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BREATHER-WAVE, MULTI-WAVE AND INTERACTION SOLUTIONS FOR THE (3+1)-DIMENSIONAL GENERALIZED BREAKING SOLITON EQUATION

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Abstract In this paper, a (3+1)-dimensional generalized breaking soliton equation in nonlinear media is investigated. The interaction solution between lump wave and N-soliton ($N = 2, 3, 4$) are derived. The interaction solution between lump wave and periodic waves is also studied. Breather-wave and multi-wave solutions are obtained. The dynamical behavior is demonstrated by some 3D graphics and density plots. Via means of mathematical induction, we also obtain the exact solution containing three arbitrary functions.

Keywords Interaction solution, dynamical behavior, nonlinear media, generalized breaking soliton equation.

MSC(2010) 35C08, 35G20, 35Q68.

1. Introduction

There are a lot of folding phenomena in nature. Bubbles and waves in manifolds are folded waves. These phenomena can be described by the breaking soliton equation, which has been applied to many fields. The research on the solution of the breaking soliton equation has became a hot spot. Many researchers used various methods to study the breaking soliton equation, such as Wronskian technique [10,31], improved Riccati equations method [44], Jacobi elliptic function expansion method [16], three-wave method [5], modified direct Clarkson-Kruskal method [41], (G'/G)-expansion method [28], truncated Painlevé analysis [4], simplified Hirota's method [19, 39], generalized unified method [29], Bell polynomial approach [40], long wave limit method [6], Riemann-Bäcklund method [45], similarity transformation method [7], etc.

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Recently, Prof. Ma studied the N-soliton solutions and analyzed the Hirota N-soliton conditions in (1+1)- and (2+1)-dimensional integrable equations [20–27]. He proposed an algorithm method to verify the Hirota conditions. This method given the condition that the integrable equations had a N-soliton solution from the Hirota bilinear formulation for the first time, which greatly promoted the development of solving the N-soliton solution of the integrable equations.

In this paper, we investigate the following (3+1)-dimensional generalized breaking soliton equation (GBSE) [8]

$$\alpha u_{xxx}+\beta u_{xxy}+\gamma u u_x+\delta u_x\int u_y\,dx+\lambda u u_y+\int u_{zt}+u_{yt}+u_{xt}\,dx=0, \quad (1.1)$$

where $u=u(x,y,z,t)$. Lump-type, multiwave, rogue wave, multicomplexiton, breather wave and positive multicomplexiton solutions of Eq. (1.1) have been studied in Refs. [8, 11]. On the basis of Ref. [8], we will further study the interaction between lump wave and N-soliton ($N=2,3,4$). Breather-wave and multi-wave solutions are also discussed.

The bilinear form of Eq. (1.1) is known as

$$\begin{aligned}&[D_xD_t+\alpha D_x^4+\beta D_x^3D_y+D_yD_t+D_zD_t]\xi\cdot\xi\\ =&\,\xi\left(\alpha\xi_{xxxx}+\beta\xi_{xxxy}+\xi_{zt}+\xi_{yt}+\xi_{xt}\right)+3\alpha\xi_{xx}^2-4\alpha\xi_x\xi_{xxx}\\ &+3\beta\xi_{xy}\xi_{xx}-3\beta\xi_x\xi_{xxy}-\beta\xi_y\xi_{xxx}-\xi_t\xi_z-\xi_t\xi_y-\xi_t\xi_x=0, \end{aligned} \quad (1.2)$$

with

$$\begin{aligned}&\lambda=\delta=3\beta,\gamma=6\alpha,\\ &u=2\left(\ln\xi\right)_{xx},\end{aligned} \quad (1.3)$$

where $\xi=\xi(x,y,z,t)$.

This paper is organized as follows: Section 2 obtains the interaction solution between lump wave and N-soliton ($N=2,3,4$). Section 3 presents the interaction solution between lump wave and periodic waves. Section 4 gives the more general breather-wave solution. Section 5 studies the multi-wave solution. Section 6 gives a conclusion.

2. Lump-N-soliton ($N=2,3,4$)

In general, the interaction solution between lump wave and 2-solitons can be assumed to be

$$\begin{aligned}&g=\mathcal{G}_5+\mathcal{G}_4t+\mathcal{G}_1x+\mathcal{G}_2y+\mathcal{G}_3z,\\ &h=\mathcal{G}_{10}+\mathcal{G}_9t+\mathcal{G}_6x+\mathcal{G}_7y+\mathcal{G}_8z,\\ &\xi=k_{12}\exp\left(\theta_1+\theta_2\right)+k_1e^{\theta_1}+k_2e^{\theta_2}+g^2+\mathcal{G}_{11}+h^2,\end{aligned} \quad (2.1)$$

where $\theta_i=a_ix+b_iy+c_iz+d_it+e_i(i=1,2)$, $\mathcal{G}_i(i=1,\cdots,11)$, k_1, k_2 and k_{12} are unknown constants. When $a_2=-a_1$, $b_2=-b_1$, $c_2=-c_1$, $d_2=-d_1$ and $e_2=-e_1$, the corresponding results have been discussed in Ref. [8]. Eq. (2.1) contains more arbitrary parameters than Eq. (19) in Ref. [8]. Substituting Eq. (2.1) into Eq. (1.2) gives

$$c_1=-a_1-b_1,c_2=-a_2-b_2,b_2=-\frac{\alpha a_2}{\beta},b_1=-\frac{\alpha a_1}{\beta},$$

$$\mathcal{G}_2 = -\frac{\alpha\mathcal{G}_1}{\beta}, \mathcal{G}_7 = -\frac{\alpha\mathcal{G}_6}{\beta}, \mathcal{G}_3 = \frac{\mathcal{G}_1(\alpha-\beta)}{\beta}, \mathcal{G}_8 = \frac{\mathcal{G}_6(\alpha-\beta)}{\beta}. \tag{2.2}$$

