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SOME INTEGRAL REPRESENTATION FORMULAS AND SCHWARZ LEMMAS RELATED TO PERTURBED DIRAC OPERATORS*

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Abstract In this paper, we first obtain some integral representations for perturbed Dirac operators by using the fundamental solutions of the modified Helmholtz equation and Clifford calculus approach. Second, based on the exhaustion of arbitrary open subsets and integral representations, we investigate generalized Cauchy type integral representation formulas. Moreover, we establish Schwarz lemmas for the null solutions of perturbed Dirac operators in $\mathbb{R}^3$. Finally, as applications, we solve a kind of Dirichlet boundary value problem for perturbed Dirac operators and give the explicit representation of the solution.

Keywords Clifford analysis, integral representations, Schwarz Lemma, Dirichlet problems

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1. Introduction

The importance of integral representations formulas in classical complex analysis was recognized at least as early as 1831, when Augustin-Louis Cauchy discovered the famous formula which carries his name. Integral representations in complex analysis, quaternion analysis and Clifford analysis have been systematically developed in [2, 5, 7, 8, 10, 11, 13–17, 19, 24, 26]. These integral representations are powerful mathematical tools for the treatment of many different types of boundary value problems for partial differential equations, for instance Dirichlet problems, Neumann problems and Riemann-Hilbert problems, and so on. We refer to [1, 3, 4, 9, 16–18, 20]. The Schwarz lemma is one of the central results in complex analysis (see [6], [12], [22, 23, 25, 27, 28, 30–34]). It concerns holomorphic self-mappings of the unit disk in the complex plane and obtains general sharp estimates of the values of bounded analytic (holomorphic) functions on the open unit disk of the complex plane. It is natural and important to generalized the classical Schwarz

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Lemma to a higher dimension. Clifford analytic functions theory can be considered as a generalization to higher dimension of the theory of holomorphic functions in the complex plane and is centered around the notion of a regular function, which is a null solution of a Dirac operator, which is factorizes the Laplace operator in the Euclidean spaces $\mathbb{R}^n$. Clifford function theory has been systematically studied in [13–15]. The Schwarz lemma for regular functions in outside the unit ball in $\mathbb{R}^n$ was established by Qian and Yang in [32]. Based on integral representations for harmonic functions and regular functions with value in Clifford algebra $Cl(V_{n,0})$ Zhang proved for the first time Schwarz lemma for harmonic functions and regular functions in inside the unit ball in $\mathbb{R}^n$ which are similar to the classical results in [33, 34].

In this article, motivated by [20, 21, 32–34], under framework of the Clifford algebra $Cl(V_{3,3})$, we obtain Borel-Pompeiu integral representations associated to perturbed Dirac operators by Clifford calculus, Stokes formula and the fundamental solution of the modified Helmholtz operators. With the help of these integral representation formula, we obtain generalized Cauchy integral formulas related to perturbed Dirac operators using a compact exhaustion technique about integral domains in $\mathbb{R}^3$. Using integral representation formulas and integral inequalities, we establish Schwarz Lemmas related to perturbed Dirac operators in $\mathbb{R}^3$. As another application about these integral representation, we solve a kind of Dirichlet boundary value problem for perturbed Dirac operators. The explicit representation of the solution is also given.

2. Preliminaries

Let $\mathcal{A} := \mathbb{R}(e_1, e_2, e_3)$ denote the free $\mathbb{R}$-algebra with n indeterminants $\{e_1, \ldots, e_3\}$. Let J be the two-sided ideal in $\mathcal{A}$ generated by the elements

$$\{e_i^2 - 1, i = 1, \ldots, s; e_i^2 + 1, i = s+1, \ldots, n; e_ie_j + e_je_i, 1 \leq i < j \leq 3\}.$$

The quotient algebra $Cl(V_{3,s}) := \mathcal{A}/J$ is called the Clifford algebra with parameters $3, s$. Without risk of ambiguity, we take the usual practice of using the same symbol to denote an indeterminant e_i in $\mathcal{A}$ and its equivalent class in $\mathcal{A}/J$. Therefore, $e_1, \cdots, e_3$ considered as elements of $\mathcal{A}/J$ have the following relations:

$$\begin{cases} e_i^2 = 1, & i = 1, \ldots, s, \\ e_i^2 = -1, & i = s+1, \ldots, 3, \\ e_ie_j + e_je_i = 0, \ i \neq j. \end{cases}$$

Set

$$e_{l_1 \ldots l_r} := e_{l_1} \cdots e_{l_r}, \ \text{ while } 1 \leq l_1 < \cdots < l_r \leq 3.$$

