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ASYMPTOTICS OF A MULTIZONAL INTERNAL LAYER SOLUTION TO A PIECEWISE-SMOOTH SINGULARLY PERTURBED EQUATION WITH A TRIPLE ROOT OF THE DEGENERATE EQUATION*

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Abstract A singularly perturbed boundary value problem for a stationary equation of reaction-diffusion type in the case when reactive term undergoes discontinuity along some curve that is independent of the small parameter is studied. This is a new class of problems with triple roots of the degenerate equation, which leads to the formation of complex multizonal internal layers in the neighborhood of the discontinuity curve. By the method of asymptotic differential inequalities and matching asymptotic expansion, the existence of a contrast structure solution is proved. Using a different modified boundary layer function method, the asymptotic representation of point itself and this solution are constructed.

Keywords Reaction-diffusion equation, multizonal internal layer, a triple root of the degenerate equation, discontinuous dynamical system.

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1. Introduction

Reaction-diffusion equation has been widely used to describe various phenomena in modern science such as synergetics, astrophysic, chemical kinetics and biology [22]. In the article [9], the author studied on stability of nonhomogeneous periodic solutions to reaction-diffusion equations. For a space–time periodic reaction–diffusion equation, [18] proved that the existence and the uniqueness of the periodic solution were determined by the sign of the periodic principal eigenvalue associated with the linearized problem. Then it comes to the singularly perturbed problem of this type. The asymptotic method of differential inequalities for investigation of periodic contrast structures in singularly perturbed reaction-diffusion equations has been developed in [23]. And [19] studied a singularly perturbed periodic problem

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for a parabolic reaction-diffusion equation in the two-dimensional case and considered the case when there was an internal layer under unbalanced nonlinearity. They discussed the continuous singularly perturbed dynamical system when the degenerate equation had isolated roots. With the development of object and spectrum of research, [1] studied the asymptotics of the solution to singularly perturbed reaction-diffusion equation

$$\begin{cases} \varepsilon^2(u_t - u_{xx}) = f(u,x,t,\varepsilon), \quad 0 < x < 1, \\ u_x(0,t,\varepsilon) = u_x(1,t,\varepsilon) = 0, \quad 0 < t \le T, \\ u(x,0,\varepsilon) = u^0(x), \quad 0 < x < 1 \end{cases} \tag{1.1}$$

in the case of a double root of the degenerate equation. In this case, the traditional asymptotic method exposed the limitations. Therefore, the authors proposed an improvement method. It was worth noting that these literatures discussed the continuous singularly perturbed dynamical systems of reaction-diffusion equations.

Currently, singularly perturbed problems for differential equations with discontinuous right-hand sides become a hot spot of research interests. The asymptotic study of singularly perturbed problem of reaction-diffusion equation in the case of discontinuous summand and isolated roots of the degenerate equation has been conducted by Nefedov and Ni [15], who initiate the development of this line of research. In the case when the discontinuous curve is a vertical straight line, much research work has been done [13–17]. What's more, such problems with double roots can be found in [27]. For the case of continuous dynamic system with multiple roots, the critical manifold is not normally hyperbolic and then the classical boundary layer function method [24] cannot be applied. Studies have shown that the asymptotic behavior of solution is qualitatively different and boundary layer functions are constructed by a modified algorithm [2–6, 25]. In particular, the case of triple root is of great interest [7, 8]. In the real world, the discontinuous curve is not vertical to the time axis [28]. Such studies with isolated roots have been done in [20, 29]. This paper is a generalization to this new class of asymptotic methods for equations of reaction-diffusion type with triple roots. In contrast to the case of double roots, it is necessary for us to reconsider the construction of asymptotics and the existence of smooth solution and there are many problems to be solved.

Consider the singularly perturbed boundary value problem:

$$\begin{cases} \mu^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} = f(y,t,\mu), \quad 0 < t < 1, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}(0,\mu) = 0, \quad \dfrac{\mathrm{d}y}{\mathrm{d}t}(1,\mu) = 0, \end{cases} \tag{1.2}$$

where

$$f(y,t,\mu) = \begin{cases} f^{(-)}(y,t,\mu), \quad (y,t) \in \bar{D}^{(-)}, \\ f^{(+)}(y,t,\mu), \quad (y,t) \in \bar{D}^{(+)}, \end{cases}$$

here

$$\bar{D}^{(-)} = \{(y,t) |\ g(t) \le y \le l,\ 0 \le t \le 1\}, \tag{1.3}$$

$$\bar{D}^{(+)} = \{(y,t) |\ -l \le y \le g(t),\ 0 \le t \le 1\} \tag{1.4}$$

and

$$f^{(\mp)}(y,t,\mu) = h^{(\mp)}(t)(y - \varphi^{(\mp)}(t))^3 - \mu f_1^{(\mp)}(y,t,\mu), \tag{1.5}$$

$\mu > 0$ is a small parameter, and $f(y,t,\mu)$ is defined everywhere in the domain $\bar{D} = \{(y,t)| -l \leq y \leq l,\, 0 \leq t \leq 1,\, 0 < \mu < \mu_0\}$, here l is a given positive number in an interval. In addition, function $y = g(t)$ is defined in the interval $0 \leq t \leq 1$.

We asssume that the following assumptions are satisfied.

Assumption 1.1. *Assume that functions $h^{(\mp)}(t)$, $\varphi^{(\mp)}(t)$ and $f_1^{(\mp)}(y,t,\mu)$ are sufficiently smooth on the sets $\bar{D}^{(\mp)}$. The function $y = g(t)$ is a smooth curve that crosses the boundaries of the rectangle $\bar{D}$ at $t = 0$ and $t = 1$. Moreover, $f(y,t,\mu)$ satisfies the following inequality*

$$f^{(-)}(y, g(t), 0) \neq f^{(+)}(y, g(t), 0), \quad 0 \leq t \leq 1.$$

Under Assumption 1.1, the degenerate equation

$$f(y,t,0) = 0 \tag{1.6}$$

has a discontinuous solution with triple roots

$$\bar{y}(t) = \begin{cases} \varphi^{(-)}(t), & (y,t) \in \bar{D}^{(-)}, \\ \varphi^{(+)}(t), & (y,t) \in \bar{D}^{(+)}. \end{cases}$$

For the sake of simplicity, the following assumption is made.

Assumption 1.2. *We assume that $\varphi^{(-)}(t) > \varphi^{(+)}(t)$, $t \in T := (q,p) \cap (0,1)$, where $t = p$ and $t = q$ are the abscissas of the points of intersection of two curves $y = \varphi^{(-)}(t)$, $y = \varphi^{(+)}(t)$ with the curve $y = g(t)$ if there exist such points.*

In order to find the leading terms in the asymptotic representations of the internal layer in the neighborhood of the discontinuous curve in the course of constructing the asymptotics of solution to problem (1.2), it is necessary to use the following assumption.

