equations, $\Phi(t, \cdot) \to W_{l,L_1(\cdot)}$ in $C^2([-l, L_1])$ as $t \to \infty$.

Denote

$$V(t, x) := v(t, x + ct) \text{ and } G(t) = g(t) - ct.$$

Then from (1.4), we can deduce that

$$\begin{cases} V_t = d_2V_{xx} + cV_x + A_2(x)V - \dfrac{V^2}{\alpha}, & t > T_0, -l < x < G(t) < L_1, \\ V(t, -l) \le M_2, V(t, G(t)) = 0, & t > T_0, \\ V(L_0, x) \le M_2, & -l < x < G(T_0), \end{cases}$$

where we have used $V \le M_2$, which is guaranteed by Theorem 2.2. We may apply the standard comparison principle to conclude that

$$\Phi(t, x) \ge V(t, x) = v(t, x + t) \text{ for } t > T_0, -l < x < G(t).$$

Since $\Phi(t, \cdot) \to W_{l,L_1(\cdot)}$ in $C^2([-l, L_1])$ as $t \to \infty$, then for the above given ε_0, there exists $T_1 > T_0$ such that

$$\phi_{c-\sigma}(x) > W_{l,L_1}(x) + \varepsilon_0 > \Phi(T_1, x) \ge v(T_1, x + cT_1) \text{ for } -l \le x \le G(T_1).$$

Define

$$\begin{aligned} &\bar{v}(t, x) = \phi_{c-\sigma}(x - cT_1 - (c - \sigma)t), \\ &\eta_1(t) = cT_1 + (c - \sigma)t - l, \eta_2(t) = cT_1 + (c - \sigma)t + L_0. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} &\bar{v}(t, \eta_1(t)) = \phi_{c-\sigma}(-l) = M_0 > M_2 \ge v(t + T_1, \eta_1(t)), t > 0, \\ &\bar{v}(t, \eta_2(t)) = \phi_{c-\sigma}(L_0) = 0, t > 0, \\ &\bar{v}(0, x) = \phi_{c-\sigma}(x - cT_1) \ge v(T_1, x), x \in (\eta_1(0), \eta_2(0)), \end{aligned}$$

$$\eta_2'(t) = c - \sigma > -\mu_2 \phi_{c-\sigma}'(L_0) = -\mu_2 \bar{v}_x(t, \eta_2(t)), t > 0,$$
$$\eta_2(0) = cT_1 + L_0 > cT_1 + L_1 > g(T_1).$$

In addition, for $t > 0$, $\eta_1(t) < x < \eta_2(t)$, we have

$$\bar{v}_t = d_2 \bar{v}_{xx} + A_2(x - cT_1 - (c-\sigma)t)\bar{v} - \frac{\bar{v}^2}{\alpha} \geq d_2 \bar{v}_{xx} + A_2(x - c(t+T_1))\bar{v} - \frac{\bar{v}^2}{\alpha}.$$

Since $u(t+T_1, x) \equiv 0$ for $t \geq 0$ and $x \geq \eta_1(t)$, from (1.4), we can see that $(\bar{v}, \eta_2)$ is a super solution of the equation satisfied by $(v(T_1+t, x), g(T_1+t))$ over the range $t > 0$, $\eta_1(t) \leq x \leq \eta_2(t)$. Thus by the comparison principle for the free boundary problems, we can deduce that

$$g(t + T_1) \leq \eta_2(t) = cT_1 + (c-\sigma)t + L_0 \text{ for } t > 0.$$

Obviously, this implies $G^* = -\infty$ and $\lim_{t\to\infty}(g(t) - ct) = -\infty$. □

Lemma 3.2. *For $M > 0$, define*

$$\epsilon(M) = \limsup_{t\to\infty}\left[\sup_{0\leq x\leq ct-M} v(t,x)\right].$$

Then $\lim_{M\to\infty}\epsilon(M) = 0$. Here we have used the convention that $v(t,x) = 0$ for $x \geq g(t)$.

Proof. Since $u(t, h(t)) \leq M_1$, then we have

$$\begin{cases} v_t \leq d_2 v_{xx} + v\left[A_2(x-ct) - \dfrac{v}{M_1+\alpha}\right], & t > 0, 0 < x < g(t), \\ v_x(t,0) = 0, v(t, gh(t)) = 0, & t > 0, \\ g'(t) = -\mu_2 v_x(t, g(t)), & t > 0. \end{cases} \tag{3.5}$$

By the comparison principle, we can establish that

$$v(t,x) \leq \bar{v}(t,x), g(t) \leq \bar{g}(t) \text{ for } t > 0, 0 \leq x \leq g(t),$$

where $(\bar{v}, \bar{g})$ is the solution of

$$\begin{cases} \bar{v}_t = d_2 \bar{v}_{xx} + \bar{v}\left[A_2(x-ct) - \dfrac{\bar{v}}{M_1+\alpha}\right], & t > T_0, -l < x < \bar{g}(t) < L_1, \\ \bar{v}(t,0) = 0, \bar{v}(t, \bar{g}(t)) = 0, & t > 0, \\ \bar{g}'(t) = -\mu_2 \bar{v}_x(t, \bar{g}(t)), & t > 0, \\ \bar{g}(0) = g_0, \bar{v}(0,x) = v_0(x), & 0 < x < g_0. \end{cases}$$

By Lemma 3.10 of [12], we have

$$\lim_{M\to\infty}\left[\limsup_{t\to\infty}\left(\sup_{0\leq x\leq ct-M}\bar{v}(t,x)\right)\right] = 0.$$

It follows that $\lim_{M\to\infty}\epsilon(M) = 0$. □

From Lemma 3.1 and Lemma 3.2, we obtain the following conclusion.

Corollary 3.1. *If $G^* < L_*$, then* $\lim\limits_{t\to\infty}\left[\max\limits_{x\in[0,g(t)]} v(t,x)\right] = 0.$

To establish the proof of Step 1, it only to obtain that $g_\infty < +\infty$, which will follow from Corollary 3.1 and the following result.

Lemma 3.3. *If* $\lim\limits_{t\to\infty}\left[\max\limits_{x\in[0,g(t)]} v(t,x)\right] = 0$, *then* $g_\infty = \lim\limits_{t\to\infty} g(t) < \infty$.

Proof. This is a simple variant of the proof of Lemma 3.11 in [12]. Here omit the details. □

Proof of Step 2. $G^* = L_*$ implies borderline spreading of v.

Denote

$$G(t) := g(t) - ct.$$

First, we claim that $\lim\limits_{t\to\infty} G(t) = L_*$. If this is not true, then

$$G_* := \liminf_{t\to\infty} G(t) < L_*.$$

For any $L \in (G_*, L_*)$, the function $G(t) - L$ changes sign infinitely many times as $t \to \infty$. In the following, we are going to drive a contradiction by using the zero number argument.

Since the fact $G^* = L_*$ implies $g_\infty = +\infty$ and $h_\infty < +\infty$, then for all large $t \geq \tilde{T} > 0$, we have that

$$g(t) > h_\infty \text{ and } u(t,x) = 0 \text{ for } x \in [h_\infty, g(t)].$$

Hence from (1.4), we have

$$\begin{cases} v_t - d_2 v_{xx} = vA_2(x-ct) - \dfrac{v^2}{\alpha}, & t > \tilde{T}, h_\infty < x < g(t), \\ v(t,h_\infty) \in (0, M_2], v(t,g(t)) = 0, & t > \tilde{T}, \\ g'(t) = -\mu_2 v_x(t,g(t)), & t > \tilde{T}, \end{cases} \tag{3.6}$$

where M_2 is given in Theorem 2.2. Now using the same argument as in the proof of Lemma 3.6 in [12], we can show that $G(t) - L$ can only change sign finitely many times for $t > \tilde{T}$. This is a contradiction. Thus

$$\lim_{t\to\infty} G(t) = L_*.$$

Next, we prove

$$\lim_{t\to\infty}\left[\max_{x\in[0,g(t)]} |v(t,x) - V_*(x - g(t) + L_*)|\right] = 0. \tag{3.7}$$

Choosing a sequence t_n satisfied $t_n \to \infty$ as $n \to \infty$. Define

$$w(t,x) := v(t, x + g(t)), w_n(t,x) = w(t + t_n, x),$$
$$G_n(t) := G(t + t_n), g_n(t) := g(t + t_n).$$