The corresponding interaction solution can be written as

$$u = 2\left(\frac{\xi_{xx}}{\xi} - \frac{\xi_x^2}{\xi^2}\right), \tag{2.3}$$

with

$$\begin{aligned}
\xi = {} & k_1 \exp[a_1 x - \frac{\alpha a_1 y}{\beta} + z(\frac{\alpha a_1}{\beta} - a_1) + d_1 t + e_1] + k_2 \exp[a_2 x - \frac{\alpha a_2 y}{\beta} \\
& + z(\frac{\alpha a_2}{\beta} - a_2) + d_2 t + e_2] + k_{12} \exp[(a_1 + a_2)x - (\frac{\alpha a_1}{\beta} + \frac{\alpha a_2}{\beta})y + e_1 \\
& + z(\frac{\alpha a_1}{\beta} - a_1 + \frac{\alpha a_2}{\beta} - a_2) + (d_1 + d_2)t + e_2] + \mathcal{G}_{11} + [\mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x \\
& - \frac{\alpha\mathcal{G}_1 y}{\beta} + \frac{\mathcal{G}_1 z(\alpha-\beta)}{\beta}]^2 + [\mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x - \frac{\alpha\mathcal{G}_6 y}{\beta} + \frac{\mathcal{G}_6 z(\alpha-\beta)}{\beta}]^2.
\end{aligned}$$

Fig. 1 shows the dynamical behavior of interaction solution (2.3) when $k_{12} = 0$. We can observe the interaction between two solitons. At the center of the interaction there is a lump wave, which propagates with two solitons, and the amplitude remains unchanged. Fig. 2 shows the dynamical behavior of interaction solution (2.3) when $k_{12} \neq 0$.

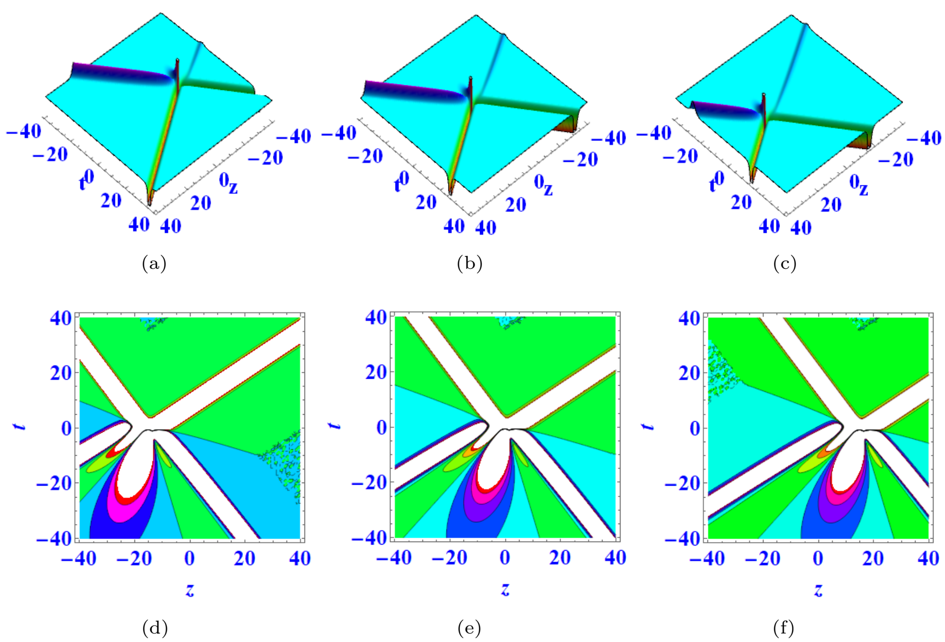


Figure 1. $\mathcal{G}_1 = \mathcal{G}_5 = \mathcal{G}_{11} = \alpha = a_1 = d_1 = d_2 = e_1 = k_1 = k_2 = 1$, $\mathcal{G}_4 = \mathcal{G}_9 = -1$, $\mathcal{G}_6 = -3$, $\beta = 3$, $\mathcal{G}_{10} = a_2 = k_{12} = -2$, $e_2 = 2$, $y = 0$, when $x = -10$ in (a) (d), $x = 0$ in (b) (e), $x = 10$ in (c) (f).