Assumption 1.3. *If $g(t) \leq y \leq \varphi^{(-)}(t)$, one has the inequalities $h^{(-)}(t) > 0$, $\bar{f}_1^{(-)}(t) \equiv f_1^{(-)}(\varphi^{(-)}(t), t, 0) < 0$, $0 \leq t \leq 1$, and if $\varphi^{(+)}(t) \leq y \leq g(t)$, we assume that $h^{(+)}(t) > 0$, $\bar{f}_1^{(+)}(t) \equiv f_1^{(+)}(\varphi^{(+)}(t), t, 0) > 0$, $0 \leq t \leq 1$.*

By virtue of Assumptions 1.1-1.3, there maybe exist a solution to problem (1.2) with internal layer in the neighborhood of some point $t = t^* \in T$ on the discontinuous curve $y = g(t)$. And the point t^* is unknown in advance.

In order to construct a smooth asymptotic solution to problem (1.2) and prove its existence, we introduce the function

$$\mathrm{I}(t) = \mathrm{I}^{(-)}(t) - \mathrm{I}^{(+)}(t), \tag{1.7}$$

where

$$\mathrm{I}^{(\mp)}(t) = -\sqrt{\frac{h^{(\mp)}(t)}{2}} \Big(g(t) - \varphi^{(\mp)}(t)\Big)^2. \tag{1.8}$$

Assumption 1.4. *Assume that the equation $\mathrm{I}(t) = 0$ has a solution $t = t_0$, $t_0 \in T$. And suppose that*

$$\mathrm{I}'(t_0) \neq 0. \tag{1.9}$$

The expression of formula for $\mathrm{I}'(t)$ is as follows

$$\begin{aligned}\mathrm{I}'(t) = &-\frac{h^{(-)'}(t)}{2\sqrt{2h^{(-)}(t)}}\left(g(t)-\varphi^{(-)}(t)\right)^2+\frac{h^{(+)'}(t)}{2\sqrt{2h^{(+)}(t)}}\left(g(t)-\varphi^{(+)}(t)\right)^2\\ &-\sqrt{2h^{(-)}(t)}\left(g(t)-\varphi^{(-)}(t)\right)\left(g'(t)-\varphi^{(-)'}(t)\right)\\ &+\sqrt{2h^{(+)}(t)}\left(g(t)-\varphi^{(+)}(t)\right)\left(g'(t)-\varphi^{(+)'}(t)\right).\end{aligned}$$

It is noted that the equation $\mathrm{I}(t)=0$ can have several solutions, which means that several transfer points can appear at the same time. At these points, there will be jumps between different roots $\varphi^{(\pm)}(t)$. If equation $\mathrm{I}(t)=0$ has no solution, there will be no steplike contrast structure solution.

In the following sections, the asymptotics is constructed by the modified boundary layer function method and the technique of differential inequality is used to prove the existence of solutions to auxiliary problems [21,26]. Based on the matching technique, the existence of a smooth solution is proved [10].

2. Construction of asymptotic solution

For problem (1.2), the asymptotic solution with internal layer in the neighborhood of point $(t^*, g(t^*))$ shall be constructed. And we will consider two problems: a boundary value problem on the left of $y=g(t)$ and a boundary value problem on the right of $y=g(t)$:

$$\begin{cases}\mu^2\dfrac{\mathrm{d}^2y^{(-)}}{\mathrm{d}t^2}=f^{(-)}(y^{(-)},t,\mu), \quad 0<t<t^*,\\ \dfrac{\mathrm{d}y^{(-)}}{\mathrm{d}t}(0,\mu)=0, \quad y^{(-)}(t^*,\mu)=g(t^*);\end{cases} \tag{2.1}$$

$$\begin{cases}\mu^2\dfrac{\mathrm{d}^2y^{(+)}}{\mathrm{d}t^2}=f^{(+)}(y^{(+)},t,\mu), \quad t^*<t<1,\\ y^{(+)}(t^*,\mu)=g(t^*), \quad \dfrac{\mathrm{d}y^{(+)}}{\mathrm{d}t}(1,\mu)=0,\end{cases} \tag{2.2}$$

where t^* is to be found afterwards.

The asymptotic representations of the solutions to two problems (2.1) and (2.2) are in the forms of three terms

$$y^{(\mp)}(t,\mu)=\bar{y}^{(\mp)}(t,\mu)+Q^{(\mp)}(\tau,\mu)+\Pi^{(\mp)}(\rho_{\mp},\mu), \tag{2.3}$$
$$\tau=\frac{t-t^*}{\mu}, \quad \rho_-=\frac{t}{\mu^{2/3}}, \quad \rho_+=\frac{1-t}{\mu^{2/3}},$$

where $\bar{y}^{(\mp)}(t,\mu)$ are the regular parts, $\Pi^{(\mp)}(\rho_{\mp},\mu)$ are the boundary layer parts in the neighborhood of $t=0$ and $t=1$, and $Q^{(\mp)}(\tau)$ are the internal layer parts near the point $(t^*, g(t^*))$ on the discontinuous curve $y=g(t)$. Each of these functions will be represented in the form of an expansion in the small parameter $\mu^{\frac{1}{3}}$

$$\bar{y}^{(\mp)}(t,\mu)=\bar{y}_0^{(\mp)}(t)+\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t)+\cdots+\mu^{\frac{k}{3}}\bar{y}_k^{(\mp)}(t)+\cdots, \tag{2.4}$$

$$Q^{(\mp)}(\tau,\mu)=Q_0^{(\mp)}(\tau)+\mu^{\frac{1}{3}}Q_1^{(\mp)}(\tau)+\cdots+\mu^{\frac{k}{3}}Q_k^{(\mp)}(\tau)+\cdots, \tag{2.5}$$

$$\Pi^{(\mp)}(\rho_\mp,\mu)=\mu^{\frac{2}{3}}\left(\Pi_0^{(\mp)}(\rho_\mp)+\mu^{\frac{1}{3}}\Pi_1^{(\mp)}(\rho_\mp)+\cdots+\mu^{\frac{k}{3}}\Pi_k^{(\mp)}(\rho_\mp)+\cdots\right). \tag{2.6}$$

And most remarkably, the expansions are in the one-third power $\mu^{1/3}$ of the singular parameter μ. The first reason to do this is that the first term of display (1.5) is a cube of $y-\varphi^{(\mp)}$ and the last term is order of μ. The second reason to tackle regular part like this is that the equation for approximation of the order of μ is unbalanced if we expand them by integer powers of μ. More importantly, the last reason to deal with internal and boundary layer parts like this is to obtain the decay nature of their coefficients, which is closely linked with our proof of existence and remainder estimation.