By (3.6), we have

$$\begin{cases} \dfrac{\partial w_n}{\partial t} - d_2 \dfrac{\partial^2 w_n}{\partial x^2} - [G_n'(t) + c] \dfrac{\partial w_n}{\partial x} = w_n A_2(x + G_n(t)) - \dfrac{w_n^2}{\alpha}, \\ \qquad\qquad\qquad\qquad\qquad\qquad t > \tilde{T} - t_n, h_\infty - g_n(t) < x < 0, \\ w_n(t,0) = 0, -\mu_2 \dfrac{\partial w_n}{\partial x}(t,0) = c + G_n'(t), \quad t > \tilde{T} - t_n. \end{cases}$$

Using the proof of Lemma 3.5 in [12], we can find a subsequence of w_n, for convenience, still denoted by w_n, such that $w_n \to \tilde{w}$ in $C_{loc}^{\frac{1+\kappa}{2}, 1+\kappa}(\mathbf{R} \times (-\infty, 0])$, $G_n'(t) \to 0$ in $C_{loc}^{\frac{\kappa}{2}}(\mathbf{R})$, and $\tilde{w}$ satisfies

$$\begin{cases} \dfrac{\partial \tilde{w}}{\partial t} - d_2 \dfrac{\partial^2 \tilde{w}}{\partial x^2} - c \dfrac{\partial \tilde{w}}{\partial x} = \tilde{w} A_2(x + L_*) - \dfrac{\tilde{w}^2}{\alpha}, \ t \in \mathbf{R}, -\infty < x < 0, \\ \tilde{w}(t,0) = 0, -\mu_2 \dfrac{\partial \tilde{w}}{\partial x}(t,0) = c, \qquad\qquad\qquad t \in \mathbf{R}. \end{cases}$$

Following the proof of Theorem 3.13 in [12], we can conclude that

$$\tilde{w}(t,x) \equiv V_*(x + L_*).$$

Thus (3.7) holds. This completes the proof of Step 2.

Proof of Step 3. $G^* > L_*$ implies spreading of v.

When $G^* > L_*$, $g_\infty = \infty$ and hence (3.6) holds. This allows us to obtain the same properties for $g(t)$ as in Section 3.1 in [12]. Here we list these properties for $g(t)$ and omit their proof.

(i) (Lemma 3.2 of [12]) $G^* = \infty$ implies $\lim_{t\to\infty} [g(t) - ct] = \infty$;

(ii) (Lemma 3.3 of [12]) $G^* = \infty$ implies $\lim_{t\to\infty} \dfrac{g(t)}{t} = c_{0,2}$;

(iii) (Lemma 3.4 of [12]) $G^* \in (L_*, \infty)$ implies $\lim_{t\to\infty} [g(t) - ct] = G^*$;

(iv) (Lemma 3.5 of [12]) $G^* < \infty$ implies $G^* \le L_*$.

From the last two properties, we can see that $G^* > L_*$ implies $G^* = \infty$. Hence by the first two properties, we can obtain the following result.

Lemma 3.4. *If $G^* > L_*$, then $\lim_{t\to\infty} [g(t) - ct] = \infty$ and $\lim_{t\to\infty} \dfrac{g(t)}{t} = c_{0,2}$.*

According to the proof of Theorem 3.9 in [12], we can establish the following lemma.

Lemma 3.5. *Assume $G^* > L_*$. Then for any given small $\epsilon > 0$,*

$$\lim_{t\to\infty} \left[\sup_{(c+\epsilon)t \le x \le (1-\epsilon)g(t)} |v(t,x) - a_2\alpha| \right] = 0.$$

To complete this step, we need to prove the following refined results.

Theorem 3.1. *Assume $G^* > L_*$. Then $\lim_{t\to\infty} [g(t) - c_{0,2}t] = G_0$ for some $G_0 \in \mathbf{R}$, and for any $\tilde{c} \in (c, c_{0,2})$,*

$$\lim_{t\to\infty} \left[\max_{[0,\tilde{c}t]} |v(t,x) - \phi(x - ct)| \right] = 0,$$

$$\lim_{t\to\infty} \left[\max_{[\tilde{c}t, g(t)]} |v(t,x) - p_2(g(t) - x)| \right] = 0.$$

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 3.6. *For any $M > 0$,*

$$\lim_{t\to\infty}\left[\max_{-M\le x-ct\le M}|v(t,x)-\phi(x-ct)|\right]=0.$$

Proof. By Proposition 1 in [17], (1.10) has a unique solution ϕ. Fix $\epsilon \in (0, \tilde{c}-c)$ and define

$$w(t,y) := v(t, y+ct) \text{ for } t>0, y\in[-\epsilon t, \epsilon t].$$

Since $h_\infty < \infty$, there exists $T > 0$ such that $u(t, y+ct) = 0$ for $t \ge T$ and $y \in [-\epsilon t, \epsilon t]$. Therefore, w satisfies

$$\begin{cases} w_t = d_2 w_{yy} + c w_y + A_2(y) w - \dfrac{w^2}{\alpha}, & t>T, -\epsilon t < y < \epsilon t,\\ w(t, \pm\epsilon t) = w(t, ct\pm\epsilon t) = 0, & t > T. \end{cases}$$

Next the discussion can be followed by the proof of (11) in [17]. So we omit the details. □

Lemma 3.7. *Let $\tilde{c} \in (c, c_{0,2})$. Then*

$$\lim_{t\to\infty}\left[\max_{[0,\tilde{c}t]}|v(t,x)-\phi(x-ct)|\right]=0.$$

Proof. For $M > 0$ and $\tilde{c} \in (c, c_{0,1})$, define

$$\tilde{\epsilon}(M) := \limsup_{t\to\infty}\left[\max_{ct+M\le x\le\tilde{c}t}|v(t,x)-a_2\alpha|\right].$$

By the proof of (13) in [17], we have

$$\lim_{M\to\infty}\tilde{\epsilon}(M)=0. \tag{3.8}$$

By Lemma 3.2 and (3.8), for any given $\epsilon > 0$, we can find $B_1 > 0$ large enough such that

$$\epsilon(M) < \frac{\epsilon}{3}, \tilde{\epsilon}(M) < \frac{\epsilon}{3} \text{ for } M \ge B_1.$$

Therefore, for each $M \ge B_1$, there exists $T_1 = T_1(M) > 0$ such that

$$|v(t,x)| < \frac{2\epsilon}{3} \text{ for } t\ge T_1, 0\le x\le ct-M,$$

and

$$|v(t,x)-a_2\alpha| < \frac{2\epsilon}{3} \text{ for } t\ge T_1, ct+M\le x\le \tilde{c}t.$$

Since $\phi(-\infty) = 0$ and $\phi(+\infty) = a_2\alpha$, we can find $B_2 > 0$ such that for $M \ge B_2$,

$$|\phi(x-ct)| < \frac{\epsilon}{3} \text{ when } x\le ct-M, |\phi(x-ct)-a_2\alpha| < \frac{\epsilon}{3} \text{ when } x\ge ct+M.$$

Therefore, for $M \ge \max\{B_1, B_2\}$ and $t \ge T_1(M) > 0$, we have

$$|v(t,x)-\phi(x-ct)| < \epsilon \text{ for } x\in[0, ct-M]\cup[ct+M, \tilde{c}t].$$

Fix $M \geq \max\{B_1, B_2\}$ and applying Lemma 3.6, we can find $T_2 = T_2(M)$ such that

$$|v(t,x) - \phi(x-ct)| < \epsilon \text{ for } t \geq T_2(M),\ x \in [ct - M, ct + M].$$

Therefore,

$$|v(t,x) - \phi(x-ct)| < \epsilon \text{ for } x \in [0, \tilde{c}t], t \geq T := \max\{T_1(M), T_2(M)\}.$$

The proof is completed. □

Lemma 3.8. *For any $\tilde{c} \in (c, c_{0,2})$, there exist $\sigma_0 > 0$ and $B > 0$ such that*

$$v(t, \tilde{c}t) \geq a_2\alpha - Be^{-\sigma_0 t} \text{ for all large } t. \tag{3.9}$$