The interaction solution between lump wave and 3-soliton can be supposed as follow

$$g = \mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x + \mathcal{G}_2 y + \mathcal{G}_3 z,$$

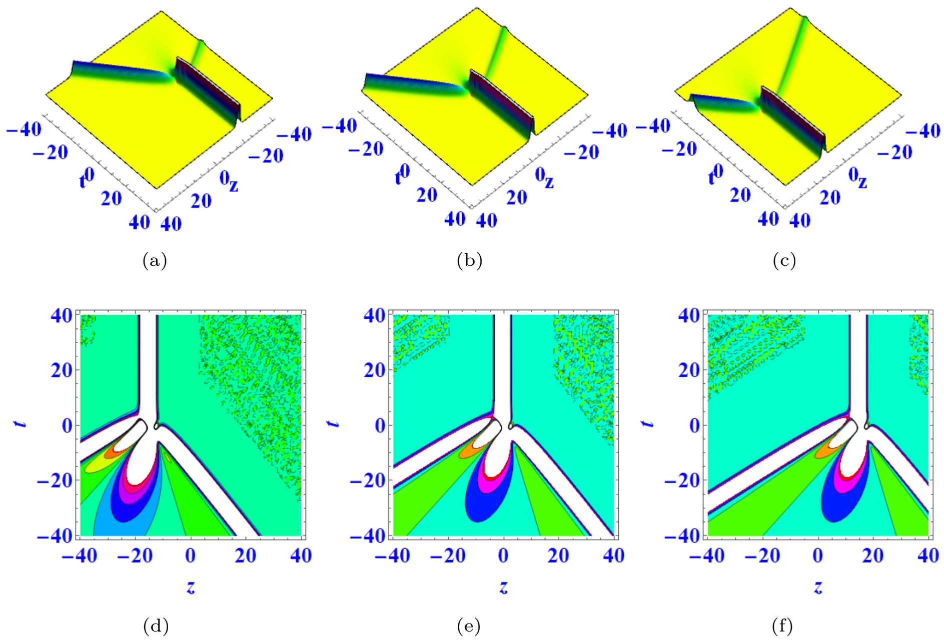


Figure 2. $\mathcal{G}_1 = \mathcal{G}_5 = \mathcal{G}_{11} = \alpha = a_1 = d_1 = d_2 = e_1 = k_1 = k_2 = 1$, $\mathcal{G}_4 = \mathcal{G}_9 = -1$, $\mathcal{G}_6 = -3$, $\beta = 3$, $\mathcal{G}_{10} = a_2 = -2$, $e_2 = 2$, $y = k_{12} = 0$, when $x = -10$ in (a) (d), $x = 0$ in (b) (e), $x = 10$ in (c) (f).

$$
\begin{aligned}
h &= \mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x + \mathcal{G}_7 y + \mathcal{G}_8 z,\\
\xi &= k_3 e^{\theta_3} + k_1 e^{\theta_1} + k_2 e^{\theta_2} + g^2 + \mathcal{G}_{11} + h^2,
\end{aligned}
\tag{2.4}
$$

where $\theta_3 = a_3 x + b_3 y + c_3 z + d_3 t + e_3$ and k_3 are unknown constants. To our knowledge, the interaction solution (2.4) is entirely new and not discussed before. Substituting Eq. (2.4) into Eq. (1.2), we have

$$
\begin{aligned}
&c_1 = -a_1 - b_1, c_2 = -a_2 - b_2, c_3 = -a_3 - b_3, b_2 = -\frac{\alpha a_2}{\beta}, \mathcal{G}_8 = \frac{\mathcal{G}_6(\alpha - \beta)}{\beta},\\
&b_3 = -\frac{\alpha a_3}{\beta}, \mathcal{G}_2 = -\frac{\alpha \mathcal{G}_1}{\beta}, \mathcal{G}_7 = -\frac{\alpha \mathcal{G}_6}{\beta}, \mathcal{G}_3 = \frac{\mathcal{G}_1(\alpha - \beta)}{\beta}, b_1 = -\frac{\alpha a_1}{\beta}.
\end{aligned}
\tag{2.5}
$$

The corresponding interaction solution can be written as

$$
u = 2\left(\frac{\xi_{xx}}{\xi} - \frac{\xi_x^2}{\xi^2}\right),
\tag{2.6}
$$

with

$$
\begin{aligned}
\xi = {} & k_1 \exp[a_1 x - \frac{\alpha a_1 y}{\beta} + z(\frac{\alpha a_1}{\beta} - a_1) + d_1 t + e_1] + k_2 \exp[a_2 x - \frac{\alpha a_2 y}{\beta}\\
& + z(\frac{\alpha a_2}{\beta} - a_2) + d_2 t + e_2] + k_3 \exp[a_3 x - \frac{\alpha a_3 y}{\beta} + z(\frac{\alpha a_3}{\beta} - a_3) + d_3 t + e_3]\\
& + \mathcal{G}_{11} + [\mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x - \frac{\alpha \mathcal{G}_1 y}{\beta} + \frac{\mathcal{G}_1 z(\alpha - \beta)}{\beta}]^2\\
& + [\mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x - \frac{\alpha \mathcal{G}_6 y}{\beta} + \frac{\mathcal{G}_6 z(\alpha - \beta)}{\beta}]^2.
\end{aligned}
$$

Fig. 3 shows the dynamical behavior of interaction solution (2.6). The interaction between three solitons can be seen. There is a lump wave, which propagates with three solitons.