Substituting (2.3) into problems (2.1)-(2.2) and using the theory of scale separation between fast and slow variables, one can obtain the problems for defining the regular parts:

$$\begin{aligned}&\mu^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{y}^{(\mp)}(t,\mu)\\ =&\bar{f}^{(\mp)}\equiv h^{(\mp)}(t)[\bar{y}^{(\mp)}(t,\mu)-\varphi^{(\mp)}(t)]^3-\mu f_1^{(\mp)}(\bar{y}^{(\mp)}(t,\mu),t,\mu),\end{aligned} \tag{2.7}$$

the problems for determining internal layer functions:

$$\begin{cases}\dfrac{\mathrm{d}^2Q^{(\mp)}}{\mathrm{d}\tau^2}=Q^{(\mp)}f,\\ Q^{(\mp)}(0,\mu)=g(t^*)-\bar{y}^{(\mp)}(t^*,\mu),\quad Q^{(\mp)}(\mp\infty,\mu)=0,\end{cases} \tag{2.8}$$

where

$$\begin{aligned}Q^{(\mp)}f=&h^{(\mp)}\left(t^*+\mu^{\frac{2}{3}}\zeta\right)\times\left[\bar{y}^{(\mp)}(t^*+\mu\tau)+Q^{(\mp)}(\tau,\mu)-\varphi^{(\mp)}(t^*+\mu\tau)\right]^2\\ &-h^{(\mp)}\left(t^*+\mu\tau\right)\left[\bar{y}^{(\mp)}(t^*+\mu\tau)-\varphi^{(\mp)}(t^*+\mu\tau)\right]^2-\mu Q^{(\mp)}f_1^{(\mp)},\end{aligned}$$

here

$$Q^{(\mp)}f_1^{(\mp)}=f_1^{(\mp)}(\bar{y}^{(\mp)}(t^*+\mu\tau)+Q^{(\mp)}(\tau,\mu),t^*+\mu\tau,\mu)-f_1^{(\mp)}(\bar{y}^{(\mp)}(t^*+\mu\tau),t^*+\mu\tau,\mu),$$

the problems for determining left boundary layer functions:

$$\begin{cases}\mu^{\frac{2}{3}}\dfrac{\mathrm{d}^2\Pi^{(-)}}{\mathrm{d}\rho_-^2}=f^{(-)}\left(\bar{y}^{(-)}(\mu^{2/3}\rho_-,\mu)+\Pi^{(-)},\mu^{2/3}\rho^{(-)},\mu\right)\\ \qquad\qquad\qquad -f^{(-)}\left(\bar{y}^{(-)}(\mu^{2/3}\rho_-,\mu),\mu^{2/3}\rho_-,\mu\right),\\ \dfrac{\mathrm{d}\bar{y}^{(-)}}{\mathrm{d}t}(0)+\dfrac{\mathrm{d}\Pi^{(-)}}{\mathrm{d}\rho_-}(0)=0,\end{cases} \tag{2.9}$$

and the problems for determining right boundary layer functions:

$$\begin{cases}\mu^{\frac{2}{3}}\dfrac{\mathrm{d}^2\Pi^{(+)}}{\mathrm{d}\rho_+^2}=f^{(+)}\left(\bar{y}^{(+)}(1-\mu^{2/3}\rho_+,\mu)+\Pi^{(+)},1-\mu^{2/3}\rho^{(+)},\mu\right)\\ \qquad\qquad\qquad -f^{(+)}\left(\bar{y}^{(+)}(1-\mu^{2/3}\rho_+,\mu),1-\mu^{2/3}\rho_+,\mu\right),\\ \dfrac{\mathrm{d}\bar{y}^{(+)}}{\mathrm{d}t}(1)-\dfrac{\mathrm{d}\Pi^{(+)}}{\mathrm{d}\rho_+}(0)=0.\end{cases} \tag{2.10}$$

Taking account of the fact that the derivative of asymptotic solution to problem (1.2) is continuous, one can obtain

$$\frac{\mathrm{d}y^{(-)}}{\mathrm{d}t}(t^*,\mu)-\frac{\mathrm{d}y^{(+)}}{\mathrm{d}t}(t^*,\mu)=0,$$

where $t^*(\mu)$ can be found in the form of expansion in powers of $\mu^{\frac{1}{3}}$

$$t^*(\mu)=t_0+\mu^{\frac{1}{3}}t_1+\cdots+\mu^{\frac{k}{3}}t_k+\cdots. \tag{2.11}$$

After substituting (2.4) into (2.7) and matching the coefficients of like powers of the small parameter in the Taylor expansion in $\mu^{\frac{1}{3}}$ in the equations (2.7), the functions $\bar{y}_i^{(\mp)}$ of the regular part are determined. It is easy to see that

$$\bar{y}_0^{(\mp)}(t)=\varphi^{(\mp)}(t).$$

In the first approximation, one can obtain

$$h^{(\mp)}(t)(\bar{y}_1^{(\mp)}(t))^3-\bar{f}_1^{(\mp)}(t)=0, \tag{2.12}$$

whose solution can be represented as

$$\bar{y}_1^{(\mp)}(t)=\left[\frac{\bar{f}_1^{(\mp)}(t)}{\bar{h}^{(\mp)}(t)}\right]^{\frac{1}{3}}\neq 0 \tag{2.13}$$

by virtue of Assumption 1.3.

From Assumption 1.3 and the formula (2.13), the remaining coefficients of the expansion (2.4) can be represented as

$$\bar{y}_k^{(\mp)}(t)=\frac{f_k^{(\mp)}(t)}{3h^{(\mp)}(t)(\bar{y}_1^{(\mp)}(t))^2},\qquad k>1,$$

where $f_k^{(\mp)}(t)$ are known functions that depend on $\bar{y}_j^{(\mp)}$, $j<k$.

Substituting (2.4),(2.5) into (2.8), we have

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(Q_0^{(\mp)}+\mu^{\frac{1}{3}}Q_1^{(\mp)}+\cdots\right)\\ =&h^{(\mp)}(t^*+\mu\tau)\times[\bar{y}_0^{(\mp)}(t^*+\mu\tau)+\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*+\mu\tau)+\cdots+Q_0^{(\mp)}+\mu^{\frac{1}{3}}Q_1^{(\mp)}+\cdots\\ &-\varphi^{(\mp)}(t^*+\mu\tau)]^3-h^{(\mp)}(t^*+\mu\tau)\times[\bar{y}_0^{(\mp)}(t^*+\mu\tau)+\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*+\mu\tau)+\cdots\\ &-\varphi^{(\mp)}(t^*+\mu\tau)]^3-\mu Q^{(\mp)}f_1^{(\mp)},\end{aligned} \tag{2.14}$$

$$\begin{aligned}&Q_0^{(\mp)}(0)+\mu^{\frac{1}{3}}Q_1^{(\mp)}(0)+\cdots=g(t^*)-\bar{y}_0^{(\mp)}(t^*)-\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*)-\cdots,\\ &Q_0^{(\mp)}(\mp\infty)+\mu^{\frac{1}{3}}Q_1^{(\mp)}(\mp\infty)+\cdots=0.\end{aligned} \tag{2.15}$$

Since the nature of solutions to problems (2.1),(2.2) are complicated in the neighborhood of discontinuity curve $y=g(t)$, i.e., for internal layer functions, there exist different scale of variable in three distinct zones. Generally speaking, it will require that all of asymptotics in distinct zones are matched to complete the construction of asymptotic solution. However, we shall construct a unified internal layer function to describe the behavior of solutions to problems (2.1),(2.2) in distinct zones. To this end, taking (2.15) into account, we need to modify the equation for finding the internal layer functions and obtain the following problem

$$\begin{cases}\dfrac{\mathrm{d}^2 Q_0^{(\mp)}}{\mathrm{d}\tau^2} = h^{(\mp)}(t^*)\left[(Q_0^{(\mp)})^3 + 3\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*)(Q_0^{(\mp)})^2 + 3\mu^{\frac{2}{3}}(\bar{y}_1^{(\mp)}(t^*))^2 Q_0^{(\mp)}\right],\\ Q_0^{(\mp)}(0) = g(t^*) - \varphi^{(\mp)}(t^*), \quad Q_0^{(\mp)}(\mp\infty,\mu) = 0.\end{cases} \tag{2.16}$$