For more information on $Cl(V_{3,s})$, we refer to [11, 15]. In this article, we only consider $s = 3$. Thus $Cl(V_{3,3})$ is a real linear non-commutative algebra. An involution is defined by

$$\begin{cases} \overline{e}_A = (-1)^{\frac{n(A)(n(A)+3)}{2}} e_A, \text{ if } A \in \mathcal{P}N, \\ \overline{\lambda} = \sum\limits_{A \in \mathcal{P}N} \lambda_A \overline{e_A}, \qquad \text{if } \lambda = \sum\limits_{A \in \mathcal{P}N} \lambda_A e_A, \end{cases}$$

where

$$\{e_A, A = \{l_1, \dots, l_r\} \in \mathcal{P}N, 1 \le l_1 < \cdots < l_r \le n\},$$

$n(A)$ is the cardinal number of the set A, N stands for the set $\{1, 2, \cdots, n\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N in the above way. The norm of λ is defined by $\|\lambda\| = (\sum\limits_{A\in\mathcal{P}N} |\lambda_A|^2)^{\frac{1}{2}}$. If $\Re e(\lambda)$ denotes the scalar portion of $\lambda \in Cl(V_{3,3})$, then it follows

$$\Re e(\lambda\overline{\lambda}) = \Re e(\overline{\lambda}\lambda) = \sum_{A\in\mathcal{P}N} |\lambda_A|^2 = \|\lambda\|^2.$$

Lemma 2.1. *Suppose that* $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{y} \in \mathbb{R}^3$ *and* $\lambda \in Cl(V_{3,3})$. *Then*

$$\|\lambda\mathbf{x}\| = \|\lambda\|\|\mathbf{x}\|, \tag{2.1}$$

$$\|\mathbf{x}\lambda\| = \|\lambda\|\|\mathbf{x}\|, \tag{2.2}$$

$$\|\lambda\mathbf{x}\mathbf{y}\| = \|\mathbf{x}\lambda\mathbf{y}\| = \|\mathbf{x}\mathbf{y}\lambda\| = \|\lambda\|\|\mathbf{x}\|\|\mathbf{y}\|. \tag{2.3}$$

Proof. The equality (2.1) can be directly proved as follows

$$\|\lambda\mathbf{x}\|^2 = \Re e(\lambda\mathbf{x}\overline{\lambda\mathbf{x}}) = \Re e(\lambda\mathbf{x}\overline{\mathbf{x}}\overline{\lambda}) = \|\mathbf{x}\|^2\Re e(\lambda\overline{\lambda}) = \|\mathbf{x}\|^2\|\lambda\|^2.$$

The result equality (2.2) can be similarly proved as (2.1). By (2.1) and (2.2), the equality (2.3) holds. The proof is finished. □

Lemma 2.2. *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ *and* $\mathbf{x} \neq \mathbf{y}$. *Then*

$$\left\|\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} - \frac{\mathbf{y}}{\|\mathbf{y}\|^3}\right\| \le \frac{2\|\mathbf{y}\| + \|\mathbf{x}\|}{\|\mathbf{y}-\mathbf{x}\|^2\|\mathbf{y}\|^2}\|\mathbf{x}\| \tag{2.4}$$

and

$$\left\|\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} - \frac{\mathbf{y}}{\|\mathbf{y}\|^2}\right\| \le \frac{1}{\|\mathbf{y}-\mathbf{x}\|\|\mathbf{y}\|}\|\mathbf{x}\|. \tag{2.5}$$

Proof. For $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ and $\mathbf{x} \neq \mathbf{y}$, by Lemma 2.1, we have

$$\Gamma \triangleq \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} - \frac{\mathbf{y}}{\|\mathbf{y}\|^3} = \frac{(\mathbf{y}-\mathbf{x})[\|\mathbf{y}\|\mathbf{y} - (\mathbf{y}-\mathbf{x})\|\mathbf{y}-\mathbf{x}\|]\mathbf{y}}{\|\mathbf{y}-\mathbf{x}\|^3\|\mathbf{y}\|^3},$$

it is easy to check that

$$\Gamma \le \frac{2\|\mathbf{y}\| + \|\mathbf{x}\|}{\|\mathbf{y}-\mathbf{x}\|^2\|\mathbf{y}\|^2}\|\mathbf{x}\|.$$

Obviously, the inequality (2.4) follows. Using the similar method, we can prove the inequality (2.5). The proof is done. □