Proof. Choosing $\tilde{\epsilon}$ such that $[\tilde{c} - 2\tilde{\epsilon}, \tilde{c} + 2\tilde{\epsilon}] \subset (c, c_{0,2})$. By Lemma 3.4 and (ii) of Theorem 1.1 , there exists $T_0 > 0$ such that

$$ct > h_\infty,\ g(t) > (\tilde{c} + 2\tilde{\epsilon})t \text{ for } t \geq T_0,$$

and

$$u(t,x) = 0 \text{ for } t \geq T_0, x \geq ct.$$

Denote

$$\Omega_T := t \geq T, (\tilde{c} - 2\tilde{\epsilon})t \leq x \leq (\tilde{c} + 2\tilde{\epsilon})t,$$

then

$$v_t = d_2 v_{xx} + vA_2(x - ct) - \frac{v^2}{\alpha} \text{ for } (t,x) \in \Omega_T, T \geq T_0.$$

By Lemma 3.5, for any small $\upsilon > 0$, there exists $T_1 \geq T_0$ such that

$$v(t,x) \geq a_2\alpha - \upsilon \quad \text{for} \quad (t,x) \in \Omega_T \quad \text{with} \quad t \geq T_1. \tag{3.10}$$

We may proceed as in the proof of Lemma 3.1 in [17] (with obvious modifications) to show that (3.9) holds. □

Lemma 3.9. *There exists $C > 0$ such that*

$$|g(t) - c_{0,2}t| \leq C \text{ for all large } t.$$

Proof. Denote $F(v) = a_2 v - \dfrac{v^2}{\alpha}$. For σ_0 given by Lemma 3.8 and $\sigma \in (0, \sigma_0)$, there exists $\eta > 0$ such that

$$\begin{cases} \sigma < -F'(v),\ a_2\alpha(1-\eta) \leq v \leq a_2\alpha(1+\eta), \\ F(v) \geq 0, \quad\ 0 \leq v \leq a_2\alpha. \end{cases} \tag{3.11}$$

For $\eta > 0$, we define $z_\eta \in (0, \infty)$ and P'_η as follows

$$p_2(z_\eta) = a_2\alpha(1 - \frac{\eta}{2}), P'_\eta = \min_{0 \leq z \leq z_\eta} p'_2(z) > 0, \tag{3.12}$$

where p_2 is the solution of (1.6).

Clearly, for small σ given above, there exists $M_0 > 0$ such that

$$\psi(t) \leq a_2\alpha(1 + M_0 e^{-\sigma t}) \quad \text{for} \quad t > 0,$$

where $\psi(t) = a_2\alpha \cdot \dfrac{e^{a_2 t}}{e^{a_2 t} + \left(\frac{a_2\alpha}{\|v_0\|_\infty} - 1\right)}$ is the unique solution of

$$\psi'(t) = a_2\psi - \frac{\psi^2}{\alpha}, \psi(0) = \|v_0\|_\infty.$$

By the comparison principle, we have that $v(t,x) \le \psi(t)$ for $t > 0$, $x \in [0, g(t)]$. It follows that $v(t,x) \le a_2\alpha(1 + M_0 e^{-\sigma t})$ for $t > 0$ and $x \in [0, g(t)]$.

In view of Lemmas 3.2 and 3.4, for $\hat{c} \in (0, c)$, we can find a $T^* > 0$ such that $v(t, \hat{c}t) < \frac{a_2\alpha}{2}$ and $g(t) > \hat{c}t$ for $t \ge T^*$. For given $B_1 > M_0$, there exists a constant $X_0 > 0$ such that

$$(1 + B_1 e^{-\sigma T^*})p_2(x) \ge a_2\alpha(1 + M_0 e^{-\sigma T^*}) \text{ and } p_2(x) \ge \frac{a_2\alpha}{2} \text{ for } x \ge X_0.$$

Define $\bar{v}(t,x)$, $\eta_1(t)$ and $\eta_2(t)$ as follows

$$\begin{aligned}
&\eta_1(t) = \hat{c}t, \eta_2(t) = c_{0,2}(t - T^*) + mB_1(e^{-\sigma T^*} - e^{-\sigma t}) + \hat{c}T^* + X_0 + g(T^*), t \ge T^*,\\
&\bar{v}(t,x) = (1 + B_1 e^{-\sigma t})p_2(\eta_2(t) - x) \text{ for } t \ge T^*, \eta_1(t) < x < \eta_2(t),
\end{aligned}$$

where $m \ge \frac{c_{0,2}}{\sigma}$ is large enough. By the analysis in the proof of Lemma 3.3 in [17], $(\bar{v}(t,x), \eta_2(t))$ is an upper solution of the free boundary problem satisfied by (v, g) over the region with $x = \eta_1(t)$ as a given left boundary. Therefore,

$$g(t) \le \eta_2(t),\ v(t,x) \le \bar{v}(t,x) \text{ for } t \ge T^*,\ \hat{c}t \le x \le g(t).$$

It follows that

$$g(t) \le c_{0,2}t + C \text{ for } t \ge T^*,$$

where $C > 0$ is some constant. Next, we construct a lower solution to show that $g(t) - c_{0,2}t$ is bounded. For $\tilde{c} \in (c, c_{0,2})$ and $\sigma \in (0, \sigma_0)$, if follows from Lemma 3.8 that

$$v(t, \tilde{c}t) > a_2\alpha(1 - M_0 e^{-\sigma t}) \text{ for } t \ge T^*.$$

We choose $T^{**} > T^*$ so that

$$M_0 e^{-\sigma t} \le \frac{\eta}{2} \text{ for } t \ge T^{**}. \tag{3.13}$$

By virtue of Lemma 3.4 and (ii) of Theorem 1.1, we can enlarge T^{**} such that

$$g(t) > \tilde{c}t \text{ for } t \ge T^{**}, \tag{3.14}$$

and

$$u(t,x) = 0 \text{ for } t > T^{**} \text{ and } x > \tilde{c}t.$$

Thus (v, g) is satisfied

$$\begin{cases}
v_t - d_2 v_{xx} = v a_2 - \dfrac{v^2}{\alpha}, & t > T^{**}, \tilde{c}t < x < g(t),\\
v(t, \tilde{c}t) > a_2\alpha(1 - M_0 e^{-\sigma t}), v(t, g(t)) = 0, & t > T^{**},\\
g'(t) = -\mu_2 v_x(t, g(t)), & t > T^{**}.
\end{cases} \tag{3.15}$$

Now, we define

$$\xi_1(t) = \tilde{c}t, \xi_2(t) = c_{0,2}(t - T^{**}) - mM_0\left(e^{-\sigma T^{**}} - e^{-\sigma t}\right) + \tilde{c}T^{**} \text{ for } t \geq T^{**},$$

and

$$\underline{v}(t,x) = (1 - M_0 e^{-\sigma t})p_2(\xi_2(t) - x) \text{ for } t \geq T^{**},\ \xi_1(t) \leq x \leq \xi_2(t),$$

where $m > \frac{c_{0,2}}{\sigma}$. Since $\xi_1(T^{**}) = \xi_2(T^{**})$, as in Lemma 3.3 of [17], we can easily check that $(\underline{v}, \xi_2)$ is a lower solution to (3.15) with $x = \xi_1(t)$ as a given boundary. It follows from the comparison principle that

$$g(t) \geq \xi_2(t), v(t,x) \geq \underline{v}(t,x) \text{ for } t \geq T^{**},\ \xi_1(t) \leq x \leq \xi_2(t).$$

Therefore,

$$g(t) - c_{0,2}t \geq -C \text{ for } t \geq T^{**},$$

where $C > 0$ is some constant. Thus we obtain that $|g(t) - c_{0,2}t| \leq C$ for all large t. □

Lemma 3.10. *$\lim_{t\to\infty} |g(t) - c_{0,2}t| = G_0$ for some $G_0 \in \mathbf{R}$.*

Proof. Define

$$\tilde{g}(t) := g(t) - c_{0,2}t, l_1(t) := (c - c_{0,2})t, W(t,y) := v(t, y + c_{0,2}t).$$

Since h_∞ is finite, there exists $T_0 > 0$ such that $ct > h_\infty$ for $t \geq T_0$ and

$$u(t, y + c_{0,2}t) = 0 \text{ for } t \geq T_0, y \in [l_1(t), \tilde{g}(t)].$$

Thus W satisfies

$$\begin{cases} W_t = d_2 W_{yy} + c_{0,2}W_y + a_2 W - \dfrac{W^2}{\alpha}, & t > T_0, l_1(t) < y < \tilde{g}(t), \\ W(t, \tilde{g}(t)) = 0, & t > T_0, \\ \tilde{g}'(t) = -\mu_2 W_y(t, \tilde{h}(t)) - c_{0,2}, & t > T_0. \end{cases}$$

Since $\lim_{t\to\infty} l_1(t) = -\infty$ for any given $l \in \mathbf{R}$, by enlarging T_0 (depending on l), we can guarantee that $l \geq l_1(t)$ for $t \geq T_0$.