Reducing this problem into a first-order initial value problem, one obtains

$$\begin{cases}\dfrac{\mathrm{d}Q_0^{(\mp)}}{\mathrm{d}\tau} = -\sqrt{\frac{h^{(\mp)}(t^*)}{2}}\sqrt{(Q_0^{(\mp)})^2 + 4\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*)Q_0^{(\mp)} + 6\mu^{\frac{2}{3}}(\bar{y}_1^{(\mp)}(t^*))^2}\,Q_0^{(\mp)},\\ Q_0^{(\mp)}(0) = g(t^*) - \varphi^{(\mp)}(t^*).\end{cases} \tag{2.17}$$

By comparing principle, there exists solutions $Q_0^{(\mp)}(\tau)$ to the above problems and these solutions satisfy the inequalities

$$\begin{aligned}&\frac{\mu^{\frac{1}{3}}2\bar{y}_1^{(-)}(t^*)\beta^{(-)}\exp(-\mu^{\frac{1}{3}}\sqrt{2h^{(-)}(t^*)}\bar{y}_1^{(-)}(t^*)\tau)}{1-\beta^{(-)}\exp(-\mu^{\frac{1}{3}}\sqrt{2h^{(-)}(t^*)}\bar{y}_1^{(-)}(t^*)\tau)}\\ \le& Q_0^{(-)}(\tau) \le \frac{\mu^{\frac{1}{3}}\sqrt{6}\bar{y}_1^{(-)}(t^*)\alpha^{(-)}\exp(-\mu^{\frac{1}{3}}\sqrt{3h^{(-)}(t^*)}\bar{y}_1^{(-)}(t^*)\tau)}{1-\alpha^{(-)}\exp(-\mu^{\frac{1}{3}}\sqrt{3h^{(-)}(t^*)}\bar{y}_1^{(-)}(t^*)\tau)},\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\frac{\mu^{\frac{1}{3}}\sqrt{6}\bar{y}_1^{(+)}(t^*)\alpha^{(+)}\exp(-\mu^{\frac{1}{3}}\sqrt{3h^{(+)}(t^*)}\bar{y}_1^{(+)}(t^*)\tau)}{1-\alpha^{(+)}\exp(-\mu^{\frac{1}{3}}\sqrt{3h^{(+)}(t^*)}\bar{y}_1^{(+)}(t^*)\tau)}\\ \le& Q_0^{(+)}(\tau) \le \frac{\mu^{\frac{1}{3}}2\bar{y}_1^{(+)}(t^*)\beta^{(+)}\exp(-\mu^{\frac{1}{3}}\sqrt{2h^{(+)}(t^*)}\bar{y}_1^{(+)}(t^*)\tau)}{1-\beta^{(+)}\exp(-\mu^{\frac{1}{3}}\sqrt{2h^{(+)}(t^*)}\bar{y}_1^{(+)}(t^*)\tau)},\end{aligned} \tag{2.19}$$

where

$$\alpha^{(\mp)} = \frac{g(t^*) - \varphi^{(\mp)}(t^*)}{g(t^*) - \varphi^{(\mp)}(t^*) + \mu^{\frac{1}{3}}\sqrt{6}\bar{y}_1^{(\mp)}(t^*)}, \quad \beta^{(\mp)} = \frac{g(t^*) - \varphi^{(\mp)}(t^*)}{g(t^*) - \varphi^{(\mp)}(t^*) + \mu^{\frac{1}{3}}2\bar{y}_1^{(\mp)}(t^*)}.$$

From (2.18)-(2.19), we have a conclusion that the decay rate of $Q_0^{(\mp)}(\tau)$ differ as τ is in distinct regions. In the first region $-\mu^{-\lambda} \le \tau \le 0 (0 \le \tau \le \mu^{-\lambda}), 0 \le \lambda < \frac{1}{3}$, $Q_0^{(\mp)}(\tau) = O(1/(1+\tau))$ decreases algebraically as $\tau \to \mp\infty$. The second region is $-\mu^{-1/3} \le \tau \le -\mu^{-\lambda}(\mu^{-\lambda} \le \tau \le \mu^{-1/3})$ where the variable and decay character of internal layer functions $Q_0^{(\mp)}(\tau)$ experience changes. In the last region $\tau \le -\mu^{-1/3}(\tau \ge \mu^{-1/3})$, $Q_0^{(\mp)}(\tau) = O(\sqrt{\mu})\exp(\pm\kappa\zeta)$, i.e., a new time scale $\zeta = \mu^{\frac{1}{3}}\tau = \frac{t-t^*}{\mu^{2/3}}$ appears and $Q_0^{(\mp)}(\tau)$ decreases exponentially as $\zeta \to \mp\infty$.

From (2.17), one has the estimates

$$|Q_0^{(\mp)}(\tau)| \le C Q_\kappa^{(\mp)}(\tau), \tag{2.20}$$

where

$$Q_\kappa^{(\mp)}(\tau) = \left(g(t^*) - \varphi^{(\mp)}(t^*)\right)\exp\left(-\sqrt{\frac{h^{(\mp)}(t^*)}{2}}\int_0^\tau J^{(\mp)}(s)\,\mathrm{d}s\right), \tag{2.21}$$

here

$$J^{(\mp)}(s) = \sqrt{(Q_0^{(\mp)}(s))^2 + 4\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*)Q_0^{(\mp)}(s) + 6\mu^{\frac{2}{3}}\left(4(\bar{y}_1^{(\mp)}(t^*))^2 + \kappa^2\right)},$$

C is a positive constant that is independent of μ, and $0 < \kappa < \sqrt{2}\bar{y}_1^{(\mp)}(t^*)$.

From (2.17), we have the equations of $\frac{\mathrm{d}Q_0^{(\mp)}}{\mathrm{d}\tau}(0)$, and expand the right side of this equation into series in integer powers of $\mu^{\frac{1}{3}}$

$$\frac{\mathrm{d}Q_0^{(\mp)}}{\mathrm{d}\tau}(0) = I^{(\mp)}(t^*) + \mu^{\frac{1}{3}}d_1^{(\mp)}(t^*) + \mu^{\frac{2}{3}}d_2^{(\mp)}(t^*) + \cdots + \mu^{\frac{k}{3}}d_k^{(\mp)}(t^*) + \cdots, \quad (2.22)$$

here $I^{(\mp)}(t^*)$ is defined from (1.8), and functions $d_k^{(\mp)}(t^*)$ are infinitely differentiable, specifically,

$$d_1^{(\mp)}(t^*) = \pm\sqrt{2h^{(\mp)}(t^*)}\bar{y}_1^{(\mp)}(t^*)(g(t^*) - \varphi^{(\mp)}(t^*)).$$

The problems for determining $Q_k^{(\mp)}(\tau)$, $k = 1, 2, \cdots$ are in the forms

$$\begin{cases} \dfrac{\mathrm{d}^2Q_k^{(\mp)}}{\mathrm{d}\tau^2} = 3h^{(\mp)}(t^*)\left(Q_0^{(\mp)} + \mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*)\right)^2 Q_k^{(\mp)} + q_k^{(\mp)}(\tau,\mu), \\ Q_k^{(\mp)}(0) = -\bar{y}_k^{(\mp)}(t^*), \quad Q_k^{(\mp)}(\mp\infty,\mu) = 0. \end{cases} \quad (2.23)$$