Suppose Ω be an open bounded non-empty subset of $\mathbb{R}^3$. We now introduce the Dirac operator $D = \sum\limits_{i=1}^{3} e_i \frac{\partial}{\partial x_i}$. In particular, we have that $DD = \Delta$ where Δ is the Laplacian over $\mathbb{R}^3$. A function $u : \Omega \mapsto Cl(V_{3,3})$ is said to be left monogenic if it satisfies the equation $D[u](\mathbf{x}) = 0$ for each $\mathbf{x} \in \Omega$. A similar definition can be given for right monogenic functions. Elementary properties of the Dirac operators and left monogenic functions can be found in References [11, 15]. In [29], the elliptic partial differential operator $H = (\Delta - \kappa^2)$, for $\kappa \geq 0$, corresponds to the modified Helmholtz equation:

$$Hu = (\Delta - \kappa^2)u = 0,$$

which has the fundamental solution as follows:

$$E_1(\mathbf{x}, \kappa^2) = \frac{e^{-\kappa\|\mathbf{x}\|}}{4\pi\|\mathbf{x}\|}. \tag{2.6}$$

Denote

$$L_\kappa u = Du + \kappa u, \qquad L_{-\kappa} u = Du - \kappa u.$$

Using the above Clifford algebra $Cl(V_{3,3})$, then the modified Helmholtz equation may be written as

$$L_\kappa L_{-\kappa} u = L_{-\kappa} L u = 0.$$

Let

$$K_1(\mathbf{y}-\mathbf{x}, \kappa) = \frac{1}{4\pi}\left(\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \frac{\kappa(\mathbf{y}-\mathbf{x})}{\|\mathbf{y}-\mathbf{x}\|^2} + \frac{\kappa}{\|\mathbf{y}-\mathbf{x}\|}\right) e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}, \tag{2.7}$$

$$K_{*1}(\mathbf{y}-\mathbf{x}, \kappa) = \frac{1}{4\pi}\left(\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \frac{\kappa(\mathbf{y}-\mathbf{x})}{\|\mathbf{y}-\mathbf{x}\|^2} - \frac{\kappa}{\|\mathbf{y}-\mathbf{x}\|}\right) e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}, \tag{2.8}$$

where $\mathbf{y}-\mathbf{x} = \sum_{i=1}^{3}(y_i - x_i)e_i$. It is clear that $K_1(\mathbf{y}-\mathbf{x}, \kappa)$ and $K_{*1}(\mathbf{y}-\mathbf{x}, \kappa)$ are fundamental solutions of $L_\kappa = \sum\limits_{i=1}^{3} e_i \frac{\partial}{\partial y_i} + \kappa$ and $L_{-\kappa} = \sum\limits_{i=1}^{3} e_i \frac{\partial}{\partial y_i} - \kappa$, respectively.

Lemma 2.3. *Let $E_1(\mathbf{y}-\mathbf{x}, \kappa^2), K_1(\mathbf{y}-\mathbf{x}, \kappa)$ and $K_{*1}(\mathbf{y}-\mathbf{x}, \kappa)$ be as in* (2.6), (2.7) *and* (2.8) *respectively. Then*

$$\begin{cases} L_{-\kappa}[E_1(\mathbf{y}-\mathbf{x}, \kappa^2)] = [E_1(\mathbf{y}-\mathbf{x}, \kappa^2)]L_{-\kappa} = K_1(\mathbf{y}-\mathbf{x}, \kappa) \\ L_{\kappa}[E_1(\mathbf{y}-\mathbf{x}, \kappa^2)] = [E_1(\mathbf{y}-\mathbf{x}, \kappa^2)]L_{\kappa} = K_{*1}(\mathbf{y}-\mathbf{x}, \kappa) \\ L_{-\kappa}[K_{*1}(\mathbf{y}-\mathbf{x}, \kappa)] = [K_{*1}(\mathbf{y}-\mathbf{x}, \kappa)]L_{-\kappa} = 0 \\ L_{\kappa}[K_1(\mathbf{y}-\mathbf{x}, \kappa)] = [K_1(\mathbf{y}-\mathbf{x}, \kappa)]L_{\kappa} = 0. \end{cases}$$

where $\mathbf{y} \in \mathbb{R}^3 \setminus \{\mathbf{x}\}$.

3. Some integral representation formulas in Clifford analysis

Suppose M be a 3-dimensional differentiable and oriented manifold contained in some open subset Ω of $\mathbb{R}^3$. The $Cl(V_{3,3})$-value 2-differential form

$$\begin{aligned} d\sigma &= e_1 dx_2 \wedge dx_3 + e_2 dx_3 \wedge dx_1 + e_3 dx_1 \wedge dx_2 \\ &= \sum_{i=1}^{3} (-1)^{i-1} e_i d\widehat{x}_i^N \end{aligned}$$

is exact, where

$$d\widehat{x}_i^N = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_3.$$

If dS stands for the classical surface element and

$$\mathbf{n} = \sum_{i=1}^{3} e_i \mathbf{n}_i,$$

where $\mathbf{n}_i$ is the i-th component of the outward pointing normal, then the Clifford-valued surface element $d\sigma$ can be written as

$$d\sigma = \mathbf{n} dS. \tag{3.1}$$

By Stokes' Theorem in Clifford analysis in [13], we can prove the following the lemma.