Further, We follow the proof of Lemma 3.4 in [17] to establish the result by a zero number argument. □

Lemma 3.11. *$\lim_{t\to\infty} g'(t) = c_{0,2}$ and for every $M > 0$,*

$$\lim_{t\to\infty}\left[\max_{g(t)-M\leq x\leq g(t)} |v(t,x) - p_2(g(t) - x)|\right] = 0.$$

Proof. Since $u(t,x) = 0$ for all large t and $x \geq ct$, the conclusions can be directly established from the proof of Lemma 3.5 in [17]. □

Lemma 3.12. *Let $\tilde{c} \in (c, c_{0,2})$. Then*

$$\lim_{t\to\infty}\left[\max_{[\tilde{c}t, g(t)]} |v(t,x) - p_2(g(t) - x)|\right] = 0.$$

Proof. Let σ, $\xi_2(t)$, $\eta_2(t)$ be given in the proof of Lemma 3.9. Defining $p_2(x) = 0$ for $x \leq 0$, we can see from the proof of Lemma 3.9 that

$$(1 - M_0 e^{-\sigma t}) p_2(\xi_2(t) - x) \leq v(t, x) \leq (1 + B_1 e^{-\sigma t}) p_2(\eta_2(t) - x) \tag{3.16}$$

for $\tilde{c}t \leq x \leq g(t)$ and $t \geq T$. By Lemma 3.9 and the definitions of $\xi_2(t)$ and $\eta_2(t)$, there exists $B_2 > 0$ such that

$$|\xi_2(t) - g(t)| \leq B_2, |\eta_2(t) - g(t)| \leq B_2 \text{ for } t \geq T.$$

Therefore, for any given $\epsilon > 0$, due to (3.16) and $p_2(\infty) = a_2\alpha$, we can find $M > 0$ and $T_1 > 0$ large enough such that

$$|v(t, x) - a_2\alpha| < \frac{\epsilon}{2}, |p_2(g(t) - x) - a_2\alpha| < \frac{\epsilon}{2}$$

for $t \geq T_1$ and $x \in [\tilde{c}t, g(t) - M]$. It follows that

$$|v(t, x) - p_2(g(t) - x)| < \epsilon \text{ for } t \geq T_1, x \in [\tilde{c}t, g(t) - M].$$

By Lemma 3.11, we can find $T_2 > 0$ such that

$$|v(t, x) - p_2(g(t) - x)| < \epsilon \text{ for } t \geq T_2, x \in [g(t) - M, g(t)].$$

Therefore,

$$|v(t, x) - p_2(g(t) - x)| < \epsilon \text{ for } t \geq \max\{T_1, T_2\}, x \in [\tilde{c}t, g(t)].$$

This completes the proof. □

Obviously, Theorem 3.1 is a consequence of Lemmas 3.7 and 3.12. Then we have completed the proof of Step 3 and the proof of Theorem 1.2.

4. Trichotomy via the variation of a parameter

In this section, we will discuss that each of the three cases described in Theorem 1.2 happens by varying a certain parameter in (1.4). For this purpose, we can choose μ_2 as the varying parameter, and keep all the others fixed. In order to stress the dependence of $c_{0,2}$ on μ_2, we will write $c_{0,2} = c_{0,2}(\mu_2)$. By Theorem 6.2 in [9] and its proof, we can deduce that $c_{0,2}(\mu_2)$ is continuous and strictly increasing in μ_2, with

$$c_{0,2}(0) = 0, c_{0,2}(\infty) = 2\sqrt{a_2 d_2}.$$

Therefore, if $c \geq 2\sqrt{a_2 d_2}$, then $c_{0,2}(\mu_2) < c$ for all $\mu_2 > 0$. By Theorem 1.1, v will vanish in this case.

Thus in this section, we assume that

$$c_{0,1} \leq c < 2\sqrt{a_2 d_2}. \tag{4.1}$$

By the properties of $c_{0,2}(\mu_2)$ described above, there exists a unique $\mu_2^* > 0$ such that

$$c_{0,2}(\mu_2^*) = c, \;\; c_{0,2}(\mu_2) < c \;\text{ for }\; 0 < \mu_2 < \mu_2^*, \;\; c_{0,2}(\mu_2) > c \;\text{ for }\; \mu_2 > \mu_2^*.$$

According to (ii) of Theorem 1.1, v will vanish for $\mu_2 \in (0, \mu_2^*]$.

Next, we examine the case $\mu_2 > \mu_2^*$. For such μ_2, we have that $c_{0,1} \le c < c_{0,2}$ holds. Hence Theorem 1.2 gives a trichotomy for the long-time behavior of v.

In order to emphasize the dependence of the solution (u, v, h, g) of (1.4) on the parameter μ_2, we denote them as $(u_{\mu_2}, v_{\mu_2}, h_{\mu_2}, g_{\mu_2})$. For the rest of this section, we always assume that all the parameters in (1.4) except μ_2 are fixed, the initial functions u_0, v_0 satisfying (1.5) are also fixed, and (4.1) holds.

In the following, we will describe the behavior of $(u_{\mu_2}, v_{\mu_2}, h_{\mu_2}, g_{\mu_2})$ as $\mu_2 \to \infty$.

Lemma 4.1. *As* $\mu_2 \to \infty$,

$$h_{\mu_2}(t) \to H(t) \ \textit{in} \ C_{loc}^{\frac{\kappa}{2}}([0, +\infty)), \tag{4.2}$$

$$g_{\mu_2}(t) \to +\infty \ \textit{for every} \ t > 0, \tag{4.3}$$

$$u_{\mu_2}(t, x) \to U(t, x) \ \textit{in} \ C_{loc}^{\frac{1+\kappa}{2}, 1+\kappa}(D), \tag{4.4}$$

$$v_{\mu_2}(t, x) \to V(t, x) \ \textit{in} \ C_{loc}^{\frac{1+\kappa}{2}, 1+\kappa}([0, +\infty) \times [0, +\infty)), \tag{4.5}$$

where $D := \{(t, x) \in \mathbf{R^2} : t \ge 0, 0 \le x \le H(t)\}$, $\kappa \in (0, 1)$ *and* (U, V, H) *is the unique solution to*

$$\begin{cases} U_t - d_1 U_{xx} = U A_1(x - ct) - U^2 - \beta U V, & t > 0, 0 < x < H(t), \\ V_t - d_2 V_{xx} = V A_2(x - ct) - \dfrac{V^2}{U + \alpha}, & t > 0, 0 < x < +\infty, \\ H'(t) = -\mu_1 U_x(t, H(t)), H(0) = H_0, & t > 0, \\ V_x(t, 0) = U_x(t, 0) = 0, U(t, x) = 0, & t > 0, x \ge H(t), \\ U(0, x) = u_0(x), & 0 \le x \le h_0, \\ V(0, x) = v_0(x), & 0 \le x \le g_0. \end{cases} \tag{4.6}$$

Proof. We will use two steps to establish the proof of the lemma.

Step 1. We deduce that $\lim\limits_{\mu_2 \to \infty} g_{\mu_2}(t) = +\infty$ for $t > 0$.