Here the coefficients of $Q_k^{(\mp)}(\tau)$ are the derivatives of the right side of equation in the problem (2.17) with respect to $Q_0^{(\mp)}(\tau)$. After the right side of (2.14) is expanded into series of $\mu^{\frac{1}{3}}$, $q_k^{(\mp)}(\tau,\mu)$ only include those terms whose modulus are no less than the coefficient of product of three terms from $Q_l^{(\mp)}(\tau), Q_m^{(\mp)}(\tau), Q_n^{(\mp)}(\tau)$, $l+m+n \le k$, $l < k$, $m < k, n < k$ in the $\mu^{\frac{k}{3}}$ approximation. In addition, $q_k^{(\mp)}(\tau,\mu)$ also include the products of $\mu^{\frac{1}{3}}$ and the coefficients in the approximation $\mu^{\frac{k+1}{3}}$ whose bounds have two such factors, as well as the products of $\mu^{\frac{2}{3}}$ and those coefficients of $\mu^{\frac{k+2}{3}}$ whose bounds have one such factor. Moreover, it is necessary to replace the variable ζ in $q_k^{(\mp)}(\tau,\mu)$ by $\mu^{\frac{1}{3}}\tau$. In particular, $q_1^{(\mp)}(\tau,\mu)$ can be represented as

$$q_1^{(\mp)}(\tau,\mu) = 3h^{(\mp)}(t^*)\bar{u}_2^{(\mp)}(t^*)\mu^{\frac{1}{3}}(Q_0^{(\mp)})^2 + \left(6\bar{u}_1^{(\mp)}(t^*)\bar{u}_2^{(\mp)}(t^*)Q_0^{(\mp)} - Q_0^{(\mp)}f_1^{(\mp)}\right)\mu^{\frac{2}{3}}.$$

Here

$$|q_k^{(\mp)}(\tau,\mu)| \le C[(Q_\kappa^{(\mp)}(\tau))^3 + \mu^{\frac{1}{3}}(Q_\kappa^{(\mp)})^2 + \mu^{\frac{2}{3}}Q_\kappa^{(\mp)}(\tau)]. \quad (2.24)$$

And $Q_k^{(\mp)}(\tau)$ can be expressed in explicit forms

$$Q_k^{(\mp)}(\tau) = -\bar{y}_k^{(\mp)}(t^*)\frac{\Psi^{(\mp)}(\tau)}{\Psi^{(\mp)}(0)} + \Psi^{(\mp)}(\tau)\int_0^\tau \frac{M_k^{(\mp)}(\eta)}{(\Psi^{(\mp)}(\eta))^2}\mathrm{d}\eta, \quad (2.25)$$

where

$$M_k^{(\mp)}(\eta) = \int_{\mp\infty}^\eta \Psi^{(\mp)}(s)q_k^{(\mp)}(s)\mathrm{d}s,$$

$$\Psi^{(\mp)}(\tau) = \frac{\mathrm{d}Q_0^{(\mp)}}{\mathrm{d}\tau}(\tau) = -\sqrt{\frac{h^{(\mp)}(t^*)}{2}}\sqrt{(Q_0^{(\mp)})^2 + 4\mu^{\frac{1}{3}}\bar{y}_1^{(\mp)}(t^*)Q_0^{(\mp)} + 6\mu^{\frac{2}{3}}(\bar{y}_1^{(\mp)}(t^*))^2}Q_0^{(\mp)}.$$

From (2.21), (2.24), (2.25), all internal layer functions $Q_k^{(\mp)}(\tau)$, $k \geq 0$ and their derivatives have the estimates

$$\left|Q_k^{(\mp)}(\tau)\right| \leq CQ_\kappa^{(\mp)}(\tau), \quad \left|\frac{\mathrm{d}Q_k^{(\mp)}}{\mathrm{d}\tau}(\tau)\right| \leq CQ_\kappa^{(\mp)}(\tau) \tag{2.26}$$

by using mathematical induction method.

In the following, we write out the problems for left boundary layer function $\Pi_k^{(-)}(\rho_-, \mu)$, $k \geq 0$, one can obtain

$$\begin{cases} \dfrac{\mathrm{d}^2\Pi_k^{(-)}}{\mathrm{d}\rho_-^2} = \left[3\bar{h}^{(-)}(0)(\bar{y}_1^{(-)}(0))^2\right]\Pi_k^{(-)} + \pi_i^{(-)}(\rho_-), \quad \rho_- > 0, \\ \dfrac{\mathrm{d}\Pi_k^{(-)}}{\mathrm{d}\rho_-}(0) = -\dfrac{\mathrm{d}\bar{y}_k^{(-)}}{\mathrm{d}t}(0), \qquad \Pi_k^{(-)}(+\infty) = 0, \end{cases} \tag{2.27}$$

where $\pi_k^{(-)}(\rho_-)$ are known functions that depend on $\Pi_j^{(-)}(\rho_-)$, $j < k$. In particular, $\pi_0^{(-)}(\rho_-) \equiv 0$.

Under Assumption 1.3, problem (2.27) can be solved sequentially. Thus, $\Pi_k^{(-)}(\rho_-)$, $k \geq 0$ are obtained. In particular,

$$\Pi_0^{(-)}(\rho_-) = \frac{\mathrm{d}\varphi^{(-)}}{\mathrm{d}t}(0)\frac{\exp\left(-\sqrt{3\bar{h}^{(-)}(0)(\bar{y}_1^{(-)}(0))^2}\rho_-\right)}{\sqrt{3\bar{h}^{(-)}(0)(\bar{y}_1^{(-)}(0))^2}}. \tag{2.28}$$

And one has the estimate

$$\left|\Pi_k^{(-)}(\rho_-)\right| \leq C\exp(-\kappa\rho_-), \quad \rho_- \geq 0, \quad k \geq 0. \tag{2.29}$$

Likewise, the right boundary layer function $\Pi^{(+)}(\rho_+, \mu)$ in the neighborhood of $x = 1$ can also be determined. And $\Pi_k^{(+)}(\rho_+, \mu)$, $k \geq 0$ satisfies the exponential estimate of the type (2.29).

So all terms of $y^{(\mp)}(t, \mu)$ are determined. However, internal layer functions $Q_k^{(\mp)}(\tau)$ $(k = 0, 1, 2, \cdots)$ depend on an unknown parameter t^*.