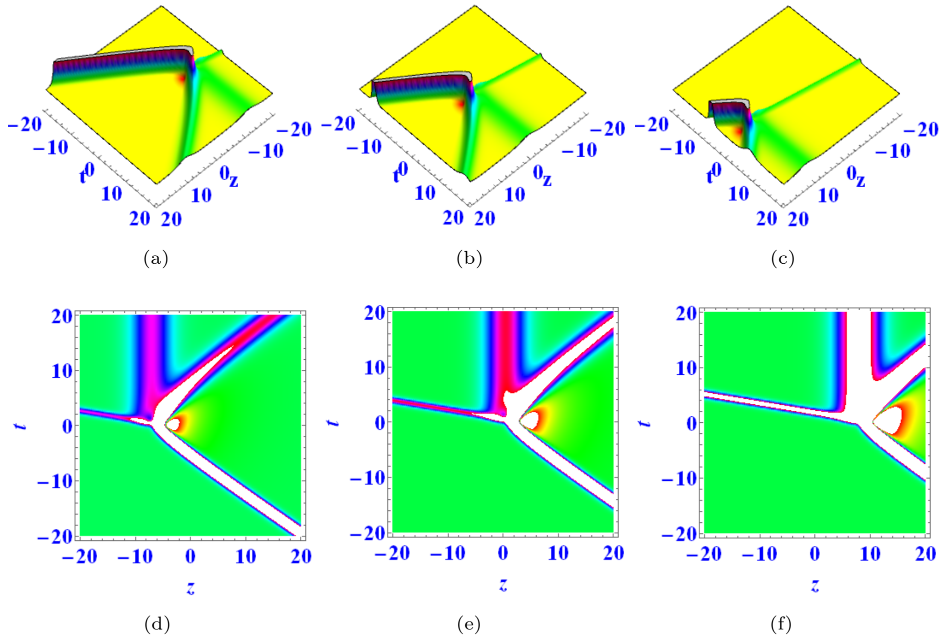


Figure 3. $\mathcal{G}_1 = \mathcal{G}_5 = \mathcal{G}_{11} = \alpha = a_1 = d_1 = d_2 = e_1 = 1$, $k_1 = e_3 = 4$, $\mathcal{G}_{10} = -2$, $\mathcal{G}_4 = \mathcal{G}_9 = -1$, $\mathcal{G}_6 = d_3 = -3$, $\beta = a_3 = k_3 = 3$, $a_2 = e_2 = k_2 = 2$, $y = 0$, when $x = -5$ in (a) (d), $x = 0$ in (b) (e), $x = 5$ in (c) (f).

The interaction solution between lump wave and 4-soliton can be written as follow

$$\begin{aligned} g &= \mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x + \mathcal{G}_2 y + \mathcal{G}_3 z, \\ h &= \mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x + \mathcal{G}_7 y + \mathcal{G}_8 z, \\ \xi &= k_4 e^{\theta_4} + k_3 e^{\theta_3} + k_1 e^{\theta_1} + k_2 e^{\theta_2} + g^2 + \mathcal{G}_{11} + h^2, \end{aligned} \tag{2.7}$$

where $\theta_4 = a_4 x + b_4 y + c_4 z + d_4 t + e_4$ and k_4 are unknown constants. The interaction solution (2.7) is not discussed in other literature. Substituting Eq. (2.7) into Eq. (1.2), we have

$$\begin{aligned} & c_1 = -a_1 - b_1, c_2 = -a_2 - b_2, c_3 = -a_3 - b_3, b_2 = -\frac{\alpha a_2}{\beta}, \\ & b_3 = -\frac{\alpha a_3}{\beta}, \mathcal{G}_2 = -\frac{\alpha \mathcal{G}_1}{\beta}, \mathcal{G}_7 = -\frac{\alpha \mathcal{G}_6}{\beta}, \mathcal{G}_3 = \frac{\mathcal{G}_1(\alpha - \beta)}{\beta}, b_1 = -\frac{\alpha a_1}{\beta}, \\ & c_4 = -a_4 - b_4, b_4 = -\frac{\alpha a_4}{\beta}, \mathcal{G}_8 = \frac{\mathcal{G}_6(\alpha - \beta)}{\beta}. \end{aligned} \tag{2.8}$$

The corresponding interaction solution can be written as

$$u = 2\left(\frac{\xi_{xx}}{\xi} - \frac{\xi_x^2}{\xi^2}\right), \tag{2.9}$$

with

$$\xi = k_1 \exp[a_1 x - \frac{\alpha a_1 y}{\beta} + z(\frac{\alpha a_1}{\beta} - a_1) + d_1 t + e_1] + k_2 \exp[a_2 x - \frac{\alpha a_2 y}{\beta} + z(\frac{\alpha a_2}{\beta} - a_2) + d_2 t + e_2] + k_3 \exp[a_3 x - \frac{\alpha a_3 y}{\beta} + z(\frac{\alpha a_3}{\beta} - a_3) + d_3 t + e_3] + \mathcal{G}_{11} + k_4 \exp[a_4 x - \frac{\alpha a_4 y}{\beta} + z\left(\frac{\alpha a_4}{\beta} - a_4\right) + d_4 t + e_4] + [\mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x - \frac{\alpha \mathcal{G}_1 y}{\beta} + \frac{\mathcal{G}_1 z(\alpha - \beta)}{\beta}]^2 + [\mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x - \frac{\alpha \mathcal{G}_6 y}{\beta} + \frac{\mathcal{G}_6 z(\alpha - \beta)}{\beta}]^2.$$