By a simple comparison consideration, we can obtain that

$$0 \le u(t, x) \le M_0, 0 \le v(t, x) \le M_0 \text{ for } t \ge 0, x \ge 0,$$

where $M_0 := \max\{\frac{a_1}{\beta}, a_2\alpha, \|u_0\|_\infty, \|v_0\|_\infty\}$. Hence, by (1.4), (v_{μ_2}, g_{μ_2}) was satisfied

$$\begin{cases} v_t - d_2 v_{xx} \ge M v, & t > 0, \ 0 < x < g(t), \\ v_x(t, 0) = 0, v(t, g(t)) = 0, & t > 0, \\ g'(t) = -\mu_2 v_x(t, g(t)), & t > 0, \\ g(0) = g_0, v(0, x) = v_0(x), & 0 \le x \le g_0, \end{cases} \tag{4.7}$$

with $M := a_{0,2} - \frac{M_0}{\alpha}$. Denote (w_{μ_2}, l_{μ_2}) as the unique positive solution of

$$\begin{cases} w_t - d_2 w_{xx} = Mw, & t > 0,\ 0 < x < l(t), \\ w_x(t,0) = 0, w(t,l(t)) = 0, & t > 0, \\ l'(t) = -\mu_2 w_x(t,l(t)), & t > 0, \\ l(0) = g_0, w(0,x) = v_0(x), & 0 \le x \le g_0. \end{cases} \tag{4.8}$$

Apply the comparison principle, we can deduce that

$$l_{\mu_2}(t) \le g_{\mu_2}(t), w_{\mu_2}(t,x) \le v_{\mu_2}(t,x) \ \text{ for } \ t > 0,\ 0 \le x \le l_{\mu_2}(t). \tag{4.9}$$

By the comparison principle, we also easily see that $l_{\mu_2}(t)$ is non-decreasing in μ_2. Therefore, we can find $l_\infty(t) \in (0,\infty]$ such that $\lim\limits_{\mu_2\to\infty} l_{\mu_2}(t) = l_\infty(t)$ for each $t > 0$.

Next, we show that $l_\infty(t) = +\infty$ for every $t > 0$. Due to the Hopf boundary lemma, it follows from (4.8) that $(w_{\mu_2})_x(t, l_{\mu_2}(t)) < 0$ for $t > 0$. This implies that $l'_{\mu_2}(t) > 0$ for $t > 0$. Hence for any given $\delta > 0$, we have $l_{\mu_2}(t) > l_{\mu_2}(\delta)$ for $t > \delta$. By the same argument used in the proof of Lemma 5.3 in [6], we can prove $l_\infty(\delta) = +\infty$. Thus $\lim\limits_{\mu_2\to\infty} l_{\mu_2}(t) = +\infty$ for $t > 0$. Then it follows from (4.9) that

$$\lim_{\mu_2\to\infty} g_{\mu_2}(t) = +\infty \text{ for } t > 0.$$

Step 2. We will prove that (4.3), (4.4) and (4.5) hold.

Let μ_2^n be an increasing positive sequence satisfying $\lim\limits_{n\to\infty} \mu_2^n = +\infty$. Denote

$$u_n(t,x) := u_{\mu_2^n}(t,x), v_n(t,x) := v_{\mu_2^n}(t,x), h_n(t) := h_{\mu_2^n}(t) \text{ and } g_n(t) := g_{\mu_2^n}(t).$$

From Step 1, $v_n(t,x) \le M_0$ for $t \ge 0$, $x \ge 0$ and all $n \ge 1$. By the comparison principle, we can deduce that

$$\begin{aligned} & h_1(\infty) \ge h_n(t) \ge h_{n+1}(t) \ge h_0,\ g_n(t) \le g_{n+1}(t) \text{ for } t > 0, \\ & u_n(t,x) \ge u_{n+1}(t,x) > 0 \ \text{ for } \ t > 0,\ x \in [0, h_{n+1}(t)], \\ & 0 < v_n(t,x) \le v_{n+1}(t,x) \le M_0 \text{ for } t > 0,\ x \ge 0. \end{aligned}$$

Therefore, there exists $H(t) > 0$ for $t > 0$, $V(t,x) > 0$ for $(t,x) \in (0,\infty) \times [0,+\infty)$ and $U(t,x) \ge 0$ for $x \in [0, H(t)]$, $t \in (0,+\infty)$ such that

$$\lim_{n\to\infty} h_n(t) = H(t) \text{ for } t > 0, \tag{4.10}$$

$$\lim_{n\to\infty} u_n(t,x) = U(t,x) \text{ for } t > 0, x \in [0, H(t)], \tag{4.11}$$

$$\lim_{n\to\infty} v_n(t,x) = V(t,x) \text{ for } t > 0, x \ge 0. \tag{4.12}$$

Since v_n is satisfied

$$\begin{cases} (v_n)_t - d_2 (v_n)_{xx} = v_n \left(A_2(x-ct) - \dfrac{v_n}{u_n + \alpha} \right), & t > 0,\ 0 < x < g_n(t), \\ (v_n)_x(t,0) = 0, v_n(t, g_n(t)) = 0, & t > 0, \end{cases}$$

in view of the conclusion of Step 1, we can apply the interior parabolic L^p estimates and Sobolev embedding theorem to the above equation for $(t,x) \in (k, k+2] \times [0, R+1)$ with every positive integer k, and the estimate up to $t = 0$ for $(t,x) \in [0,2] \times [0, R+1)$. We can conclude that, for some $\iota \in (0,1)$ and $C^*(R) > 0$ independent of n,

$$\|v_n(t,x)\|_{C^{\frac{1+\iota}{2},1+\iota}([0,\infty)\times[0,R])} \leq C^*(R) \text{ for all large } n \text{ and every } R > 0.$$

Let $\kappa \in (0, \iota)$. By a compactness consideration, we can see that the convergence in (4.12) can be strengthened to

$$v_n \to V \text{ in } C_{loc}^{\frac{1+\kappa}{2},1+\kappa}([0,\infty)\times[0,\infty)). \tag{4.13}$$

Let $y = \dfrac{x}{h_n(t)}$ and $w_n(t,y) = u_n(t, h_n(t)y)$. Then w_n satisfies

$$\begin{cases} \partial_t w_n - \dfrac{d_1}{h_n^2}\partial_{yy} w_n - \dfrac{h_n'}{h_n}\partial_y w_n = w_n(A_1(h_n(t)y - ct) - w_n - \beta v_n(t, h_n(t)y)), \\ \qquad\qquad\qquad\qquad\qquad\qquad t > 0, 0 < y < 1, \\ \partial_y w_n(t,0) = 0, \ w_n(t,1) = 0, \quad t > 0. \end{cases}$$

By the estimate (1.11) in [15], we can find $M_4 > 0$ independent of n so that

$$0 < h_n'(t) \leq M_4 \quad \text{for} \ \ t > 0, \ n \geq 1.$$

Applying the L^p estimates and Sobolev embedding theorem, we can deduce that

$$\|w_n\|_{C^{\frac{1+\iota}{2},1+\iota}([0,\infty)\times[0,1])} \leq C^*, \tag{4.14}$$

where $C^* > 0$ is independent of n, and $\iota \in (0,1)$.

By (4.14) and $h_n'(t) = -\dfrac{\mu_1}{h_n(t)}(w_n)_y(t,1)$, we can obtain that

$$\|h_n\|_{C^{1+\frac{\iota}{2}}}([0,\infty)) \leq C^{**} \tag{4.15}$$

for some $C^{**} > 0$ independent of n.

Let $\kappa \in (0,\iota)$. Due to (4.14) and (4.15), we see that the convergence in (4.10) can be strengthened to

$$h_n \to H \text{ in } C_{loc}^{1+\frac{\kappa}{2}}([0,\infty) \text{ as } n \to \infty. \tag{4.16}$$

By passing to a subsequence, we can deduce

$$w_n \to W \text{ in } C_{loc}^{\frac{1+\kappa}{2},1+\kappa}([0,\infty)\times[0,1]) \text{ as } n \to \infty.$$

Denote $Z(t,x) := W(t, \frac{x}{H(t)})$. We can easily see that

$$u_n \to Z \text{ in } C_{loc}^{\frac{1+\kappa}{2},1+\kappa}(D) \text{ as } n \to \infty, \text{ with } D := (t,x) \in \mathbf{R^2} : t \geq 0, x \in [0, H(t)], \tag{4.17}$$

and

$$H'(t) = -\frac{\mu_1}{H(t)} W_y(t,1) = -\mu_1 Z_x(t, H(t)).$$

In view of (4.11), we have $U \equiv Z$. Furthermore, (U,V,H) is satisfied (4.6). From the proof of Section 2 of [8], (U,V,H) is the unique solution of (4.6). Together with the uniqueness of (U,V,H), (4.16), (4.17) and (4.13) can imply (4.2), (4.4) and (4.5). This is completed the proof. □

Lemma 4.2. *For all sufficiently large $\mu_2 > 0$, the prey v_{μ_2} will spread in the environment.*

Proof. Let $\lambda_1^{[a,b]}$ be the principle eigenvalue of

$$\begin{cases} -c\vartheta' - d_2\vartheta'' = A_2(x)\vartheta + \lambda\vartheta, \, a < x < b, \\ \vartheta(a) = \vartheta(b) = 0. \end{cases}$$

Assume $\mu_2 > \mu_2^*$. Thus $c_{0,1} < c < c_{0,2}(\mu_2)$. For this case, we can use Lemma 2.2 to deduce that (2.4) has a unique positive solution V_0 with $l = 0$ and $L = L(0)$ satisfied

$$-\mu_2 V_0'(L(0)) = c.$$

Since V_0 and $L(0)$ depend on μ_2, we denote them by V_{0,μ_2} and $L_{\mu_2}(0)$, respectively. It follows that $\lambda_1^{[0,L_{\mu_2}(0)]} < 0$.