Let

$$G(t^*(\mu)) = \mu\frac{\mathrm{d}y^{(-)}}{\mathrm{d}t}(t^*, \mu) - \mu\frac{\mathrm{d}y^{(+)}}{\mathrm{d}t}(t^*, \mu).$$

Substituting the asymptotic representations of (2.3), (2.11) into $y^{(\mp)}(t^*, \mu)$, t^*, we have

$$\begin{aligned} G(t^*(\mu)) =& \mu\Big(\frac{\mathrm{d}\varphi^{(-)}}{\mathrm{d}t}(t^*) + \mu^{\frac{1}{3}}\frac{\mathrm{d}\bar{y}_1^{(-)}}{\mathrm{d}t}(t^*) + \cdots\Big) \\ &+ \Big(\frac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau}(0) + \mu^{\frac{1}{3}}\frac{\mathrm{d}Q_1^{(-)}}{\mathrm{d}\tau}(0) + \mu^{\frac{2}{3}}\frac{\mathrm{d}Q_2^{(-)}}{\mathrm{d}\tau}(0) + \cdots\Big) \\ &- \mu\Big(\frac{\mathrm{d}\varphi^{(+)}}{\mathrm{d}t}(t^*) + \mu^{\frac{1}{3}}\frac{\mathrm{d}\bar{y}_1^{(+)}}{\mathrm{d}t}(t^*) + \cdots\Big) \\ &- \Big(\frac{\mathrm{d}Q_0^{(+)}}{\mathrm{d}\tau}(0) + \mu^{\frac{1}{3}}\frac{\mathrm{d}Q_1^{(+)}}{\mathrm{d}\tau}(0) + \mu^{\frac{2}{3}}\frac{\mathrm{d}Q_2^{(+)}}{\mathrm{d}\tau}(0) + \cdots\Big) \end{aligned}$$

$$=I(t_0)+\mu^{\frac{1}{3}}[I'(t_0)t_1-w_1]+\cdots+\mu^{\frac{k}{3}}[I'(t_0)t_k-w_k]+\cdots=0. \quad (2.30)$$

By Assumption 1.4, the equation $I(t)=0$ has a solution $t=t_0$ and t_k $(k=1,2,\cdots)$ can be uniquely determined by the linear equations

$$I'(t_0)t_k=w_k, \quad k=1,2,\cdots, \quad (2.31)$$

where w_k depend on t_j, $j<k$. Specifically,

$$\begin{aligned}
w_1=&\frac{\bar{y}_1^{(-)}(t_0)h^{(-)}(t_0)(g(t_0)-\varphi^{(-)}(t_0))^3-M_1^{(-)}(0)}{I^{(-)}(t_0)}-d_1^{(-)}(t_0)\\
&-\frac{\bar{y}_1^{(+)}(t_0)h^{(+)}(t_0)(g(t_0)-\varphi^{(+)}(t_0))^3-M_1^{(+)}(0)}{I^{(-)}(t_0)}+d_1^{(+)}(t_0),\\
w_2=&-\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2}(t_0)t_1^2-(d_1^{(-)'}(t_0)-d_1^{(+)'}(t_0))t_1-d_2^{(-)}(t_0)+d_2^{(+)}(t_0)\\
&+\frac{3\bar{y}_1^{(-)}(t_0)h^{(-)}(t_0)\bar{y}_1^{(-)}(t_0)(g(t_0)-\varphi^{(-)}(t_0))^2}{I^{(-)}(t_0)}\\
&-\frac{3\bar{y}_1^{(+)}(t_0)h^{(+)}(t_0)\bar{y}_1^{(+)}(t_0)(g(t_0)-\varphi^{(+)}(t_0))^2}{I^{(+)}(t_0)}\\
&+\frac{\bar{y}_2^{(-)}(t_0)h^{(+)}(t_0)(g(t_0)-\varphi^{(+)}(t_0))^3-M_2^{(-)}(0)}{I^{(-)}(t_0)}\\
&-\frac{\bar{y}_2^{(+)}(t_0)h^{(+)}(t_0)(g(t_0)-\varphi^{(+)}(t_0))^3-M_2^{(+)}(0)}{I^{(-)}(t_0)}\\
&+\frac{I^{(-)'}(t_0)t_1+d_1^{(-)}(t_0)}{\left(I^{(-)}(t_0)\right)^2}\left[-\bar{y}_1^{(-)}(t_0)h^{(-)}(t_0)(g(t_0)-\varphi^{(-)}(t_0))^3+M_1^{(-)}(0)\right]\\
&-\frac{I^{(-)'}(t_0)t_1+d_1^{(+)}(t_0)}{\left(I^{(+)}(t_0)\right)^2}\left[-\bar{y}_1^{(+)}(t_0)h^{(+)}(t_0)(g(t_0)-\varphi^{(+)}(t_0))^3+M_1^{(+)}(0)\right]\\
&+\frac{t_1(g(t_0)-\varphi^{(-)}(t_0))^2\Big[\Big(\bar{y}_1^{(-)'}(t_0)h^{(-)}(t_0)+\bar{y}_1^{(-)}(t_0)h^{(-)'}(t_0)\Big)(g(t_0)-\varphi^{(-)}(t_0))\Big]}{I^{(-)}(t_0)}\\
&-\frac{t_1(g(t_0)-\varphi^{(+)}(t_0))^2\Big[\Big(\bar{y}_1^{(+)'}(t_0)h^{(+)}(t_0)+\bar{y}_1^{(+)}(t_0)h^{(+)'}(t_0)\Big)(g(t_0)-\varphi^{(+)}(t_0))\Big]}{I^{(-)}(t_0)}\\
&+\frac{3t_1(g(t_0)-\varphi^{(-)}(t_0))^2\bar{y}_1^{(-)}(t_0)h^{(-)}(t_0)\left(g'(t_0)-\varphi^{(-)'}(t_0)\right)}{I^{(-)}(t_0)}\\
&-\frac{3t_1(g(t_0)-\varphi^{(+)}(t_0))^2\bar{y}_1^{(+)}(t_0)h^{(+)}(t_0)\left(g'(t_0)-\varphi^{(+)'}(t_0)\right)}{I^{(-)}(t_0)}.
\end{aligned}$$

Thus, the asymptotics of $y(t,\mu)$ to the original problem (1.2) is constructed completely.

3. Existence of a smooth solution of contrast structure type

To prove the existence of a solution to the original problem (1.2) , it is necessary to show that solutions to two auxiliary problems (2.1)-(2.2) exist. Then we need to show that there exists $t^* \in T$ whose asymptotic representation can be also obtained on the discontinuous curve $y = g(t)$ such that the asymptotic solutions to these two problems (2.1)-(2.2) are connected at the point t^* smoothly. It is noted that regular terms $\bar{y}_k^{(\mp)}(t)$ and boundary layer functions $\Pi_k^{(\mp)}(\rho_\mp)$ do not rely on t^*, but internal layer functions $Q_k^{(\mp)}(\tau,\mu)$ depend upon t^*.

3.1. Existence of solutions to auxiliary problems (2.1)-(2.2)

Lemma 3.1. *Under Assumptions 1.1-1.3, for any $t^* \in T$, in the respective regions $N^{(-)} = [0,t^*] \times [g(t^*), l]$ and $N^{(+)} = [t^*, 1] \times [-l, g(t^*)]$, two auxiliary problems* (2.1)-(2.2) *have solutions $y^{(\mp)}(t,\mu)$, whose asymptotic representations are*

$$\begin{cases} y^{(-)}(t,\mu) = Y_n^{(-)}(t,\mu) + O(\mu^{\frac{n+1}{3}}), & 0 \le t \le t^*, \\ y^{(+)}(t,\mu) = Y_n^{(+)}(t,\mu) + O(\mu^{\frac{n+1}{3}}), & t^* \le t \le 1, \end{cases} \tag{3.1}$$

where

$$Y_n^{(\mp)}(t,\mu) = \sum_{i=0}^{n} \mu^{\frac{i}{3}} \left[\bar{y}_i^{(\mp)}(t) + Q_i^{(\mp)}(\tau) + \mu^{\frac{2}{3}} \Pi_i^{(\mp)}(\rho_\mp) \right]. \tag{3.2}$$

In view of the fact that the similar result in case of triple roots in [8] is aimed at a different partial differential equation, the proof of Lemma 3.1 is given with the problem (2.2). Let us introduce the definition of upper and lower solutions to this problem.