Lemma 3.1. *Let $u,\ v \in C^1(\Omega, Cl(V_{3,3}))$. Then for any 3-chain C on $M \subset \Omega$,*

$$\begin{aligned}\int\limits_{\partial C} v d\sigma u &= \int\limits_{C} [v] L_\kappa u dV + \int\limits_{C} v L_{-\kappa}[u] dV\\ &= \int\limits_{C} [v] L_{-\kappa} u dV + \int\limits_{C} v L_{\kappa}[u] dV.\end{aligned}$$

where dV denotes the Lebesgue volume measure.

Proof. By Theorem 2 in [13], we have

$$\begin{aligned}\int\limits_{C} [v] L_\kappa u dV + \int\limits_{C} v L_{-\kappa}[u] dV &= \int\limits_{C} [v] L_{-\kappa} u dV + \int\limits_{C} v L_{\kappa}[u] dV\\ &= \int\limits_{C} ([v] D u + v D[u]) dV\\ &= \int\limits_{\partial C} v d\sigma u.\end{aligned}$$

$\square$

Theorem 3.1. *Let $\widetilde{\Omega}$ be an open nonempty of $\mathbb{R}^3$ and $\Omega \subset \widetilde{\Omega}$ be a domain with piecewise C^1 boundary. Then, for $u \in C^1(\widetilde{\Omega}, Cl(V_{3,3}))$*

$$\begin{aligned}&\int\limits_{\partial\Omega} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}) - \int\limits_{\Omega} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) L_\kappa[u](\mathbf{y}) dV\\ &= \begin{cases} u(\mathbf{x}), \mathbf{x} \in \Omega, \\ 0, \qquad \mathbf{x} \in \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases}\end{aligned} \tag{3.2}$$

where $K_{*1}(\mathbf{y}-\mathbf{x},\kappa)$ is as in (2.8).

Proof. Let $\mathbf{x} \in \widetilde{\Omega} \setminus \overline{\Omega}$. By Lemma 3.1 and Lemma 2.3, the left hand side of the stated formula (3.2) reduces to

$$\int\limits_{\Omega} [K_{*1}(\mathbf{y}-\mathbf{x},\kappa)] L_{\kappa} u(\mathbf{y}) dV = \int\limits_{\Omega} [K_{*1}(\mathbf{y}-\mathbf{x},\kappa)] L_{-\kappa} u(\mathbf{y}) dV$$

which apparently equals to zero.

Now take $\mathbf{x} \in \Omega$ and take $r > 0$ such that $B(\mathbf{x}, r) \subset \Omega$. Invoking the previous case, we may then write

$$\int\limits_{\partial(\Omega\setminus B(\mathbf{x},r))} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}) = \int\limits_{\Omega\setminus B(\mathbf{x},r)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) L_{\kappa}[u](\mathbf{y}) dV. \quad (3.3)$$

Denote

$$F(r) \triangleq \int\limits_{\Omega\setminus B(\mathbf{x},r)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) L_{\kappa}[u](\mathbf{y}) dV, \quad (3.4)$$

applying spherical coordinates, $0 < r_1 < r_2$, we find

$$\begin{aligned}
\|F(r_2) - F(r_1)\| &\le \int\limits_{B(\mathbf{x},r_2)\setminus B(\mathbf{x},r_1)} \frac{1}{4\pi}\left(\frac{1}{\|\mathbf{y}-\mathbf{x}\|^2} + \frac{2\kappa}{\|\mathbf{y}-\mathbf{x}\|}\right)\|L_{\kappa}[u](\mathbf{y})\| dV \\
&= \frac{1}{4\pi}\int_{S^2}\int_{r_1}^{r_2} (1+r)\|L_{\kappa}[u](\mathbf{y})\| dr d\omega \\
&\le C(r_2 - r_1).
\end{aligned}$$

Hence, we obtain that

$$\lim_{r\to 0} \int\limits_{\Omega\setminus B(\mathbf{x},r)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) L_{\kappa}[u](\mathbf{y}) dV = \int\limits_{\Omega} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) L_{\kappa}[u](\mathbf{y}) dV.$$