Now we establish the limits of $L_{\mu_2}(0)$ and V_{0,μ_2} as $\mu_2 \to \infty$. We can claim that $\lim\limits_{\mu_2\to\infty} L_{\mu_2}(0) = L^*$, $\lim\limits_{\mu_2\to\infty} \|V_{0,\mu_2}\|_\infty = 0$, where $L^* > 0$ is uniquely determined by

$$\lambda_1^{[0,L^*]} = 0.$$

Since $A_2(x) = a_2$ for $x \geq 0$, $\tilde{V}_{\mu_2}(x) := V_{0,\mu_2}(x - L_{\mu_2}(0))$ satisfies

$$\begin{cases} -c\tilde{V}_{\mu_2}' - d_2\tilde{V}_{\mu_2}'' = a_2\tilde{V}_{\mu_2} - \dfrac{\tilde{V}_{\mu_2}^2}{\alpha}, \ -L_{\mu_2}(0) < x < 0, \\ \tilde{V}_{\mu_2}(-L_{\mu_2}(0)) = \tilde{V}_{\mu_2}(0) = 0, \ \tilde{V}_{\mu_2}'(0) = -\dfrac{c}{\mu_2}. \end{cases}$$

By the comparison principle, we can deduce that $L_{\mu_2}(0)$ is strictly decreasing in μ_2 for $\mu_2 > \mu_2^*$, and $\tilde{V}_{\mu_2}(x) > \tilde{V}_{\tilde{\mu}_2}(x)$ for $x \in [-L_{\mu_2}(0), 0)$ when $\mu_2^* < \mu_2 < \tilde{\mu}_2$. It follows that

$$L^* := \lim_{\mu_2\to\infty} L_{\mu_2}(0) \in [0,\infty), \tilde{V}(x) := \lim_{\mu_2\to\infty} \tilde{V}_{\mu_2}(x) \text{ for } x \in (-L^*, 0]$$

are both existed, and

$$\lambda_1^{[0,L^*]} \leq 0, \tilde{V}(x) \geq 0.$$

Moreover, by elliptic regularity theory, we easily see that $\tilde{V}_{\mu_2} \to \tilde{V}$ in $C^2_{loc}((-L^*, 0])$ as $\mu_2 \to \infty$, and $\tilde{V}$ is satisfied

$$-c\tilde{V}' - d_2\tilde{V}'' = a_2\tilde{V} - \frac{\tilde{V}^2}{\alpha} \text{ for } x \in (-L^*, 0), \tilde{V}(0) = \tilde{V}'(0) = 0.$$

By the uniqueness of the initial value problem for the above ODE problem, we can obtain $\tilde{V} \equiv 0$. Then we can deduce $\lambda_1^{[0,L^*]} = 0$. Otherwise $\lambda_1^{[0,L^*]} < 0$. By the comparison principle, we have that $\tilde{V}_{\mu_2} \geq V > 0$ for all large μ_2, where V is the unique positive solution of the problem

$$-cV' - d_2V'' = a_2V - \frac{V^2}{\alpha} \text{ for } x \in (-L^*, 0), V(-L^*) = V(0) = 0.$$

It is a contradiction. Then the claim is right.

For some $\sigma > 0$, $\lambda_1^{[-\sigma,L^*+\sigma]} < \lambda_1^{[0,L^*]} = 0$. Then the elliptic problem

$$\begin{cases} -c\Phi' - d_2\Phi'' = A_2(x)\Phi - \dfrac{\Phi^2}{\alpha}, & -\sigma < x < L^* + \sigma, \\ \Phi(-\sigma) = \Phi(L^* + \sigma) = 0, \end{cases} \tag{4.18}$$

has a unique positive solution Φ.

For any $\varphi_0 \in C^2([-\sigma, L^* + \sigma])$ satisfied $\varphi_0(-\sigma) = \varphi_0(L^* + \sigma) = 0$ and $\varphi_0(x) > 0$ in $(-\sigma, L^* + \sigma)$, we consider the initial-boundary value problem

$$\begin{cases} \varphi_t - c\varphi_x - d_2\varphi_{xx} = A_2(x)\varphi - \dfrac{\varphi^2}{\alpha}, & t > 0, -\sigma < x < L^* + \sigma, \\ \varphi(t, -\sigma) = \varphi(t, L^* + \sigma) = 0, & t > 0, \\ \varphi(0, x) = \varphi_0(x), & -\sigma < x < L^* + \sigma. \end{cases}$$

It is well known that

$$\lim_{t\to\infty} \varphi(t, x) = \Phi(x) \text{ uniformly in } [-\sigma, L^* + \sigma]. \tag{4.19}$$

Let (U, V, H) be the unique solution of (4.6). Set

$$z(t, x) = U(t, x + ct) \text{ and } w(t, x) = V(t, x + ct).$$

Then w satisfies

$$w_t - d_2w_{xx} - cw_x = w\left(A_2(x) - \frac{w}{z + \alpha}\right) \text{ for } t > 0, -ct < x < +\infty.$$

From the proof of Lemma 4.1, $H(\infty) < +\infty$. We fix some $\tilde{\mu}_0 > \mu_2^*$ and recall from the proof of Lemma 4.1 that $H(\infty) \le h_{\tilde{\mu}_0}(\infty) < \infty$ (due to $c > c_{0,1}$). Then we can find a constant $T_0 > 0$ such that

$$ct > h_{\tilde{\mu}_0}(\infty) + \sigma \ge H(\infty) + \sigma \text{ for } t \ge T_0. \tag{4.20}$$

This implies that $z(t, x) \equiv 0$ for $t \ge T_0$, $x \ge -\sigma$. Hence, we have that

$$w_t - d_2w_{xx} - cw_x = w\left(A_2(x) - \frac{w}{\alpha}\right) \text{ for } t \ge T_0, -\sigma < x < +\infty.$$

By the strong maximum principle, we know that $w(t, x) > 0$ for $t \ge T_0$ and $x \in [-\sigma, +\infty)$. If we have chosen φ_0 small enough such that $\varphi_0(x) \le w(T_0, x)$ in $[-\sigma, L^* + \sigma]$, then the comparison principle would yield

$$\varphi(t, x) \le w(t + T_0, x) \text{ for } t \ge T_0, -\sigma \le x \le \sigma + L^*.$$

Choosing φ_0 such that the above inequality holds and denote $\varepsilon := \dfrac{1}{2}\min\limits_{[0,L^*]} \Phi(x)$. By (4.19), we can find $T_1 > 0$ such that

$$\varphi(T_1, x) > \Phi(x) - \frac{\varepsilon}{2} \text{ in } [-\sigma, \sigma + L^*].$$

Let $T := T_0 + T_1$. Then it follows that

$$V(T,x) = w(T, x - cT) > \Phi(x - cT) - \frac{\varepsilon}{2} \text{ in } [cT - \sigma, L^* + \sigma + cT].$$

By (4.2) and (4.4), we can find $\mu_0 \geq \tilde{\mu}_0$ such that

$$g_{\mu_2}(T) > cT + L^* + \sigma, \tag{4.21}$$

and

$$v_{\mu_2}(T,x) > V(T,x) - \frac{\varepsilon}{2} > \Phi(x - cT) - \varepsilon, x \in [cT - \sigma, L^* + \sigma + cT]$$

for $\mu_2 \geq \mu_0$. By the definition of ε, we can see that $\Phi(x - cT) - \varepsilon > 0$ for $x \in [cT, L^* + cT]$. Therefore we can find $\sigma_0 \in (0, \sigma)$ small enough such that

$$\Phi(x - cT) - \varepsilon > \sigma_0 \text{ for } x \in [cT - \sigma_0, L^* + \sigma_0 + cT].$$

By the conclusion in the above claim, we can find $\hat{\mu}_0 > \mu_0$ so that

$$L_{\mu_2}(0) < L^* + \sigma_0, V_{0,\mu_2}(x) \leq \sigma_0 \text{ for } x \in [0, L_{\mu_2}(0)] \text{ and } \mu_2 \geq \hat{\mu}_0.$$

Thus we obtain

$$h_{\mu_2}(T) > cT + L_{\mu_2}(0), v_{\mu_2}(T,x) > V_{0,\mu_2}(x - cT)$$

for $\mu_2 \geq \hat{\mu}_0$ and $x \in [cT, cT + L_{\mu_2}(0)] \subset [cT - \sigma_0, L^* + cT + \sigma_0]$.