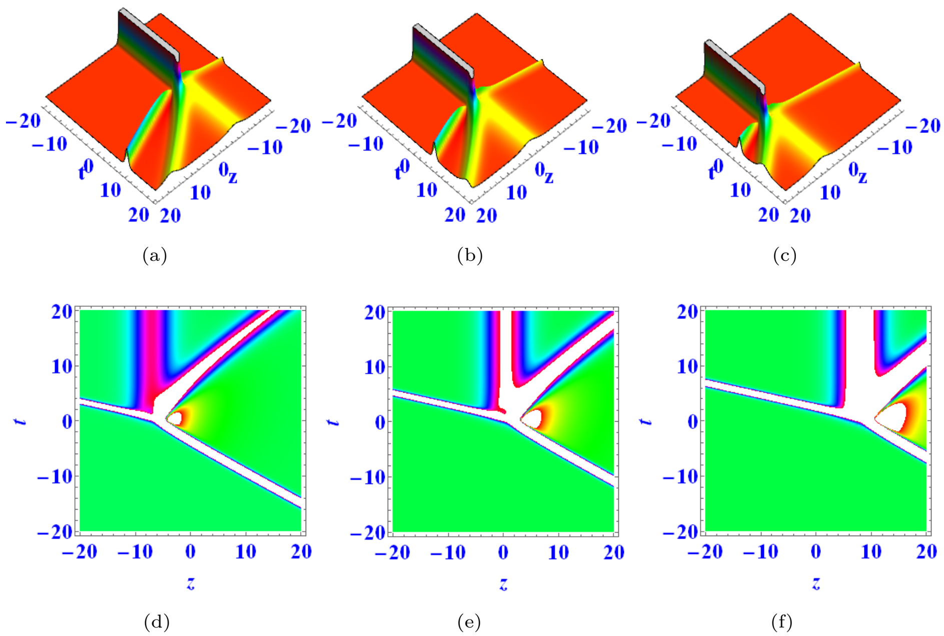


Figure 4. $\mathcal{G}_1 = \mathcal{G}_5 = \mathcal{G}_{11} = \alpha = a_1 = d_1 = d_2 = e_1 = 1$, $k_1 = e_3 = 4$, $k_4 = 6$, $y = 0$, $a_2 = e_2 = k_2 = 2$, $\mathcal{G}_4 = \mathcal{G}_9 = e_4 = -1$, $\mathcal{G}_6 = d_3 = d_4 = -3$, $\beta = a_3 = k_3 = 3$, $\mathcal{G}_{10} = a_4 = -2$, when $x = -5$ in (a) (d), $x = 0$ in (b) (e), $x = 5$ in (c) (f).

Fig. 4 shows the dynamical behavior of interaction solution (2.9). The interaction between four solitons can be seen. There is a lump wave, which propagates with four solitons.

3. Lump-periodic waves

The interaction solution between lump wave and periodic waves can be read as

$$\begin{aligned} g &= \mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x + \mathcal{G}_2 y + \mathcal{G}_3 z, \\ h &= \mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x + \mathcal{G}_7 y + \mathcal{G}_8 z, \\ \xi &= k_1 \cos\theta_1 + k_2 \sin\theta_2 + g^2 + \mathcal{G}_{11} + h^2. \end{aligned} \tag{3.1}$$

The interaction solution (3.1) is not discussed in other literature. Substituting Eq. (3.1) into Eq. (1.2), we have

$$\begin{aligned} &c_1 = -a_1 - b_1,\ c_2 = -a_2 - b_2,\ b_2 = -\frac{\alpha a_2}{\beta},\ b_1 = -\frac{\alpha a_1}{\beta}, \\ &\mathcal{G}_2 = -\frac{\alpha \mathcal{G}_1}{\beta},\ \mathcal{G}_7 = -\frac{\alpha \mathcal{G}_6}{\beta},\ \mathcal{G}_3 = \frac{\mathcal{G}_1(\alpha-\beta)}{\beta},\ \mathcal{G}_8 = \frac{\mathcal{G}_6(\alpha-\beta)}{\beta}. \end{aligned} \tag{3.2}$$

The corresponding interaction solution can be written as

$$u = 2\left(\frac{\xi_{xx}}{\xi} - \frac{\xi_x^2}{\xi^2}\right), \tag{3.3}$$

with

$$\begin{aligned} \xi = {} & k_1 \cos[a_1 x - \frac{\alpha a_1 y}{\beta} + z(\frac{\alpha a_1}{\beta} - a_1) + d_1 t + e_1] + k_2 \sin[a_2 x - \frac{\alpha a_2 y}{\beta} \\ & + z(\frac{\alpha a_2}{\beta} - a_2) + d_2 t + e_2] + \mathcal{G}_{11} + [\mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x \\ & - \frac{\alpha \mathcal{G}_1 y}{\beta} + \frac{\mathcal{G}_1 z(\alpha-\beta)}{\beta}]^2 + [\mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x - \frac{\alpha \mathcal{G}_6 y}{\beta} + \frac{\mathcal{G}_6 z(\alpha-\beta)}{\beta}]^2. \end{aligned}$$

Fig. 5 shows the dynamical behavior of interaction solution (3.3). We can observe the interaction between a lump wave and two periodic waves.