As to the left hand side of (3.3), we can write the following form

$$\int\limits_{\partial\Omega} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}) - \int\limits_{\partial B(\mathbf{x},r)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}), \quad (3.5)$$

we denote

$$\Theta(\mathbf{x}) \triangleq \int\limits_{\partial B(\mathbf{x},r)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}),$$

it follows from the classical Stokes formula that

$$\begin{aligned}
\Theta(\mathbf{x}) =& \frac{3e^{-\kappa r}}{4\pi r^3} \int\limits_{B(\mathbf{x},r)} u(\mathbf{y}) dV + \frac{3\kappa e^{-\kappa r}}{4\pi r^2} \int\limits_{B(\mathbf{x},r)} u(\mathbf{y}) dV \\
&+ \frac{e^{-\kappa r}}{4\pi r^3} \int\limits_{B(\mathbf{x},r)} (\mathbf{y}-\mathbf{x}) D[u](\mathbf{y}) dV + \frac{\kappa e^{-\kappa r}}{4\pi r^2} \int\limits_{B(\mathbf{x},r)} (\mathbf{y}-\mathbf{x}) D[u](\mathbf{y}) dV \\
&- \frac{\kappa e^{-\kappa r}}{4\pi r} \int\limits_{B(\mathbf{x},r)} D[u](\mathbf{y}) dV,
\end{aligned}$$

in view of the Lebesgue differential theorem, we have

$$\lim_{r\to 0}\Theta(\mathbf{x}) = u(\mathbf{x}). \tag{3.6}$$

Using (3.3)-(3.6), the result follows. The proof is done. □

Theorem 3.2. *Let $\widetilde{\Omega}$ be an open nonempty of $\mathbb{R}^3$ and $\Omega \subset \widetilde{\Omega}$ be a domain with piecewise C^1 boundary. Then, for $u \in C^1(\widetilde{\Omega}, Cl(V_{3,3}))$*

$$\int_{\partial\Omega} K_1(\mathbf{y}-\mathbf{x},\kappa)d\sigma_{\mathbf{y}}u(\mathbf{y}) - \int_{\Omega} K_1(\mathbf{y}-\mathbf{x},\kappa)L_{-\kappa}[u](\mathbf{y})dV$$
$$= \begin{cases} u(\mathbf{x}), \ \mathbf{x}\in\Omega \\ 0, \qquad \mathbf{x}\in\widetilde{\Omega}\setminus\overline{\Omega}, \end{cases}$$

where $K_1(\mathbf{y}-\mathbf{x},\kappa)$ is as in (2.7).

Proof. The result can be similarly proved as Theorem 3.1. □

From Theorems 3.1 − 3.2, the following results can be directly proved.

Theorem 3.3. *Suppose that $\widetilde{\Omega}$ is an open nonempty of $\mathbb{R}^3$, $\Omega \subset \widetilde{\Omega}$ is a domain with piecewise C^1 boundary and $L_{\kappa}[u] = 0$ in $\widetilde{\Omega}$. Then*

$$\int_{\partial\Omega} K_{*1}(\mathbf{y}-\mathbf{x},\kappa)d\sigma_{\mathbf{y}}u(\mathbf{y}) = \begin{cases} u(\mathbf{x}), \ \mathbf{x}\in\Omega \\ 0, \qquad \mathbf{x}\in\widetilde{\Omega}\setminus\overline{\Omega}, \end{cases}$$

where $K_{*1}(\mathbf{y}-\mathbf{x},\kappa)$ is as in (2.8).

Theorem 3.4. *Suppose that $\widetilde{\Omega}$ is an open nonempty of $\mathbb{R}^3$, $\Omega \subset \widetilde{\Omega}$ is a domain with piecewise C^1 boundary and $L_{-\kappa}[u] = 0$ in $\widetilde{\Omega}$. Then*

$$\int_{\partial\Omega} K_1(\mathbf{y}-\mathbf{x},\kappa)d\sigma_{\mathbf{y}}u(\mathbf{y}) = \begin{cases} u(\mathbf{x}), \ \mathbf{x}\in\Omega \\ 0, \qquad \mathbf{x}\in\widetilde{\Omega}\setminus\overline{\Omega}, \end{cases}$$

where $K_1(\mathbf{y}-\mathbf{x},\kappa)$ is as in (2.7).

In order to prove the generalized Cauchy integral formula, we need the following topological technique. Fix an increasing sequence of compact sets $\{K_l\}$, such that

1. $K_1 \subset\subset intK_2 \subset\subset \ldots K_l \subset\subset intK_{l+1} \subset\subset \ldots \subset \Omega$ i.e., K_l is contained in the interior of K_{l+1} for all $l = 1, 2, \ldots$.
2. $\bigcup_{l=1}^{+\infty} K_l = \Omega$.

A sequence $\{K_l\}$ which satisfies the above conditions (1) and (2) is called a normal exhaustion of Ω. For more details, we refer to [26].