Definition 3.1. The functions $\overline{Y}(t,\mu)$ and $\underline{Y}(t,\mu)$ are called upper and lower solutions to the problem (2.2), if they satisfy the following conditions

(i) $\underline{Y}(t,\mu) \le \overline{Y}(t,\mu)$, $t^* \le t \le 1$.

(ii) $L_\mu \underline{Y} \ge 0 \ge L_\mu \overline{Y}$, $t^* < t < 1$, where

$$L_\mu \underline{Y} = \mu^2 \frac{\mathrm{d}^2 \underline{Y}}{\mathrm{d}t^2} - f^{(+)}(\underline{Y}, t, \mu),$$
$$L_\mu \overline{Y} = \mu^2 \frac{\mathrm{d}^2 \overline{Y}}{\mathrm{d}t^2} - f^{(+)}(\overline{Y}, t, \mu).$$

(iii) $\underline{Y}'(1,\mu) \ge 0 \ge \overline{Y}'(1,\mu)$, $\underline{Y}'(t^*,\mu) \le g(t^*) \le \overline{Y}'(t^*,\mu)$.

Notably, it is stated that t^* in this definition is allowed to depend on μ but is fixed.

Proof. To prove Lemma 3.1 by asymptotic differential inequalities, we shall construct the upper and lower solutions as follows

$$\begin{aligned} \overline{Y}(t,\mu) &= Y_n^{(+)}(t,\mu) + \mu^{\frac{n+1}{3}} \gamma, \\ \underline{Y}(t,\mu) &= Y_n^{(+)}(t,\mu) - \mu^{\frac{n+1}{3}} \gamma, \end{aligned} \tag{3.3}$$

where γ is a positive number that is independent of μ.

When μ is sufficiently small and γ is sufficiently large, $\overline{Y}(t,\mu)$ and $\underline{Y}(t,\mu)$ meet Nagumo conditions (i) and (iii).

The Nagumo condition (ii) shall be verified below. Let us take for example $\underline{Y}(x,\mu)$. It follows from the algorithm of constructing asymptotic solution to problem (2.2) that

$$L_\varepsilon Y_n^{(+)} = O(\mu^{\frac{n}{3}+1})(1+\exp(-\kappa\rho_+)) + O(\mu^{\frac{n+1}{3}})Q_\kappa^{(+)}(\tau), \quad t^* \le t \le 1.$$

Thus,

$$\begin{aligned} L_\mu \underline{Y}(t,\mu) =& L_\mu Y_n^{(+)} + h^{(+)}(t)\left[(Y_n^{(+)} - \varphi^{(+)}(t))^2 - (Y_n^{(+)} - \varphi^{(+)}(t) - \mu^{\frac{n+1}{3}}\gamma)^2\right] \\ & - \mu\left[f_1(Y_n^{(+)}, t, \mu) - f_1(Y_n^{(+)} - \mu^{\frac{n+1}{3}}\gamma, t, \mu)\right] \\ =& O(\mu^{\frac{n+1}{3}})\left(Q_\kappa^{(+)} + 3\gamma h^{(+)}(t) Q_0^{(+)}\right) + O(\varepsilon^{\frac{n}{3}+1})(1+\exp(-\kappa\rho_+) \\ & + 3\gamma h^{(+)}(t)\exp(-\kappa\rho_+)). \end{aligned}$$

Taking $Q_0^{(+)}(\tau) > 0$ and Assumption 1.3 into account, if γ is sufficiently large, $L_\mu \underline{Y}(t,\varepsilon) > 0$.

Likewise, for sufficiently large γ, if μ is sufficiently small, then we have $L_\mu \overline{Y}(t,\varepsilon) < 0$.

According to Nagumo theorem, problem (2.2) has a solution $y^{(+)}(t,\mu)$ that satisfies the inequality

$$\underline{Y}(t,\varepsilon) \le y(t,\mu) \le \overline{Y}(t,\mu), \quad t^* \le t \le 1.$$

From (3.3), we have

$$\begin{aligned} \underline{Y}(t,\mu) &= Y_n^{(+)}(t,\mu) + O(\mu^{\frac{n+1}{3}}), \\ \overline{Y}(t,\varepsilon) &= Y_n^{(+)}(t,\mu) + O(\mu^{\frac{n+1}{3}}). \end{aligned}$$

Therefore, the second formula of (3.1) is true. □

Lemma 3.2. *Under Assumptions 1.1-1.3, for sufficiently small parameter $\mu > 0$, the derivatives of $y^{(\mp)}(t,\mu)$ can be written as*

$$\frac{\mathrm{d}y^{(\mp)}}{\mathrm{d}t}(t,\mu) = \frac{\mathrm{d}Y_n^{(\mp)}}{\mathrm{d}t}(t,\mu) + O(\mu^{\frac{n-2}{3}}). \tag{3.4}$$

The proof of asymptotic representation of (3.4) is similar to the proof of corollary in the paper [3] .

3.2. Existence of solution to the original problem (1.2)

In the following, we shall proceed to show that there exists $t^* \in T$ that satisfies the matching condition

$$\frac{\mathrm{d}y^{(-)}}{\mathrm{d}t}(t^*,\mu) = \frac{\mathrm{d}y^{(+)}}{\mathrm{d}t}(t^*,\mu).$$

To this end, we represent t^* as

$$t^* = t_\delta = t_0 + \mu^{\frac{1}{3}} t_1 + \mu^{\frac{2}{3}} t_2 + \cdots + \mu^{\frac{n+1}{3}}(t_n + \delta), \tag{3.5}$$

where δ is a parameter.

Denote

$$\Delta(t^*,\mu)=\mu\left(\frac{\mathrm{d}y^{(-)}}{\mathrm{d}x}(t^*,\mu)-\frac{\mathrm{d}y^{(+)}}{\mathrm{d}t}(t^*,\mu)\right).$$

From $I(x_0)=0$, (2.31) and Lemmas 3.1-3.2, we have

$$\Delta(t^*,\mu)=\mu^{\frac{n}{3}}I'(t_0)\delta+O(\mu^{\frac{n+1}{3}}). \tag{3.6}$$

If μ is sufficiently small, there exists $\bar{\delta}\in(-1,1)$, such that $\Delta(t^*,\mu)=0$. It is proved that there exist t^* such that asymptotic solutions to problems (2.1)-(2.2) are connected at the point $(t^*,g(t^*))$ smoothly. Thus, by Lemma 3.1, the main theorem can be derived.