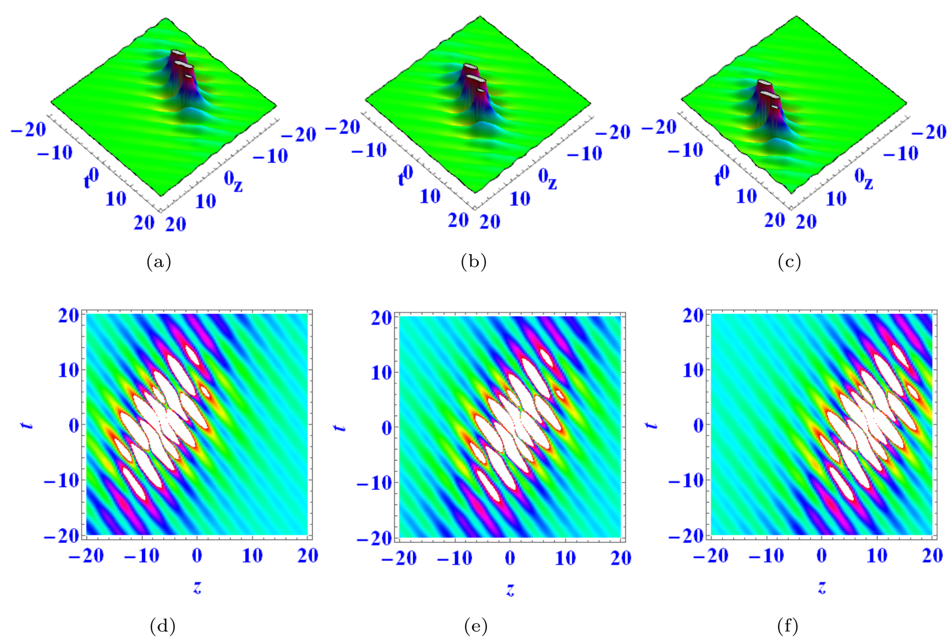


Figure 5. $\mathcal{G}_1 = \mathcal{G}_5 = \alpha = a_1 = d_1 = d_2 = e_1 = 1$, $\mathcal{G}_{11} = 10$, $\mathcal{G}_4 = \mathcal{G}_9 = -1$, $\mathcal{G}_6 = -3$, $\beta = 3$, $\mathcal{G}_{10} = a_2 = -2$, $e_2 = k_1 = 2$, $k_2 = 6$, $y = 0$, when $x = -5$ in (a) (d), $x = 0$ in (b) (e), $x = 5$ in (c) (f).

4. Breather-wave solution

We suppose that Eq. (1.1) has the following breather-wave solution

$$\xi \ = k_1 e^{\theta_1} + e^{-\theta_1} + k_2 \sin \theta_2 + k_3 \cos \theta_3. \tag{4.1}$$

When $k_2 = 0$, the corrsponding breather-wave solutions have been obtained in Ref. [8]. When $k_2 \neq 0$, the breather-wave solution (4.1) is not discussed in other literature. Substituting Eq. (4.1) into Eq. (1.2), we have

$$\begin{aligned} &c_1 = -a_1 - b_1, c_2 = -a_2 - b_2, c_3 = -a_3 - b_3, \\ &b_2 = -\frac{\alpha a_2}{\beta}, b_1 = -\frac{\alpha a_1}{\beta}, b_3 = -\frac{\alpha a_3}{\beta}. \end{aligned} \tag{4.2}$$

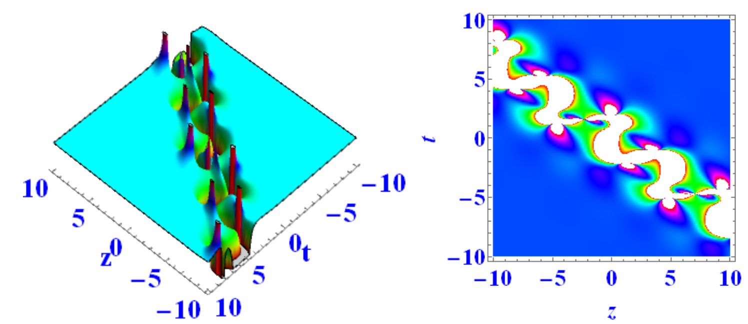
The corresponding breather-wave solution can be written as

$$u = 2\left(\frac{\xi_{xx}}{\xi} - \frac{\xi_x^2}{\xi^2}\right), \tag{4.3}$$

with

$$\begin{aligned} \xi = {} & k_1 \exp[a_1 x - \frac{\alpha a_1 y}{\beta} + z\left(\frac{\alpha a_1}{\beta} - a_1\right) + d_1 t + e_1] + \exp[-a_1 x + \frac{\alpha a_1 y}{\beta} \\ & -z\left(\frac{\alpha a_1}{\beta} - a_1\right) - d_1 t - e_1] + k_2 \sin[a_2 x - \frac{\alpha a_2 y}{\beta} + z\left(\frac{\alpha a_2}{\beta} - a_2\right) + d_2 t \\ & + e_2] + k_3 \cos[a_3 x - \frac{\alpha a_3 y}{\beta} + z\left(\frac{\alpha a_3}{\beta} - a_3\right) + d_3 t + e_3]. \end{aligned}$$

Fig. 6 shows the dynamical behavior of breather solution (4.3). We can observe the interaction between the breather wave and two periodic waves.



(a) 3D plot (b) Density plot

Figure 6. $k_1 = d_4 = -1$, $a_1 = \alpha = a_2 = d_2 = d_3 = 1$, $\beta = 3$, $a_3 = -2$, $k_2 = -6$, $k_3 = 2$, $x = y = 0$.