Lemma 3.2. *Any bounded domain Ω with piecewise C^1 boundary in $\mathbb{R}^3$ has an exhaustion.*

Proof. Let K_l denote the set of all points in Ω at distance $\geq \frac{1}{l}$ from the boundary of Ω. □

For the general case about Theorem 3.3 and Theorem 3.4, by the help of Lemma 3.2, apply Theorem 3.3 and Theorem 3.4 to a suitable exhaustion of Ω by domain $\Omega_l \subset\subset \Omega$, $l = 1, 2, \ldots$, and pass to the limit $k \to +\infty$, we then have the following results:

Theorem 3.5. *If $u(\mathbf{x}) \in C^1(\Omega, Cl(V_{3,3})) \bigcap C(\overline{\Omega}, Cl(V_{3,3}))$ and $L_\kappa[u] = 0$ in Ω, then*

$$\int\limits_{\partial\Omega} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}) = \begin{cases} u(\mathbf{x}), \ \mathbf{x} \in \Omega \\ 0, \qquad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

where $K_{*1}(\mathbf{y}-\mathbf{x},\kappa)$ is as in (2.8).

Theorem 3.6. *If $u(\mathbf{x}) \in C^1(\Omega, Cl(V_{3,3})) \bigcap C(\overline{\Omega}, Cl(V_{3,3}))$ and $L_{-\kappa}[u] = 0$ in Ω, then*

$$\int\limits_{\partial\Omega} K_{1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}) = \begin{cases} u(\mathbf{x}), \ \mathbf{x} \in \Omega \\ 0, \qquad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega} \end{cases}$$

where $K_1(\mathbf{y}-\mathbf{x},\kappa)$ is as in (2.7).

Remark 3.1. For $\kappa = 0$, Theorem 3.3 and Theorem 3.4 are the classical generalized Cauchy formula in Clifford analysis.

4. Schwarz lemmas related to perturbed Dirac operators in $\mathbb{R}^3$

In this section, we shall consider the Schwarz-type lemma for null solutions of $L_{-\kappa}$ and $L_{-\kappa}$ by the above integral representations in Clifford analysis.

Theorem 4.1. *Let $B(0,R_1)$ be an open ball with center origin and radius R_1 in $\mathbb{R}^3$, $u \in C^1(B(0,R_1), Cl(V_{3,3}))$, $L_\kappa[u] = 0$ in $B(0,R_1)$, $\|u(\mathbf{x})\| \leq R_2$, $\forall \mathbf{x} \in B(0,R_1)$ and $u(0) = 0$. Then for $\forall \mathbf{x} \in B(0,R_1)$,*

$$\|u(\mathbf{x})\| \leq \begin{cases} \frac{2\phi(\kappa)}{\varphi(\kappa)-\sqrt{\varphi^2(\kappa)-4\phi(\kappa)}} \frac{R_2}{R_1}\|\mathbf{x}\|, \ 0 \leq \kappa < \frac{1}{2}, \\ \frac{2\phi(\kappa)}{\varphi(\kappa)-\sqrt{\varphi^2(\kappa)-4\phi(\kappa)}} \frac{R_2}{R_1}\|\mathbf{x}\|, \ \frac{1}{2} < \kappa < +\infty, \\ 5\frac{R_2}{R_1}\|\mathbf{x}\|, \qquad \kappa = \frac{1}{2}, \end{cases} \tag{4.1}$$

where $\phi(\kappa) = 2\kappa^2 + \kappa - 1$, $\varphi(\kappa) = 2\kappa^2 + 3\kappa + 3$.

Proof. 1. We first investigate a special case when $R_1 = R_2 = 1$. For $\forall \mathbf{x} \in B(0,r)$, $0 < r < 1$, by Theorem 3.3

$$u(\mathbf{x}) = \int\limits_{\partial B(0,r)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}).$$

In view of $u(0) = 0$, we have

$$
\begin{aligned}
u(\mathbf{x}) =& \int\limits_{\partial B(0,r)} [K_{*1}(\mathbf{y}-\mathbf{x},\kappa) - K_{*1}(\mathbf{y},\kappa)] d\sigma_{\mathbf{y}} u(\mathbf{y}) \\
=& \frac{1}{4\pi} \int\limits_{\partial B(0,r)} [\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|^3} e^{-\kappa\|\mathbf{y}\|}] d\sigma_{\mathbf{y}} u(\mathbf{y}) \\
&+ \frac{\kappa}{4\pi} \int\limits_{\partial B(0,r)} [\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|^2} e^{-\kappa\|\mathbf{y}\|}] d\sigma_{\mathbf{y}} u(\mathbf{y}) \\
&- \frac{\kappa}{4\pi} \int\limits_{\partial B(0,r)} [\frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}}{\|\mathbf{y}-\mathbf{x}\|} - \frac{e^{-\kappa\|\mathbf{y}\|}}{\|\mathbf{y}\|}] d\sigma_{\mathbf{y}} u(\mathbf{y}) \triangleq \mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3.
\end{aligned}
$$