Theorem 3.1. *If Assumptions 1.1-1.4 hold, then, for sufficiently small $\mu>0$, problem (1.2) has a solution $y(t,\mu)$ whose asymptotic representation is in the form*

$$y(t,\mu)=\begin{cases}\displaystyle\sum_{i=0}^{n}\mu^{\frac{i}{3}}\left[\bar{y}_i^{(-)}(t)+Q_i^{(-)}(\tau)+\mu^{\frac{2}{3}}\Pi_i^{(-)}(\rho_-)\right]+O(\mu^{\frac{n+1}{3}}), & 0\le t\le t^*,\\ \displaystyle\sum_{i=0}^{n}\mu^{\frac{i}{3}}\left[\bar{y}_i^{(+)}(t)+Q_i^{(+)}(\tau)+\mu^{\frac{2}{3}}\Pi_i^{(+)}(\rho_+)\right]+O(\mu^{\frac{n+1}{3}}), & t^*\le t\le 1,\end{cases} \tag{3.7}$$

where

$$t^*=t_0+\mu^{\frac{1}{3}}t_1+\cdots+\mu^{\frac{n}{3}}(t_n+\delta),\quad \tau=\frac{t-t^*}{\mu}.$$

It is worth pointing out that this theorem is proved by Lemma 3.1 and the exsitence of transfer point $(t^*,g(t^*))$. Meanwhile, one obtains the expansion of t^*. The obtained smooth solution has an internal layer in the neighborhood of $y=g(t)$.

4. Numerical Example

Consider Neumann boundary value problem

$$\begin{cases}\mu^2y''=\begin{cases}2\left(y+\dfrac{1}{2}t^2-1\right)^3+2\mu, & (y,t)\in D^{(-)},\\ 2\left(y+\dfrac{1}{4}t^2-\dfrac{2}{9}\right)^3-16\mu, & (y,t)\in D^{(+)},\end{cases}\\ y'(0,\mu)=0,\quad y'(1,\mu)=0,\end{cases} \tag{4.1}$$

where $g(t)=t^2$. Here the degenerate equation $f(y,t,\mu)=0$ has triple roots

$$\bar{y}_0(t)=\begin{cases}-\dfrac{1}{2}t^2+1, & (y,t)\in\bar{D}^{(-)},\\ -\dfrac{1}{4}t^2+\dfrac{2}{9}, & (y,t)\in\bar{D}^{(+)}.\end{cases}$$

By solving the equation

$$I(t_0)=-\left(\frac{3}{2}t_0^2-1\right)^2+\left(\frac{5}{4}t_0^2-\frac{2}{9}\right)^2=0, \tag{4.2}$$

we have $t_0 = \frac{2}{3}$. Here $I(t)$ is defined by (1.7). The computation shows that $I'(t_0) = \frac{22}{9} \neq 0$.

From (2.13), we have

$$\bar{y}_1^{(-)}(t) = -1, \quad \bar{y}_1^{(+)}(t) = 2.$$

It follows from (2.28) that

$$\Pi_0^{(-)}(\rho_-) = 0, \quad \Pi_0^{(+)}(\rho_+) = -\left(4\sqrt{6}\right)^{-1} \exp\left(-2\sqrt{6}\rho_+\right).$$

The problems for determining $Q_0^{(\mp)}(\tau)$ are in the forms

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau} = -\sqrt{(Q_0^{(-)})^2 - 4\mu^{\frac{1}{3}}Q_0^{(-)} + 6\mu^{\frac{2}{3}}}\,Q_0^{(-)}, \quad \tau < 0, \\ Q_0^{(-)}(0) = -\frac{1}{3}; \end{cases} \tag{4.3}$$

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(+)}}{\mathrm{d}\tau} = -\sqrt{(Q_0^{(+)})^2 + 8\mu^{\frac{1}{3}}Q_0^{(+)} + 24\mu^{\frac{2}{3}}}\,Q_0^{(+)}, \quad \tau > 0, \\ Q_0^{(+)}(0) = \frac{1}{3}. \end{cases} \tag{4.4}$$

And they satisfy the inequalities

$$\frac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau} \leq -\sqrt{(Q_0^{(-)})^2 - 2\sqrt{6}\mu^{\frac{1}{3}}Q_0^{(-)} + 6\mu^{\frac{2}{3}}}\,Q_0^{(-)};$$
$$\frac{\mathrm{d}Q_0^{(+)}}{\mathrm{d}\tau} \leq -\sqrt{(Q_0^{(+)})^2 + 8\mu^{\frac{1}{3}}Q_0^{(+)} + 16\mu^{\frac{2}{3}}}\,Q_0^{(+)}.$$

According to comparison principle,

$$Q_0^{(-)}(\tau) \leq \bar{Q}^{(-)}(\tau), \quad Q_0^{(+)}(\tau) \leq \bar{Q}^{(+)}(\tau),$$

here

$$\bar{Q}^{(-)}(\tau) = -\frac{\sqrt{6}\mu^{\frac{1}{3}}\alpha^{(-)}\exp(\sqrt{6}\mu^{\frac{1}{3}}\tau)}{1 - \alpha^{(-)}\exp(\sqrt{6}\mu^{\frac{1}{3}}\tau)}, \quad \alpha^{(-)} = \frac{1}{1 + 3\sqrt{6}\mu^{\frac{1}{3}}},$$

and

$$\bar{Q}^{(+)}(\tau) = \frac{4\mu^{\frac{1}{3}}\beta^{(+)}\exp(-4\mu^{\frac{1}{3}}\tau)}{1 - \beta^{(+)}\exp(-4\mu^{\frac{1}{3}}\tau)}, \quad \beta^{(+)} = \frac{1}{1 + 12\mu^{\frac{1}{3}}}.$$

By Theorem 3.1, one has the following asymptotic representation

$$y(t,\mu) = \begin{cases} -\dfrac{1}{2}t^2 + 1 + \bar{Q}_0^{(-)}(\tau) + O(\mu^{\frac{1}{3}}), \quad 0 \leq t \leq \dfrac{2}{3}, \\ -\dfrac{1}{4}t^2 + \dfrac{2}{9} + \bar{Q}_0^{(+)}(\tau) - \mu^{\frac{2}{3}}\dfrac{\exp\left(-2\sqrt{6}\rho_+\right)}{4\sqrt{6}} + O(\mu^{\frac{1}{3}}), \quad \dfrac{2}{3} \leq t \leq 1. \end{cases}$$

Fig.1 shows the zero-order asymptotic solution to problem (4.1), here $\mu = 0.001$.

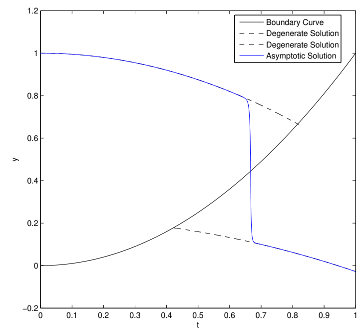

Figure 1. Zero-order asymptotic solution to problem (4.1).

5. Conclusion

Such problems with discontinuous reaction terms along a discontinuity curve in the case of isolated roots of the degenerate equations are studied in [20]. This paper discusses the case of triple roots of the degenerate equation on the left and on the right of the discontinuity point. Here the decay property of internal layer is analyzed by the comparison principle and the existence of solutions to auxiliary problems need to be reconsidered. Finally, the asymptotics of a smooth solution and transition point is constructed. Then we show that there exists a smooth solution with internal layers in the neighborhood of some point on the discontinuous curve. Our results can be used to propose an efficient numerical algorithm for coefficient inverse problems for a nonlinear singularly perturbed reaction-diffusion-advection equation [12].

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