5. Multi-wave solution

We suppose that Eq. (1.1) has the following multi-wave solution

$$\xi \ = k_1 \cosh \theta_1 + k_2 \cos \theta_2 + k_3 \cosh \theta_3. \tag{5.1}$$

Substituting Eq. (5.1) into Eq. (1.2), we have

$$c_1 = -a_1 - b_1, c_2 = -a_2 - b_2, c_3 = -a_3 - b_3,$$
$$b_2 = -\frac{\alpha a_2}{\beta}, b_1 = -\frac{\alpha a_1}{\beta}, b_3 = -\frac{\alpha a_3}{\beta}. \tag{5.2}$$

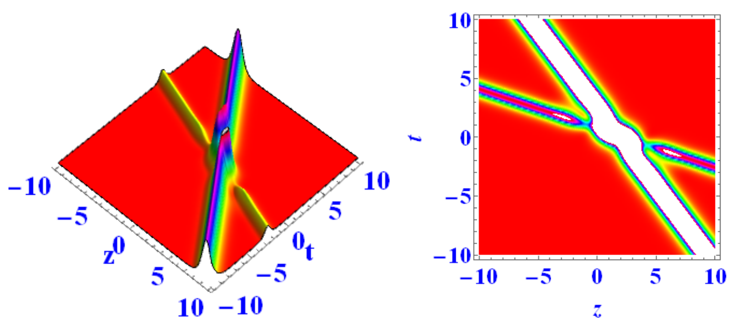
The corresponding multi-wave solution can be written as

$$u = 2\left(\frac{\xi_{xx}}{\xi} - \frac{\xi_x^2}{\xi^2}\right), \tag{5.3}$$

with

$$\begin{aligned}\xi = {} & k_2 \cos[a_2 x - \frac{\alpha a_2 y}{\beta} + z\left(\frac{\alpha a_2}{\beta} - a_2\right) + d_2 t + e_2] \\ & + k_1 \cosh[a_1 x - \frac{\alpha a_1 y}{\beta} + z\left(\frac{\alpha a_1}{\beta} - a_1\right) + d_1 t + e_1] \\ & + k_3 \cosh[a_3 x - \frac{\alpha a_3 y}{\beta} + z\left(\frac{\alpha a_3}{\beta} - a_3\right) + d_3 t + e_3].\end{aligned}$$

Fig. 7 and Fig. 8 show the dynamical behavior of multi-wave solution (5.3).



(a) 3D plot (b) Density plot

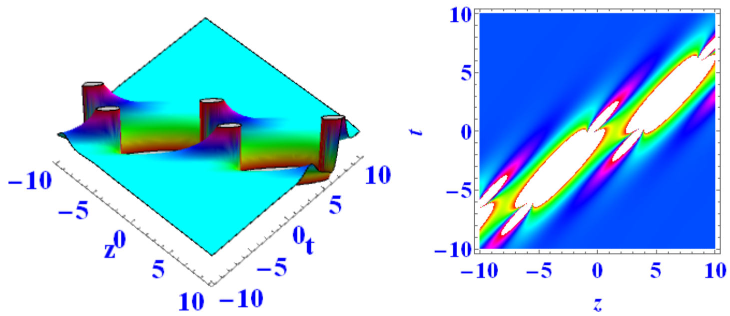
Figure 7. $k_1 = k_2 = k_3 = a_1 = \alpha = a_1 = d_1 = d_2 = e_1 = 1$, $\beta = a_3 = 3$, $e_3 = 4$, $a_2 = e_2 = 2$, $d_3 = -3$, $x = y = 0$.

6. Conclusion

In this paper, we investigate a (3 + 1)-dimensional GBSE. Based on the Hirota bilinear form and symbolic computation [1–3, 9, 12–15, 17, 18, 30, 32–38, 42, 43], abundant exact solutions are obtained, including breather-wave, multi-wave and interaction solutions between lump wave, periodic waves and N-soliton ($N = 2, 3, 4$). Dynamical behavior is shown in Figs. 1-8.

According to our observation of interaction solutions (2.3), (2.6) and (2.9), we can conclude the following interaction solutions between lump wave and N-soliton

$$\xi = (\mathcal{G}_5 + \mathcal{G}_4 t + \mathcal{G}_1 x + \mathcal{G}_2 y + \mathcal{G}_3 z)^2 + (\mathcal{G}_{10} + \mathcal{G}_9 t + \mathcal{G}_6 x + \mathcal{G}_7 y + \mathcal{G}_8 z)^2$$



(a) 3D plot (b) Density plot

Figure 8. $k_1 = a_1 = \alpha = a_1 = d_1 = d_2 = e_1 = 1$, $\beta = a_3 = 3$, $d_3 = -3$, $e_3 = 4$,$a_2 = e_2 = k_2 = 2$, $x = y = k_3 = 0$.

$$
\begin{aligned}
&+\mathcal{G}_{11} + \sum_{i=1}^{N} k_i \exp(a_i x + b_i y + c_i z + d_i t + e_i), \\
&u = 2\left(\xi_{xx}/\xi - \xi_x^2/\xi^2\right),
\end{aligned} \tag{6.1}
$$

with

$$
\begin{aligned}
&c_i = -a_i - b_i, b_i = -\frac{\alpha a_i}{\beta}, \mathcal{G}_2 = -\frac{\alpha \mathcal{G}_1}{\beta}, \mathcal{G}_7 = -\frac{\alpha \mathcal{G}_6}{\beta}, \\
&\mathcal{G}_3 = \frac{\mathcal{G}_1(\alpha - \beta)}{\beta}, \mathcal{G}_8 = \frac{\mathcal{G}_6(\alpha - \beta)}{\beta}.
\end{aligned} \tag{6.2}
$$