We treat $\mathbb{I}_1$ first:

$$
\begin{aligned}
\mathbb{I}_1 =& \frac{1}{4\pi} \int\limits_{\partial B(0,r)} [\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} - \frac{\mathbf{y}}{\|\mathbf{y}\|^3}] e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} d\sigma_{\mathbf{y}} u(\mathbf{y}) \\
&+ \frac{1}{4\pi} \int\limits_{\partial B(0,r)} \frac{\mathbf{y}}{\|\mathbf{y}\|^3} [e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - e^{-\kappa\|\mathbf{y}\|}] d\sigma_{\mathbf{y}} u(\mathbf{y}).
\end{aligned} \tag{4.2}
$$

Using (3.1), (4.2), (2.4) in Lemma 2.2 and Poisson integral formula, we have

$$
\begin{aligned}
\|\mathbb{I}_1\| \leq& \frac{\|\mathbf{x}\|(2r+\|\mathbf{x}\|)}{4\pi r^2} \int\limits_{\partial B(0,r)} \frac{1}{\|\mathbf{y}-\mathbf{x}\|^2} dS + \kappa\|\mathbf{x}\| \\
\leq& \frac{\|\mathbf{x}\|(2r+\|\mathbf{x}\|)}{r(r-\|\mathbf{x}\|)} \frac{1}{4\pi r} \int\limits_{\partial B(0,r)} \frac{r^2-\|\mathbf{x}\|^2}{\|\mathbf{y}-\mathbf{x}\|^3} dS + \kappa\|\mathbf{x}\| \\
=& \left[\frac{2r+\|\mathbf{x}\|}{r(r-\|\mathbf{x}\|)} + \kappa\right] \|\mathbf{x}\|.
\end{aligned} \tag{4.3}
$$

To evaluate $\mathbb{I}_2$ second:

$$
\begin{aligned}
\mathbb{I}_2 =& \frac{\kappa}{4\pi} \int\limits_{\partial B(0,r)} [\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} - \frac{\mathbf{y}}{\|\mathbf{y}\|^2}] e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} d\sigma_{\mathbf{y}} u(\mathbf{y}) \\
&+ \frac{\kappa}{4\pi} \int\limits_{\partial B(0,r)} \frac{\mathbf{y}}{\|\mathbf{y}\|^2} [e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - e^{-\kappa\|\mathbf{y}\|}] d\sigma_{\mathbf{y}} u(\mathbf{y}).
\end{aligned}
$$

In view of (2.5) in Lemma 2.2, we obtain that

$$
\begin{aligned}
\|\mathbb{I}_2\| \leq& \left[\frac{\kappa}{4\pi r} \int\limits_{\partial B(0,r)} \frac{1}{\|\mathbf{y}-\mathbf{x}\|} dS + \kappa^2 r\right] \|\mathbf{x}\| \\
\leq& \left[\frac{\kappa r}{r-\|\mathbf{x}\|} + \kappa^2 r\right] \|\mathbf{x}\|.
\end{aligned} \tag{4.4}
$$

Finally, using the similar method in the proof of $\mathbb{I}_2$, we obtain

$$\|\mathbb{I}_3\| \leq \left[\frac{r\kappa}{r-\|\mathbf{x}\|} + \kappa^2 r\right]\|\mathbf{x}\|. \tag{4.5}$$

Combining (4.3), (4.4) with (4.5) and taking $r \to 1$, we have

$$\|u(\mathbf{x})\| \leq \left[\frac{2+\|\mathbf{x}\|}{1-\|\mathbf{x}\|} + \frac{2\kappa}{1-\|\mathbf{x}\|} + 2\kappa^2 + \kappa\right]\|\mathbf{x}\|. \tag{4.6}$$

From known conditions, we get

$$\|u(\mathbf{x})\| \leq \|\mathbf{x}\|\frac{1}{\|\mathbf{x}\|} \text{ when } 0 < \|\mathbf{x}\| < 1. \tag{4.7}$$

Denote $\Phi(t) = \frac{2+t}{1-t} + \frac{2\kappa}{1-t} + 2\kappa^2 + \kappa$ and $\Psi(t) = \frac{1}{t}$, where $t = \|\mathbf{x}\|$, because $\kappa \geq 0$, then

$$\Phi'(t) = \frac{2\kappa+3}{(1-t)^2} > 0 \text{ for } 0 \leq t < 1$$

and

$$\Psi'(t) = -\frac{1}{t^2} < 0 \text{ for } 0 < t \leq 1.$$

Since $\lim\limits_{t\to 1^-}\Phi(t) = +\infty$ and $\lim\limits_{t\to 0^+}\Psi(t) = +\infty$, it can be prove that

$$\begin{aligned} &\sup_{0\leq t<1} \min\{\Psi(t), \Phi(t)\} \\ &= \begin{cases} \frac{2(2\kappa^2+\kappa-1)}{2\kappa^2+3\kappa+3-\sqrt{(2\kappa^2+3\kappa+3)^2-4(2\kappa^2+\kappa-1)}}, & 0 \leq \kappa < \frac{1}{2}, \\ \frac{2(2\kappa^2+\kappa^2-1)}{2\kappa^2+3\kappa+3+\sqrt{(2\kappa^2+3\kappa+3)^2-4(2\kappa^2+\kappa-1)}}, & \frac{1}{2} < \kappa < +\infty, \\ 5, & \kappa = \frac{1}{2}, \end{cases} \end{aligned} \tag{4.8}$$

in view of (4.6), (4.7) and (4.8), the result follows.