Using the comparison principle, we can find that

$$g_{\mu_2}(T) > ct \text{ for } t \geq T.$$

Indeed, if this is not true, then there exists $T_1 > T$ such that $g_{\mu_2}(T_1) = cT_1$ and $g_{\mu_2}(t) > ct$ for $T < t < T_1$. Due to (4.20) and $h_{\mu_2}(\infty) \leq h_{\widetilde{\mu}_0}(\infty) < \infty$, we can see that $u_{\mu_2}(t,x) = 0$ for $T \leq t \leq T_1$ and $x \in [ct, g_{\mu_2}(t)]$. Hence $(v_{\mu_2}(t,x), g_{\mu_2}(t))$ is satisfied

$$\begin{cases} v_t - d_2 v_{xx} = a_2 v - \dfrac{v^2}{\alpha}, & T < t < T_1, ct < x < g(t), \\ v(t, ct) > 0, v(t, g(t)) = 0, & T < t < T_1, \\ g'(t) = -\mu_2 v_x(t, g(t)), & T < t < T_1, \\ g(T) > cT + L_{\mu_2}(0), v(T,x) > V_{0,\mu_2}(x - cT), & cT \leq x \leq cT + L_{\mu_2}(0). \end{cases} \tag{4.22}$$

Fix μ_2 such that (4.22) holds, and define

$$\underline{V}(t,x) := V_{0,\mu_2}(x - ct), \underline{g}(t) := ct + L_{\mu_2}(0).$$

It is easily checked that

$$\begin{cases} \underline{V}_t - d_2 \underline{V}_{xx} = a_2 \underline{V} - \frac{\underline{V}^2}{\alpha}, & t > 0, ct < x < \underline{g}(t), \\ \underline{V}(t, ct) = 0, \underline{V}(t, \underline{g}(t)) = 0, & t > 0, \\ \underline{g}'(t) = -\mu_2 \underline{V}_x(t, \underline{g}(t)), & t > 0. \end{cases}$$

Hence we can apply the comparison principle to conclude that

$$g(t) \geq \underline{g}(t), v(t,x) \geq \underline{V}(t,x) \text{ for } T < t \leq T_1, x \in [ct, \underline{g}(t)].$$

It follows that $g_{\mu_2}(T_1) = g(T_1) \geq cT_1 + L_{\mu_2}(0)$, which is contradicted with our choice of T_1. This implies that $g_{\mu_2}(t) = g(t) > ct$ for $t \geq T$. Then we can repeat the above argument to conclude that

$$g(t) \geq \underline{g}(t), v(t,x) \geq \underline{V}(t,x) \text{ for } t > T, x \in [ct, \underline{g}(t)].$$

Next we want to modify $(\underline{V}, \underline{g})$ by a small perturbation of c. For this purpose, with fixed μ_2 and T such that (4.22) holds, we rewrite $(V_{0,\mu_2}, L_{\mu_2}(0))$ as (V^c, L^c). Due to the continuous dependence of (V^c, L^c) on c, the last inequalities in (4.22), which can be rewritten as

$$g(T) > cT + L^c, v(T,x) > V^c(x - cT) \text{ for } x \in [cT, cT + L^c],$$

still hold if c is replaced by some $\tilde{c} > c$ but very close to c. Fix $\tilde{c}$ and define

$$\underline{\tilde{V}}(t,x) := V^{\tilde{c}}(x - \tilde{c}t), \underline{\tilde{g}}(t) := \tilde{c}t + L^{\tilde{c}}.$$

Then we can still obtain by the comparison principle that

$$g(t) \geq \underline{\tilde{g}}(t),\ v(t,x) \geq \underline{\tilde{V}}(t,x) \text{ for } t > T, x \in [\tilde{c}t, \underline{\tilde{g}}(t)].$$

Hence

$$g(t) - ct \geq (\tilde{c} - c)t - L^{\tilde{c}} \to \infty \text{ as } t \to \infty.$$

By Step 3 in the proof of Theorem 1.2, we can conclude that spreading happens for v_{μ_2}, when $\mu_2 \geq \hat{\mu}_0$. This completes the proof. □

Theorem 1.3 clearly follows from (ii) of Theorem 1.1 and the following result.

Theorem 4.1. *There exists $\tilde{\mu} \in (\mu_2^*, +\infty)$ such that*
(i) vanishing of v_{μ_2} happens if $\mu_2 \in (0, \tilde{\mu})$;
(ii) borderline spreading of v_{μ_2} happens if $\mu_2 = \tilde{\mu}$;
(iii) spreading of v_{μ_2} happens if $\mu_2 > \tilde{\mu}$.

Proof. Define

$$\mu_* := \sup\{S_1\}, S_1 := \{\mu_2 > 0 : \text{vanishing happens to } v_{\mu_2}\},$$

and

$$\mu^* := \inf\{S_2\}, S_2 := \{\mu_2 > 0 : \text{spreading happens to } v_{\mu_2}\}.$$

By our analysis, we know that S_1 and S_2 are both nonempty sets, and $\mu_* \geq \mu_2^*$. In view of the comparison principle, it is easily seen that

$$S_1 \supset (0, \mu_*), S_2 \supset (\mu^*, \infty) \ \text{ and } \ \mu_* \leq \mu^*.$$

We divide the proof below into three steps.

Step 1. We show that $\mu_* \notin S_1$ and $\mu_* > \mu_2^*$.

For any $\mu_0 \in S_1$, due to Lemma 3.3, for some large $T_0 > 0$, we have

$$3g_{\mu_0}(T_0) < cT_0 - (l_0 + 1) \text{ and } v_{\mu_0}(T_0, x) < \frac{\sqrt{2}m}{4} \text{ in } [0, g_{\mu_0}(T_0)],$$

where $m = \dfrac{d_2\pi}{9(\mu_0+1)}$ and l_0 appears in the definition of $A_2(x)$. By the continuity of the solution with respect to μ_2, we can find a small $\varepsilon > 0$ such that (v_μ, g_μ) of the solution $(u_\mu, v_\mu, h_\mu, g_\mu)$ of (1.4) satisfies

$$3g_\mu(T_0) < cT_0 - l_0 \text{ and } v_\mu(T_0, x) < \frac{\sqrt{2}m}{2} \quad \text{for} \quad x \in [0, g_\mu(T_0)], \ \mu \in [\mu_0, \mu_0+\varepsilon].$$

Define $\kappa = \dfrac{d_2\pi^2}{36 g_\mu^2(T_0)}$ and

$$\bar{v}_\mu(t,x) = me^{-\kappa t}\cos\left(\frac{\pi x}{2\xi(t)}\right), \ \xi(t) = g_\mu(T_0)(3 - e^{-\kappa t}).$$

Then the same argument as in the proof of Lemma 3.3 can imply that

$$g_\mu(t+T_0) \le \xi(t) \le 3g_\mu(T_0) \text{ for } t > 0.$$

By Lemma 3.2, we can see that vanishing happens to v_μ. It follows that $(0, \mu_0+\varepsilon] \subset S_1$. This clearly implies $\mu_* \notin S_1$.
Step 2. We prove that $\mu_* \notin S_2$.