According to our observation of solutions (4.2) and (5.2), we can find the following more general form of the solution of Eq. (1.1)

$$
\begin{aligned}
\xi &= k_1\rho_1\left(a_1 x + b_1 y + c_1 z + d_1 t + e_1\right) + k_2\rho_2\left(a_2 x + b_2 y + c_2 z + d_2 t + e_2\right) \\
&\quad + k_3\rho_3\left(a_3 x + b_3 y + c_3 z + d_3 t + e_3\right), \\
u &= 2\left(\xi_{xx}/\xi - \xi_x^2/\xi^2\right),
\end{aligned} \tag{6.3}
$$

with

$$
c_i = -a_i - b_i, b_i = -\frac{\alpha a_i}{\beta} (i = 1, 2, 3), \tag{6.4}
$$

where ρ_1, ρ_2 and ρ_3 are arbitrary functions about $a_1 x + b_1 y + c_1 z + d_1 t + e_1$, $a_2 x + b_2 y + c_2 z + d_2 t + e_2$ and $a_3 x + b_3 y + c_3 z + d_3 t + e_3$, respectively. By choosing different expressions of the functions ρ_1, ρ_2 and ρ_3, we can get various solutions of Eq. (1.1). Even the following more general expressions are true

$$
\begin{aligned}
&\rho_i = \rho_i(a_i x + b_i y + c_i z + d_i t + e_i), \\
&\xi = \sum_{i=1}^{N} k_i \rho_i, \\
&u = 2\left(\xi_{xx}/\xi - \xi_x^2/\xi^2\right),
\end{aligned} \tag{6.5}
$$

with

$$c_i = -a_i - b_i, b_i = -\frac{\alpha a_i}{\beta}(i = 1, \cdots, N). \tag{6.6}$$

For example, in Eq. (6.3), we select

$$\begin{aligned}
\rho_1 &= Sech^2\left(a_1 x + b_1 y + c_1 z + d_1 t + e_1\right),\\
\rho_2 &= Sech^2\left(a_2 x + b_2 y + c_2 z + d_2 t + e_2\right),\\
\rho_3 &= Sech^2\left(a_3 x + b_3 y + c_3 z + d_3 t + e_3\right).
\end{aligned} \tag{6.7}$$

Substituting Eq. (6.7) into Eq. (6.3), we can obtain the following multi-wave solution

$$\begin{aligned}
\xi &= k_1 Sech^2\left(a_1 x + b_1 y + c_1 z + d_1 t + e_1\right) + k_2 Sech^2(a_2 x + b_2 y + c_2 z\\
&\quad + d_2 t + e_2) + k_3 Sech^2\left(a_3 x + b_3 y + c_3 z + d_3 t + e_3\right),\\
u &= 2\left(\xi_{xx}/\xi - \xi_x^2/\xi^2\right),
\end{aligned} \tag{6.8}$$

with

$$c_i = -a_i - b_i, b_i = -\frac{\alpha a_i}{\beta}(i = 1, 2, 3).$$

Dynamical behavior of solution (6.8) is shown in Fig. 9. Similarly, we can also choose more functions to construct the solution of Eq. (1.1).

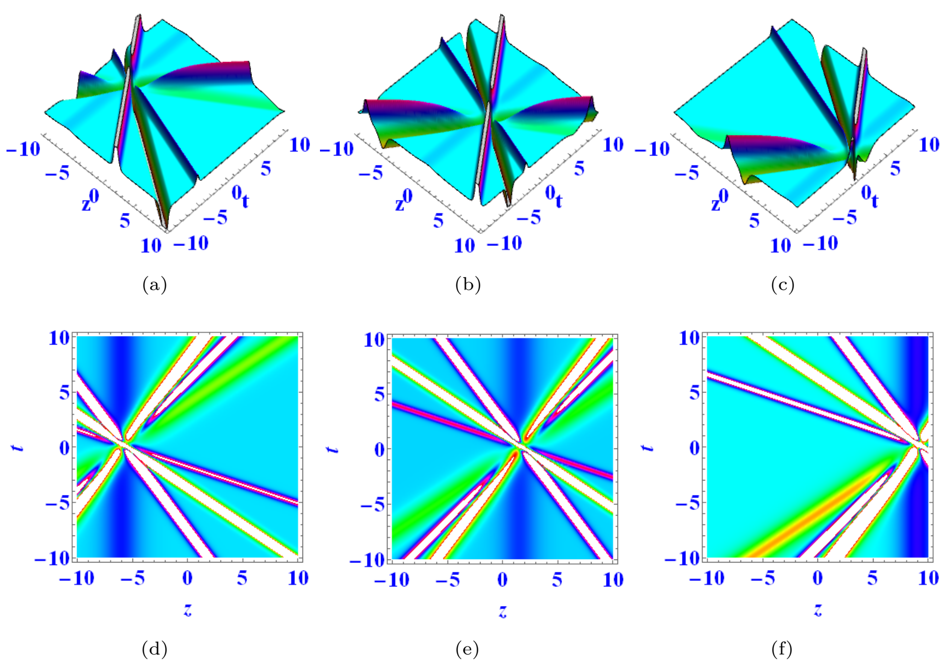


Figure 9. $k_1 = k_2 = k_3 = a_1 = \alpha = a_1 = d_1 = d_2 = e_1 = 1$, $\beta = a_3 = 3$, $e_3 = 4$, $a_2 = e_2 = 2$, $d_3 = -3$, $y = 0$, when $x = -5$ in (a) (d), $x = 0$ in (b) (e), $x = 5$ in (c) (f).

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