2. For the general case, let $f(\mathbf{y}) := \frac{u(R_1\mathbf{y})}{R_2}$, $\|\mathbf{y}\| \leq 1$, it is obviously that $L_{-\kappa}[f] = 0$ in $B(0,1)$, $f(0) = 0$ and $\|f(\mathbf{y})\| \leq 1$, $\forall \mathbf{y} \in B(0,1)$. In view of the result of the step 1, the inequality (4.1) follows. The proof is done. □

By Lemma 2.2 and Theorem 3.4, the following theorem can be similarly proved as Theorem 4.1.

Theorem 4.2. *Let $B(0,R_1)$ be an open ball with center origin and radius R_1 in $\mathbb{R}^3$, $u \in C^1(B(0,R_1), Cl(V_{3,3}))$, $L_{-\kappa}[u] = 0$ in $B(0,R_1)$, $\|u(\mathbf{x})\| \leq R_2$, $\forall \mathbf{x} \in B(0,R_1)$ and $u(0) = 0$. Then for $\forall \mathbf{x} \in B(0,R_1)$,*

$$\|u(\mathbf{x})\| \leq \begin{cases} \frac{2\phi(\kappa)}{\varphi(\kappa)-\sqrt{\varphi^2(\kappa)-4\phi(\kappa)}}\frac{R_2}{R_1}\|\mathbf{x}\|, & 0 \leq \kappa < \frac{1}{2}, \\ \frac{2\phi(\kappa)}{\varphi(\kappa)-\sqrt{\varphi^2(\kappa)-4\phi(\kappa)}}\frac{R_2}{R_1}\|\mathbf{x}\|, & \frac{1}{2} < \kappa < +\infty, \\ 5\frac{R_2}{R_1}\|\mathbf{x}\|, & \kappa = \frac{1}{2}, \end{cases}$$

where $\phi(\kappa) = 2\kappa^2 + \kappa - 1$, $\varphi(\kappa) = 2\kappa^2 + 3\kappa + 3$.

5. An application of the generalized integral representation formulas

Integral representation formulas play a very important role in classical complex analysis and Clifford analysis and are powerful tools to solve many different types of boundary value problems, for instance, Dirichlet problem, Riemann-Hilbert problems and so on. As other applications of the above integral representation formulas, we consider the following Dirichlet boundary value problems:

Theorem 5.1. *Let $g(\mathbf{x}) \in C_c^1(B(0,1), Cl(V_{3,3}))$. The Dirichlet boundary value problems:*

$$\begin{cases} L_\kappa[u] = g \ \textit{in}\ B(0,1) \\ u = 0 \qquad \textit{on}\ \partial B(0,1) \end{cases} \tag{5.1}$$

has the unique solution

$$u(\mathbf{x}) = -\int\limits_{B(0,1)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) g(\mathbf{y}) dV.$$

Proof. By using Theorem 3.1, the solution of (5.1) is formulated as

$$u(\mathbf{x}) = \int\limits_{\partial B(0,1)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) d\sigma_{\mathbf{y}} u(\mathbf{y}) - \int\limits_{B(0,1)} K_{*1}(\mathbf{y}-\mathbf{x},\kappa) L_\kappa[u](\mathbf{y}) dV,$$

since $u = 0$ on $\partial B(0,1)$.For the uniqueness we can prove that the problem

$$\begin{cases} L_\kappa[\widetilde{u}] = 0 \text{ in } B(0,1) \\ \widetilde{u} = 0 \qquad \text{on } \partial B(0,1) \end{cases}$$

has the only solution $\widetilde{u} = 0$. The result follows. □

From Theorem 3.2, we also have the following result which can be similarly proved to Theorem 5.2.

Theorem 5.2. *Let $g(\mathbf{x}) \in C_c^1(B(0,1), Cl(V_{3,3}))$. The solution of the Dirichlet boundary value problems:*

$$\begin{cases} L_{-\kappa}[u] = g \ \textit{in}\ B(0,1) \\ u = 0 \qquad \textit{on}\ \partial B(0,1) \end{cases}$$

is

$$u(\mathbf{x}) = -\int\limits_{B(0,1)} K_1(\mathbf{y}-\mathbf{x},\kappa) g(\mathbf{y}) dV.$$

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