For contradiction, suppose $\mu^* \in S_2$. Since $\mu^* \ge \mu_* > \mu_2^*$, we have $c_{0,1} < c < c_{0,2}(\mu^*)$. Then we can apply Theorem 1.2 to conclude that

$$\lim_{t\to\infty}[g_{\mu^*}(t) - c_{0,2}(\mu^*)t] = G_0 \in \mathbf{R}, \lim_{t\to\infty}\left[\max_{x\in[0,\tilde{c}t]}|v_{\mu^*}(t,x) - \phi(x-ct)|\right] = 0 \quad (4.23)$$

for any $\tilde{c} \in (c, c_{0,2}(\mu^*))$. Therefore, for any $M > 0$

$$\lim_{t\to\infty}\left[\max_{x\in[\tilde{c}t-M,\tilde{c}t+M]}|v_{\mu^*}(t,x) - a_2\alpha|\right] = 0. \quad (4.24)$$

Let $V_{\mu^*}(x)$ be the unique positive solution of (2.4) with $l = 0$, $c = \tilde{c}$ and $\mu_2 = \mu^*$. Denote the corresponding $L(0)$ by L_{μ^*}, then

$$\begin{cases} -d_2V''_{\mu^*} - \tilde{c}V'_{\mu^*} = a_2V_{\mu^*} - \dfrac{V^2_{\mu^*}}{\alpha}, & 0 < x < L_{\mu^*}, \\ V_{\mu^*}(0) = V_{\mu^*}(L_{\mu^*}) = 0, \quad -\mu^*V'_{\mu^*}(L_{\mu^*}) = \tilde{c}. \end{cases}$$

By the comparison principle, we can easily see that $V_{\mu^*}(x) < a_2\alpha$ in $[0, L_{\mu^*}]$. Therefore we can find $\epsilon > 0$ such that

$$V_{\mu^*}(x) + \epsilon < a_2\alpha \quad \text{in} \quad [0, L_{\mu^*}].$$

By (4.23) and (4.24), we can find $T > 0$ such that

$$\tilde{c}T > h_{\mu_2^*}(\infty) \ge h_{\mu^*}(\infty), \quad g_{\mu^*}(T) > \tilde{c}T + L_{\mu^*}, \quad (4.25)$$

and

$$v_{\mu^*}(T,x) + \epsilon > a_2\alpha \quad \text{for} \quad x \in [\tilde{c}T, \tilde{c}T + L_{\mu^*}].$$

Therefore,

$$v_{\mu^*}(T,x) > V_{\mu^*}(x - \tilde{c}T) \quad \text{for} \quad x \in [\tilde{c}T, \tilde{c}T + L_{\mu^*}]. \quad (4.26)$$

Due to the continuous dependence of (v_{μ_2}, g_{μ_2}) and (V_{μ_2}, L_{μ_2}) on μ_2, we can see that (4.25) and (4.26) still hold if we replace μ^* by some $\mu < \mu^*$ but very close to μ^*. Fix such μ, we can find that (v_μ, g_μ) satisfies

$$\begin{cases} v_t - d_2 v_{xx} = a_2 v - \dfrac{v^2}{\alpha}, & t > T, \tilde{c}t < x < g(t), \\ v(t, \tilde{c}t) > 0, v(t, g(t)) = 0, & t > T, \\ g'(t) = -\mu v_x(t, g(t)), & t > T, \\ g(T) > \tilde{c}T + L_\mu, v(T, x) > V_\mu(x - \tilde{c}T), & \tilde{c}T \le x \le \tilde{c}T + L_\mu. \end{cases} \tag{4.27}$$

Define

$$\underline{V}(t, x) := V_\mu(x - \tilde{c}t), \underline{g}(t) := \tilde{c}t + L_\mu.$$

It is easily checked that

$$\begin{cases} \underline{V}_t - d_2 \underline{V}_{xx} = a_2 \underline{V} - \dfrac{\underline{V}^2}{\alpha}, & t > 0, \tilde{c}t < x < \underline{g}(t), \\ \underline{V}(t, \tilde{c}t) = 0, \underline{V}(t, \underline{g}(t)) = 0, & t > 0, \\ \underline{g}'(t) = -\mu \underline{V}_x(t, \underline{g}(t)), & t > 0. \end{cases}$$

Hence we can apply the comparison principle to conclude that

$$g_\mu(t) \ge \underline{g}(t), v_\mu(t, x) \ge \underline{V}(t, x) \text{ for } t > T, x \in [\tilde{c}t, \underline{g}(t)].$$

Therefore,

$$g_\mu(t) - ct \ge (\tilde{c} - c)t - L_\mu \to \infty \text{ as } t \to \infty.$$

By Step 3 in the proof of Theorem 1.2, we can obtain that spreading happens for v_μ. Thus $\mu \in S_2$. Since $\mu < \mu^*$, this is a contradiction. Step 2 is completed.
Step 3. We prove that $\mu_* = \mu^*$. Otherwise, we have $\mu_* < \mu^*$. Denote

$$(u_*, v_*, h_*, g_*) = (u_{\mu_*}, v_{\mu_*}, h_{\mu_*}, g_{\mu_*}) \text{ and } (u^*, v^*, h^*, g^*) = (u_{\mu^*}, v_{\mu^*}, h_{\mu^*}, g_{\mu^*}).$$

By Step 1, Step 2 and Theorem 1.2, we can know that borderline spreading happens to v^* and v_*. Moreover, in view of the comparison principle and the strong maximum principle, we can easily deduce that

$$g_*(t) < g^*(t) \quad \text{for} \quad t > 0, \tag{4.28}$$
$$v_*(t, x) < v^*(t, x) \quad \text{for} \quad t > 0, \ 0 \le x \le g_*(t). \tag{4.29}$$

Due to (ii) of Theorem 1.1, there exists $T > 0$ such that

$$ct - 2l_0 > h_{\mu_*}(\infty) \ge h_{\mu^*}(\infty) > h_0 \text{ for } t \ge T.$$

Then $u^*(t, x) = u_*(t, x) \equiv 0$ for $t \ge T$, $x \ge ct - 2l_0$. Thus (v^*, g^*) satisfies

$$\begin{cases} v_t - d_2 v_{xx} = A_2(x - ct) v - \dfrac{v^2}{\alpha}, & t > T, \ ct - 2l_0 < x < g(t), \\ v(t, g(t)) = 0, g'(t) = -\mu^* v_x(t, g(t)), & t > T, \end{cases}$$

and (v_*, g_*) solves

$$\begin{cases} v_t - d_2 v_{xx} = A_2(x-ct)v - \dfrac{v^2}{\alpha}, & t > T, ct - 2l_0 < x < g(t), \\ v(t, g(t)) = 0, g'(t) = -\mu_* v_x(t, g(t)), & t > T. \end{cases}$$

Using the similar argument as in Step 3 of the proof of Theorem 4.3 in [12], we can show that

$$\liminf_{t\to\infty}[g^*(t) - ct] > L_*.$$

It is a contradiction with the fact that borderline spreading happens to v^*. This contradiction proves $\mu_* = \mu^*$. The proof of the theorem is completed. □

5. Conclusions

In this paper, we considered a Leslie-Gower predator-prey model in shifting environments. The model is studied the invasive predator that initially occupy the region $[0, g_0]$ and has a tendency to expand its territory. We establish several results in this setting.

(I) Theorem 1.1 is provided that the prey and predator will vanish when the speed of the shifting habitat edge c is more than the asymptotic speed of the prey and predator without the shifting environment.

(II) By Theorem 1.2, we can establish a spreading-borderline spreading-vanishing trichotomy for the predator with $c < c_{0,2}$.

(III) Finally, Theorem 1.3 is revealed that for $c < 2\sqrt{a_2 d_2}$, there exists a $\tilde{\mu} \in (0, +\infty)$ such that vanishing of v happens if $\mu_2 \in (0, \tilde{\mu})$; borderline spreading of v happens if $\mu_2 = \tilde{\mu}$; spreading of v happens if $\mu_2 > \tilde{\mu}$.

By our discussions, we can provide that the invasive predator can knock aquatic ecosystems right out of balance and the environment will affect the invasive specie spread. Studying the spread of an invasive predator, we can give some guidelines, especially ones that encourage the trade of less invasive and aggressive species, or protect the prey as food for the predator.

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